# Endomorphisms of varieties
## Burt Totaro

(Joint with T. Kawakami.)

Q Which proj var $X$ have endomorphisms $f: X \to X$ of $\deg > 1$?

Eg. $X$ AV, have the map $m: X \to X$ for $m \in \mathbb{Z}$, $m \geq 2$.

$X$ any (proj) toric var, ($(\mathbb{G}_m)^a$ acts on $X$ of dim $n$).

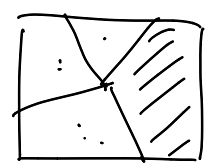
Let $m \in \mathbb{Z}$, $m \geq 2$.

Then the map $(\mathbb{G}_m)^n \longrightarrow (\mathbb{G}_m^n)$ via $x \mapsto x^m$

extends to a morphism $X \longrightarrow X$.

(It has deg $m^n > 1$.)

$X = (\mathbb{G}_m)^n \cup (\mathbb{G}_m)^{n-1} \cup \cdots$



$\circlearrowleft$ multi by $m$.

Let $X = \mathbb{P}^1 \times$ (any proj var $Y$).

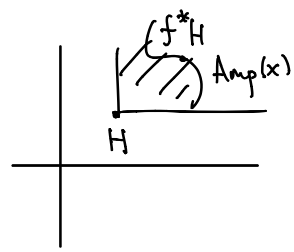
note: $\mathbb{P}^n$ is a toric var and the endomorph above is

$[x_0, \ldots, x_n] \longmapsto [x_0^m, \ldots, x_n^m]$.

Def (Zhang) An endomorph $f: X \to X$ of a proj var is <u>int-amplified</u> if there is an ample Cartier divisor $H$ on $X$ s.t. $f^*(H) - H$ is ample.

Eg. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and take $f, g: \mathbb{P}^1 \to \mathbb{P}^1$

$\rightsquigarrow f \times g: \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.



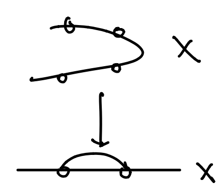

Note $\operatorname{Pic} X = \mathbb{Z}^2$, $(f \times g)^* \mathcal{O}(1,1) = \mathcal{O}(\deg f, \deg g)$

**Fact** $f \times g$ is int-amplified $\iff \deg f > 1,\ \deg g > 1$.

**Conj** (Fakhruddin, Zhang, Meng, Zhong)
$X$ sm proj var w/ an int-amplified endom.
Then $X \cong$ [toric fibration over an AV]$/G$,
for a finite gp $G$ acting freely.

**Def** An endo $f : X \to X$ is polarized if
$$f^* H \sim_{\mathbb{Q}} aH \quad \text{for some } a \in \mathbb{Q}_{>1}.$$
**Fact** polarized $\Rightarrow$ int-amplified.

**Q** What can we say if $X$ has a (separable) polarized endo?
**Note** Over $\mathbb{F}_q$, every proj var $X$ has the Frob endo
which satisfies $F^* L \cong q \cdot L$. (inseparable)




**A** Use that we can pullback diff forms of $f$.
That is, we have a map of sheaves
$$f^* : \Omega_X^i \longrightarrow f_* \Omega_X^i$$
or equivalently, a map
$$\alpha : f^* \Omega_X^n \longrightarrow \Omega_X^n = K_X \quad (n = \dim X.)$$
Since $f$ is separable, the deriv of $f$ is geometrically étale.
$\Rightarrow \alpha \not\equiv 0 \rightsquigarrow \alpha \in H^0(X, K_X - f^* K_X)$.
Assume $f$ polarized $\Rightarrow f^* K_X = m K_X \ (\exists m > 1)$
$\Rightarrow 0 \neq \alpha \in H^0(X, -(m-1) K_X)$.
$\Rightarrow -K_X$ is $\mathbb{Q}$-effective

$\Rightarrow$ $X$ is <u>not</u> of general type.

<u>Thm</u> (Zhang, Nakayama, Meng)
If $f: X \to X$ is an int-amplified endo,
$X$ normal proj var,
and if $X$ not uniruled, then
$$X \cong [\text{Some AV}]/G$$
for $G$ a finite gp acting freely in codim 1.

<u>Conj</u> If a RC var $/\mathbb{C}$ has an int-amplified endo,
then it is a toric var.

- True in $\dim X \leq 2$, by Nakayama.
- True in $\dim X = 3$ for $X$ sm Fano var.
- If a sm Fano 3-fold $/\mathbb{C}$ has an int-amplified endo,
  then it is toric.
- True in higher dim if $X^n \subset \mathbb{P}^{n+1}$.

<u>Thm A</u> (Kawakami-Totaro, Totaro)
If $X$ sm Fano 3-fold $/ k = \bar{k}$
& if $f: X \to X$ int-amplified endo of $\deg \in k^\times$,
then $X$ is toric.

<u>Thm C</u> (KT) $X$ normal proj var / perfect field $k$.
Suppose $X$ has int-amplified endo of $\deg \in k^\times$.

Then $X$ satisfies <u>Bott vanishing</u>,

i.e. $\forall j>0,\ i\geq 0$, $A$ ample Weil divisor,

$$H^j(X, \Omega^i_X(A)) = 0.$$

$\Omega^i_X \otimes A$ if $X$ sm

or $(\Omega^i_X \otimes \mathcal{O}(A))^{**}$ in general.

What does BV tell us?

<u>Ex.</u> If $i=n$ ($=\dim X$), this is just Kodaira vanishing (true for all $X/\mathbb{C}$).

If $i=0$, this says that

$$H^{>0}(X, A) = 0, \quad A \text{ ample}.$$

This is true, e.g., for all Fano varieties (by Kodaira vanishing).

Suppose $X$ sm Fano, satisfying BV. Then

$$H^1(X, TX) = H^1(X, \Omega^{n-1}_X(\underbrace{-K_X}_{\text{ample}})) = 0.$$

So $X$ is rigid.

<u>Cor</u> Over $\mathbb{C}$, only finitely many sm Fano $n$-folds have an int-amplified endo.

pf. By Kollár-Miyaka-Mori, sm Fano $n$-folds form a bounded family.

If $X$ has an int-amplified endo, Thm C + $X$ satisfies BV

$\Rightarrow X$ rigid. $\square$

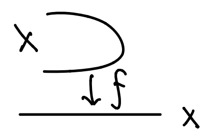
<u>Pf of Thm C</u> Assume $X$ sm.

Given $f: X \to X$ int-amplified endo,

to show BV: $H^j(X, \Omega^i_X(A)) = 0$, $j > 0$, $i \geq 0$,

$A$ ample line bundle.

<u>Idea</u> Use that we can pullback diff forms.



Firstly: assume $\exists$ ample Cartier divisor $H$

s.t. $f^*H - H$ ample.

In particular, $f^*H$ ample

$$\Rightarrow \quad (f^*H)\cdot C > 0 \text{ for every curve } C \subseteq X.$$

$$\overset{\shortparallel}{H \cdot f_* C}$$

$\Rightarrow$ $f$ does not contract any curve

$\Rightarrow$ $f$ is a finite morphism.

Have a natural map ($\forall\, i \geq 0$)

$$\Omega^i_X \longrightarrow f_* \Omega^i_X \longrightarrow \Omega^i_X$$

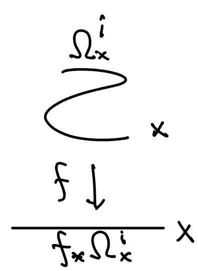
pullback $\quad$ pushforward (or "trace")

(We are assuming $\deg f \in k^\times$.)

The trace map was def'd by Garel (1984) & Kunz (1986).

The composition $\Omega^i_X \to f_*\Omega^i_X \to \Omega^i_X$

$\cong$ of deg $f$.



$\Rightarrow$ $f_*\Omega^i_X \cong \Omega^i_X \oplus (\text{sth.})$

$\Rightarrow$ $H^j(X, \Omega^i_X) \longrightarrow H^j(X, f_*\Omega^i_X)$

(let $j > 0$) $\qquad \overset{\shortparallel}{H^j(X, \Omega^i_X)}$

is (split) injective

For an ample line bundle $A$ on $X$, have

$$(f_* \Omega_X^i) \otimes A \cong (\Omega_X^i) \otimes A$$

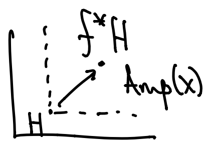
$$\parallel$$

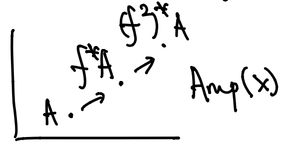
$$f_*(\Omega_X^i \otimes f^* A).$$

So $\quad H^j(X, \Omega_X^i(A)) \hookrightarrow H^j(X, \Omega_X^i(f^*A))$.

Likewise, for $e \in \mathbb{Z}^+$,

$$H^j(X, \Omega_X^i(A)) \hookrightarrow H^j(X, \Omega_X^i((f^e)^*A))$$




By Fujita vanishing,

by $(f^e)^*A$ gets arbitrarily ample as $e \to \infty$

$\Rightarrow \quad H^j(X, \Omega_X^i((f^e)^*A)) = 0, \quad j > 0.$

$\Rightarrow \quad H^j(X, \Omega_X^i(A)) = 0.$ $\square$