A prismatic-étale comparison theorem in the semi-stable case

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§1 Introduction

$K/\mathbb{Q}_p$ fin, $K \supseteq \mathcal{O}_K \supseteq m_K \twoheadrightarrow k = \mathcal{O}_K/m_K$, $W = W(k)$.

Thm (Cst-conj) X proper and semi-stable sch $/\mathcal{O}_K$.

Then $\exists$ canonical

$$H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \simeq H^i_{log\text{-}cr}(X_k/W) \otimes_W B_{st}$$

Compatible w/ $Gal(\bar{K}/K)$, φ, N-action.

N: nilp, $N\varphi = p \cdot \varphi$.

- Compatible w/ Fontaine-Janssen.
- Proved by Tsuji (1999), Faltings (2002). Niziol, Beilinson, ...

Q Does there exist a p-adic comparison thm for more general loc systems on $X_{K,ét}$?

Partial answers: Faltings (1990, 1999),

Tsuji (secret notes, ~2006).

H. Guo - Reinecke (Crystalline case, 2022).

Let X proper sm $/\mathrm{Spf}\, \mathcal{O}_K$.

$$\begin{pmatrix} \text{étale coh of crystalline} \\ \mathbb{Z}_p\text{-loc system on } X^{ad}_{K,ét} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{crystalline coh of F-crystals} \\ \text{on } (X_k/W)_{crys} \end{pmatrix}$$

$$\updownarrow \qquad\qquad\qquad \nearrow\!\!\swarrow$$

$$\left(\text{Prismatic coh of F-crystals on } (X_{\mathcal{O}_C}/A_{inf})_{\Delta}\right)$$

Here $(\mathfrak{X}_{\mathcal{O}_C}/A_{inf})_{\Delta}$ by Bhatt-Scholze

$C = \hat{\bar{K}} \supseteq \mathcal{O}_C \rightsquigarrow A_{inf} = W(\mathcal{O}_C^b)$.

<u>note</u> Crystalline comparison

= prism-ét comparison + prism-Crys comparison.

<u>Log-prismatic site</u>

<u>Bhatt-Scholze</u> A bounded prism is a pair (A, I)

- A is $\mathbb{Z}_p$-alg, $\delta: A \to A$ s.t. $\varphi(x) := x^p + p\delta(x)$ is a lift of Frob.
- $I \subseteq A$ ideal loc gen'd by a nonzero divisor $d \in A^\times$ with $\delta(d) \in A^\times$.
- A has bounded p^∞-torsion and (p, I)-adic complete.

<u>Koshiwara</u> A bounded (pre)-log prism is a tuple

$$(A, I, \alpha: M \to A, \delta_{log}):$$

- (A, I) a bounded prism,
- $\alpha: M \to A$ is a (pre-)log prism,
- $\delta_{log}: M \to A$ (s.t. $\delta(\alpha(m)) = \alpha(m)^p \delta_{log}(m)$.)

<u>Ex</u> (1) $A = W$, $I = (p)$,

$\delta(p) = \frac{p - p^p}{p} = 1 - p^{p-1}$ ($\Leftrightarrow \varphi(p) = p$).

$\alpha: \mathbb{N} \to \mathbb{N}$ via $\alpha(1) = 0$, $\delta_{log} = 0$.

(2) $\mathcal{O}_C^b := \varprojlim_{x \mapsto x^p} (\mathcal{O}_C/p)$, $A_{inf} = W(\mathcal{O}_C^b)$

$\delta([x]) = 0$ ($\Leftrightarrow \varphi([x]) = [x]^p$.)

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_C^b$,

$\mu = [\varepsilon] - 1 \in A_{inf}$, $\tilde{\xi} := \frac{\varphi(\mu)}{\mu} = \sum_{i=0}^{p-1} [\varepsilon]^i \in A_{inf}$.

$\rightsquigarrow (A_{inf}, (\tilde{\xi}))$ is a prism.

$$\alpha : M_{\mathcal{O}_C^\flat} := \mathcal{O}_C^\flat \setminus \{0\} \longrightarrow A_{\mathrm{inf}}, \quad \delta_{\log} = 0.$$
$$x \longmapsto [x]$$

$\rightsquigarrow$ $(A_{\mathrm{inf}}, (\tilde{\xi}), \alpha, \delta_{\log})$ bounded (pre)-log prism.

<u>Rmk</u> $(A, I) \longrightarrow (B, J)$ morph of bounded prisms

$\Rightarrow \quad J = IB.$

Now X semi-stable formal sch $/ \operatorname{Spf} \mathcal{O}_C$.

(étale loc, X is covered by $\operatorname{Spf} R$.
with R admitting an étale map toward it.)

$\mathcal{O}_C \langle T_0, \cdots, T_r, T_{r+1}^{\pm 1}, \cdots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^a)$.

$M_X^{\mathrm{can}} := \mathcal{O}_{X_{\mathrm{et}}} \cap \mathcal{O}_{X_{\mathrm{et}}}[\frac{1}{p}]^\times \hookrightarrow \mathcal{O}_{X_{\mathrm{et}}}$ canonical log str.

<u>Def</u> $(X/A_{\mathrm{inf}})_{\Delta}^{\log}$:= site of bounded log prisms (B, J, M_B)
over $(A_{\mathrm{inf}}, (\tilde{\xi}), M_{\mathcal{O}_C^\flat})$ with morph
$$(\operatorname{Spf}(B/J), M_B)^a \xrightarrow{f} (X, M_X^{\mathrm{can}}).$$

- Coverings = flat covers.

Denote $\mathcal{O}_\Delta : (B, J, M_B) \longmapsto B$,
$\mathcal{O}_\Delta[\frac{1}{\xi}]^\wedge : (B, J, M_B) \longmapsto (B[\frac{1}{\xi}])^\wedge_{p\text{-adic}}$.

<u>Def</u> $\mathrm{Vect}((X/A_{\mathrm{inf}})_\Delta^{\log}, \mathcal{O}_\Delta) :=$

$$\left\{ \begin{array}{l} \mathcal{O}_\Delta\text{-mod } \mathcal{E} \text{ s.t.} \\ \text{(i) } \forall (B, J, M_B) \in \mathrm{Ob}(X/A_{\mathrm{inf}})_\Delta^{\log},\ \mathcal{E}(B, J, M_B) \text{ is a fin proj } B\text{-mod} \\ \text{(ii) } \forall (B, J, M_B) \to (C, JC, M_C),\ \mathcal{E}(B, J, M_B) \otimes_B C \xrightarrow{\sim} \mathcal{E}(C, JC, M_C) \end{array} \right\}.$$

Thm (Bhatt-Scholze + ε)

$\exists$ equiv of cats

$$\underbrace{\mathrm{Vect}^{\varphi}((X/A_{inf})^{\log}_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{\xi}]^{\wedge})}_{\text{cat of Laurent F-crystals.}} \simeq \mathrm{Loc}_{\mathbb{Z}_p}(X^{ad}_{C,\acute{e}t}).$$

$$\rightsquigarrow \begin{tikzcd} \mathrm{Vect}^{\varphi}((X/A_{inf})^{\log}_{\Delta}, \mathcal{O}_{\Delta}) \arrow[r] \arrow[rrd] & & \mathrm{Vect}^{\varphi}((X/A_{inf})^{\log}_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{\xi}]^{\wedge}) \arrow[d, "\cong"] \\ \mathcal{E} \arrow[r, mapsto, "T"] & T(\mathcal{E}) & \mathrm{Loc}_{\mathbb{Z}_p}(X^{ad}_{C,\acute{e}t}) \end{tikzcd}$$

& Correspondingly,

$$\begin{tikzcd} & \mathcal{E} \arrow[d, dash] \\ & (X/A_{inf})^{\log}_{\Delta} \arrow[d, "u"] \\ T(\mathcal{E}) \arrow[d, dash] & \\ X_{C,pro\acute{e}t} \arrow[r, "\nu"] & X_{\acute{e}t} \end{tikzcd}$$

Thm (Prismatic - étale)

$\forall\, \mathcal{E} \in \mathrm{Vect}^{\varphi}((X/A_{inf})^{\log}_{\Delta}, \mathcal{O}_{\Delta})$, $\exists$ a canonical isom

$$Ru_*(\mathcal{E})[\tfrac{1}{\mu}] \xrightarrow{\sim} R\nu_*(T(\mathcal{E}) \otimes_{\mathbb{Z}_p} A_{inf,X}[\tfrac{1}{\mu}]).$$

where $\mu = [\varepsilon_p] - 1$, $\varepsilon_p = (1, \zeta_p, \zeta_{p^2}, \cdots) \in A_{inf}$

and $A_{inf,X} : \mathrm{Spa}(S, S^+) \longmapsto A_{inf}(S^+)$.

Cor. When X is moreover proper (and already semi-stable),

$\exists$ a canonical isom

$$R\Gamma((X/A_{inf})^{\log}_{\Delta}, \mathcal{E})[\tfrac{1}{\mu}] \cong R\Gamma(X^{ad}_{C,\acute{e}t}, T(\mathcal{E})) \otimes A_{inf}[\tfrac{1}{\mu}].$$

Ideas Construction of the map

$$\begin{tikzcd}
 & (X/A_{\mathrm{inf}})^{\log}_{\Delta} \arrow[d, "\lambda"] \arrow[dd, bend left, "u"] \\
X^{\diamond}_{C,v} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X^{\log}_{\mathrm{qSYN}} \arrow[d, "\gamma"] \\
X_{C,\mathrm{pro\acute{e}t}} \arrow[r, "\nu"] & X_{\acute{e}t}
\end{tikzcd}$$

Prop (Guo-Reineke) $\exists$ an isom

$$\beta^*(T(\mathcal{E}) \otimes \mathbb{A}_{\mathrm{inf},X}[\tfrac{1}{\mu}]) \cong \alpha^*(\lambda_* \mathcal{E})[\tfrac{1}{\mu}].$$

Consequently,

$$R\nu_* R\beta_*(\beta^* T(\mathcal{E}) \otimes \mathbb{A}_{\mathrm{inf},X}[\tfrac{1}{\mu}])$$
$$\cong R\gamma_* R\alpha_*(\alpha^*(\lambda_* \mathcal{E})[\tfrac{1}{\mu}])$$

adjoint of $R\gamma_*(\lambda_* \mathcal{E}[\tfrac{1}{\mu}]) = Ru_*(\mathcal{E})[\tfrac{1}{\mu}]$.

& $LHS = R\nu_*(T(\mathcal{E}) \otimes \mathbb{A}_{\mathrm{inf},X}[\tfrac{1}{\mu}])$.

$\rightsquigarrow$ get $R\nu_*(\mathcal{E})[\tfrac{1}{\mu}] \xrightarrow{(*)} R\nu_*(T(\mathcal{E}) \otimes \mathbb{A}_{\mathrm{inf},X}[\tfrac{1}{\mu}])$.

The next step:

to show $(*)$ is an isom.

Lem $R\alpha_* \alpha^*(\lambda_* \mathcal{E}[\tfrac{1}{\mu}]) = \lambda_*(\mathcal{E} \otimes_{\mathcal{O}_\Delta} \mathcal{O}^{\mathrm{perf}}_{\Delta}[\tfrac{1}{\mu}])$

$\rightsquigarrow$ reduces to show

$$Ru_*(\mathcal{E})[\tfrac{1}{\mu}] \xrightarrow{\sim} Ru_*(\mathcal{E} \otimes_{\mathcal{O}_\Delta} \mathcal{O}^{\mathrm{perf}}_{\Delta}[\tfrac{1}{\mu}])$$

is an isom, which is under control.