A p-ADIC ARITHMETIC INNER PRODUCT FORMULA

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Abstract. Fix a prime number p and let E/F be a CM extension of number fields in which p splits relatively. Let π be an automorphic representation of a quasi-split unitary group of even rank with respect to E/F such that π is ordinary above p with respect to the Siegel parabolic subgroup. We construct the cyclotomic p-adic L-function of π, and a certain generating series of Selmer classes of special cycles on Shimura varieties. We show, under some conditions, that if the vanishing order of the p-adic L-function is 1, then our generating series is modular and yields explicit nonzero classes (called Selmer theta lifts) in the Selmer group of the Galois representation of E associated with π; in particular, the rank of this Selmer group is at least 1. In fact, we prove a precise formula relating the p-adic heights of Selmer theta lifts to the derivative of the p-adic L-function. In parallel to Perrin-Riou's p-adic analogue of the Gross–Zagier formula, our formula is the p-adic analogue of the arithmetic inner product formula recently established by Chao Li and the second author.

Contents

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1. INTRODUCTION

In 1986, Gross and Zagier published a groundbreaking formula relating the heights of Heegner points on modular curves to derivatives of L-functions, known as the Gross–Zagier formula [GZ86]. For a cuspidal eigenform f of weight 2, an imaginary quadratic field K and an unramified Dirichlet character ξ of K, the formula shows, under the so-called Heegner condition (which implies that the Rankin–Selberg L-function $L(s,f,\xi)$ vanishes at the center 1), that up to some explicit constant, $L'(1,f,\xi)$ equals the Néron–Tate height of $H_\xi(f)$ – the f-isotypic component of the K-Heegner point weighted by ξ on a modular curve. Shortly after, Perrin-Riou found an analogous result in the p-adic world [PR87]. Namely, she constructed a p-adic analogue of the (complex) L-function as a p-adic measure $\mathscr{L}_p(f,\xi)$ in the Iwasawa algebra that interpolates $L(1,f\otimes\chi,\xi)$ where χ is a Dirichlet character ramified only at p, assuming that f is ordinary at p and p splits in K. Then she proved that under the same Heegner condition, up to some explicit constant, the derivative of the p-adic L-function $\mathscr{L}_p(f,\xi)$ at the trivial character equals the p-adic height of $H_\xi(f)$ – this is known as the *p-adic Gross–Zagier formula*.

Since the original work of Gross and Zagier, the Gross–Zagier formula and its p-adic avatar have been extended to various settings but all (essentially) for curves or fibrations/local systems over curves (see Remark 1.10 below for a brief review of the p-adic results), until the very recent works by Chao Li and one of us [LL21, LL22]. There, the authors proved a formula computing central L-derivatives for unitary groups of higher ranks in terms of the Beilinson–Bloch heights of special cycles. This originates from a program initiated by Kudla [Kud02, Kud03, Kud04] and can be regarded as a Gross–Zagier formula in higher dimensions, as well as an arithmetic analogue of Rallis' inner product formula in the theory of the theta correspondence [Ral82]. The current work contains a *p-adic* avatar of the *arithmetic inner product formula* in [LL21, LL22]; this is likewise the first generalization of the p-adic Gross–Zagier formula to genuinely higher dimensional varieties. A secondary aim of this article is to develop some foundational results in the theory of p-adic heights of algebraic cycles (in the two appendices); in particular, we prove a crystalline property of bi-extensions, which generalizes the fact that p-adic regulators take values in Selmer groups.

In the rest of this introduction, we explain our results in more detail. Throughout the article, we fix a prime number p, an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and a CM extension E/F of number fields such that *every p-adic place of F splits in E*. Denote by

- $\mathtt{c}\in\mathrm{Gal}(E/F)$ the Galois involution,
- $\mathtt{V}_F^{(\diamond)}$ the set of places of F above a finite set $\diamond$ of places of $\mathbb{Q}$,[1]
- $\mathtt{V}_F^{\mathrm{fin}}$ the set of non-archimedean places of F,
- $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$ and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert and ramified in E, respectively.

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[1] When $\diamond=\{w\}$ is a singleton, we simply write $\mathtt{V}_F^{(w)}$ for $\mathtt{V}_F^{(\{w\})}$.

For every number field K, we denote by $\Gamma_{K,p}$ the maximal Hausdorff quotient of

$$K^\times\backslash\mathbb{A}_K^{\infty,\times}\Big/\!\!\Big/\left(O_K\otimes\prod_{w\neq p}\mathbb{Z}_w\right)^\times,$$

which is naturally a finitely generated $\mathbb{Z}_p$-module; and let $\mathscr{X}_{K,p}$ be the rigid analytic space over $\mathbb{Q}_p$ such that for every complete topological $\mathbb{Q}_p$-ring R, $\mathscr{X}_{K,p}(R)$ is the set of continuous characters from $\Gamma_{K,p}$ to $R^\times$.

1.1. **Cyclotomic p-adic L-function.** Take an integer $r\geqslant 1$ and put $n=2r$. We equip $W_r:=E^n$ with the skew-hermitian form (with respect to $\mathtt{c}$) given by the matrix $\mathtt{w}_r:=\left(\begin{smallmatrix} & 1_r\\ -1_r & \end{smallmatrix}\right)$. Put $G_r:=\mathrm{U}(W_r)$, the unitary group of W_r, which is a quasi-split reductive group over F. Denote by $\dagger$ the involution of G_r given by the conjugation by the element $\left(\begin{smallmatrix} 1_r & \\ & -1_r\end{smallmatrix}\right)$ inside $\mathrm{Res}_{E/F}\mathrm{GL}_n$. For $v\in\mathtt{V}_F^{\mathrm{fin}}$, let $K_{r,v}\subseteq G_r(F_v)$ be the stabilizer of the lattice $O_{E_v}^n$.

Definition 1.1. Let $\mathbb{L}$ be a field embeddable into $\mathbb{C}$. A *relevant* $\mathbb{L}$*-representation* of $G_r(\mathbb{A}_F^\infty)$ is a representation π with coefficients in $\mathbb{L}$ satisfying that for every embedding $\iota\colon\mathbb{L}\to\mathbb{C}$,

$${}^\iota\pi:=\left(\otimes_{v\in\mathtt{V}_F^{(\infty)}}\pi_v^{[r]}\right)\otimes\iota\pi$$

is a tempered cuspidal automorphic representation of $G_r(\mathbb{A}_F)$. Here, for $v\in\mathtt{V}_F^{(\infty)}$, $\pi_v^{[r]}$ denotes the unique holomorphic discrete series representation of $G_r(F_v)=G_r(\mathbb{R})$ with the Harish–Chandra parameter $\{\frac{1-n}{2},\frac{3-n}{2},\dots,\frac{n-3}{2},\frac{n-1}{2}\}$. In particular, π is admissible and absolutely irreducible.

We consider a finite extension $\mathbb{L}/\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and a relevant $\mathbb{L}$-representation π of $G_r(\mathbb{A}_F^\infty)$. By Lemma 3.14, $\hat{\pi}:=(\pi^\vee)^\dagger$ is a relevant $\mathbb{L}$-representation of $G_r(\mathbb{A}_F^\infty)$ as well.

Definition 1.2. For $v\in\mathtt{V}_F^{(p)}$, let $\mathtt{P}_v$ be the set (of two elements) of places of E above v. For $u\in\mathtt{P}_v$, we have the representation π_u of $\mathrm{GL}_n(F_v)$ as a local component of π via the isomorphism $G_r(F_v)\simeq\mathrm{GL}_n(E_u)=\mathrm{GL}_n(F_v)$. In particular, $\pi_u^\vee\simeq\pi_{u^\mathtt{c}}$. We say that π_u is *Panchishkin unramified* if

(1) π_u is unramified;
(2) if we write the Satake polynomial of π_u, which makes sense by (1), as
$$\mathsf{P}_{\pi_u}(T)=T^n+\beta_{u,1}\cdot T^{n-1}+\beta_{u,2}\cdot q_v\cdot T^{n-2}+\cdots+\beta_{u,r}\cdot q_v^{\frac{r(r-1)}{2}}\cdot T^r+\cdots+\beta_{u,n}\cdot q_v^{\frac{n(n-1)}{2}}\in\mathbb{L}[T]$$
(see Definition 3.18 for more details), then $\beta_{u,r}\in O_\mathbb{L}^\times$, where q_v is the residue cardinality of F_v.

We collect two important facts about Panchishkin unramified representations:

- The representation π_u is Panchishkin unramified if and only if $\pi_{u^\mathtt{c}}$ is (Lemma 3.22). In particular, it makes sense to say that π_v is Panchishkin unramified.
- If π_u is Panchishkin unramified, then there is a unique polynomial $\mathsf{Q}_{\pi_u}(T)\in\mathbb{L}[T]$ that divides $\mathsf{P}_{\pi_u}(T)$ and has the form
$$\mathsf{Q}_{\pi_u}(T)=T^r+\gamma_{u,1}\cdot T^{r-1}+\gamma_{u,2}\cdot q_v\cdot T^{r-2}+\cdots+\gamma_{u,r}\cdot q_v^{\frac{r(r-1)}{2}}$$
with $\gamma_{u,r}\in O_\mathbb{L}^\times$ (Proposition 3.25). In particular, we have an unramified principal series $\underline{\pi_u}$ of $\mathrm{GL}_r(F_v)$ defined over $\mathbb{L}$ whose Satake polynomial is $\mathsf{Q}_{\pi_u}(T)$.

Remark 1.3. In fact, π_v is Panchishkin unramified if and only if π_v is unramified and π is ordinary at v with respect to the standard Siegel parabolic subgroup of G_r in the sense of Hida [Hid98].

Theorem 1.4. *Under the above setup, suppose that π_v is Panchishkin unramified for every $v\in\mathtt{V}_F^{(p)}$. For every finite set $\Diamond$ of places of $\mathbb{Q}$ containing $\{\infty,p\}$ such that π_v is unramified for every $v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(\Diamond)}$, there is a unique bounded analytic function $\mathscr{L}_p^\Diamond(\pi)$ on the rigid analytic space $\mathscr{X}_{F,p}\otimes_{\mathbb{Q}_p}\mathbb{L}$ such that for every finite (continuous) character $\chi\colon\Gamma_{F,p}\to\overline{\mathbb{Q}}_p^\times$ and every embedding $\iota\colon\overline{\mathbb{Q}}_p\to\mathbb{C}$, we have*

$$\iota\mathscr{L}_p^\Diamond(\pi)(\chi)=\frac{1}{\mathrm{P}_\pi^\iota}\cdot\frac{Z_r^{[F:\mathbb{Q}]}}{b_{2r}^\Diamond(\mathbf{1})}\cdot\prod_{v\in\mathtt{V}_F^{(p)}}\prod_{u\in\mathtt{P}_v}\gamma(\tfrac{1+r}{2},\iota(\underline{\pi_u}\otimes\chi_v),\psi_{F,v})^{-1}\cdot L(\tfrac{1}{2},\mathrm{BC}(\iota\pi^\Diamond)\otimes(\iota\chi^\Diamond\circ\mathrm{Nm}_{E/F})),$$

where

- $\mathrm{P}^{\iota}_{\pi} \in \mathbb{C}^{\times}$ *is a certain period for* π *with respect to* ι *for every embedding* $\iota\colon \mathbb{L} \to \mathbb{C}$*, satisfying* $\mathrm{P}^{\iota}_{\pi} = \mathrm{P}^{\iota}_{\hat{\pi}}$*;*
- $Z_r := (-1)^r 2^{-r^2-r}\boldsymbol{\pi}^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}$ *is the value of a certain explicit archimedean local doubling zeta integral;*
- $b^{\diamondsuit}_{2r}(\mathbf{1}) = \prod_{i=1}^{2r} L^{\diamondsuit}(i, \eta^{i}_{E/F})$ *is defined in §2.1(F4);*
- $\gamma(s, \iota(\underline{\pi_u} \otimes \chi_v), \psi_{F,v})$ *is the gamma factor* [Jac79] *in which* $\psi_F := \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}}$ *with* $\psi_{\mathbb{Q}}\colon \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^{\times}$ *the standard automorphic additive character;*
- $L(s, \mathrm{BC}(\iota\pi^{\diamondsuit}) \otimes (\iota\chi^{\diamondsuit} \circ \mathrm{Nm}_{E/F}))$ *is the complex L-function of the (complex) representation* $\mathrm{BC}(\iota\pi^{\diamondsuit}) \otimes (\iota\chi^{\diamondsuit} \circ \mathrm{Nm}_{E/F})$ *of* $\mathrm{GL}_n(\mathbb{A}^{\diamondsuit}_E)$*, hence is an Euler product away from* $\diamondsuit$*.*

By definition, for every $v \in \mathtt{V}^{(p)}_F$ and $u \in \mathtt{P}_v$, $\iota\underline{\pi_u} \otimes |\,|^{\frac{r}{2}}_{F_v}$ is tempered so that $\gamma(\frac{1+r}{2}, \iota(\underline{\pi_u} \otimes \chi_v), \psi_{F,v}) \in \mathbb{C}^{\times}$.

Remark 1.5. We have the following remarks concerning Theorem 1.4.

(1) A bounded analytic function on the rigid analytic space $\mathscr{X}_{F,p} \otimes_{\mathbb{Q}_p} \mathbb{L}$ is equivalent to an element in $\mathbb{Z}_p[[\Gamma_{F,p}]] \otimes_{\mathbb{Z}_p} \mathbb{L}$, that is, an $\mathbb{L}$-valued p-adic measure on $\Gamma_{F,p}$. In particular, the uniqueness of $\mathscr{L}^{\diamondsuit}_p(\pi)$ is clear.

(2) The collection of periods $(\mathrm{P}^{\iota}_{\pi})_{\iota}$ is only well-defined up to a common factor in $\mathbb{L}^{\times}$ (see Notation 3.15). In particular, the p-adic L-function $\mathscr{L}^{\diamondsuit}_p(\pi)$ is only well-defined up to a factor in $\mathbb{L}^{\times}$.

(3) The vanishing order of $\mathscr{L}^{\diamondsuit}_p(\pi)$ at $\mathbf{1}$ does not depend on $\diamondsuit$. From the interpolation formula, we have $\mathscr{L}^{\diamondsuit}_p(\pi) = \mathscr{L}^{\diamondsuit}_p(\hat{\pi})$.

(4) Our p-adic L-function is defined over the p-adic field of definition of the representation and interpolates complex L-values along *all* isomorphisms $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$; this is a rationality property stronger than the one under a fixed isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ as in the setup of many previous works in this field.

(5) Among other technical assumptions, at least when π is ordinary at p in the usual sense (that is, for every $u \in \mathtt{P}$, $\beta_{u,m} \in O^{\times}_{\mathbb{L}}$ for every $1 \leqslant m \leqslant n$ in the Satake polynomial of π_u), our p-adic L-function has already been constructed in [EHLS20] up to some constant (and with a weaker rationality property). In fact, in [EHLS20] the authors construct more generally a multi-variable p-adic L-function in which π is allowed to vary in an ordinary Hida family as well. In this article, we will give a (relatively) self-contained construction of our p-adic L-function independent of [EHLS20] since first, the process of the construction itself is an ingredient for the p-adic height formula; and second, our construction is technically much simpler to follow.

1.2. **Modularity of generating functions in Selmer groups.** In this subsection, we construct a Selmer group analogue of Kudla's generating functions and state a theorem on its modularity. We now suppose that $F \neq \mathbb{Q}$. Fix an embedding $E \hookrightarrow \mathbb{C}$ and regard E as a subfield of $\mathbb{C}$. For the simplicity of the introduction, we fix an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and will not pay attention to the rationality of the constructions below, while the full details with full generality can be found in §4.2 and §4.3.

Let V be a hermitian space over E of rank $n = 2r$ that has signature $(n-1, 1)$ along the induced inclusion $F \subseteq \mathbb{R}$ and signature $(n, 0)$ at other archimedean places of F. Put $H := \mathrm{U}(V)$. We then have a system of Shimura varieties $\{X_L\}_L$ indexed by neat open compact subgroups L of $H(\mathbb{A}^{\infty}_F)$, which are smooth projective schemes over E of dimension $n-1$. Take a neat open compact subgroup $L \subseteq H(\mathbb{A}^{\infty}_F)$. Let $\mathrm{V}_{\pi,L}$ be the $\theta(\pi)$-isotypic subspace of $\mathrm{H}^{2r-1}(X_L \otimes_E \overline{E}, \overline{\mathbb{Q}}_p(r))$ (which could be zero), where $\theta(\pi)$ denotes the (product of) local theta lifting of π. We have a canonical map $\wp_{\pi}\colon \mathrm{H}^{2r}(X_L, \overline{\mathbb{Q}}_p(r)) \to \mathrm{H}^1(E, \mathrm{V}_{\pi,L})$ from Lemma 4.7.

For every Schwartz function $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}^{\infty}_F)^L$ and every $g \in G_r(\mathbb{A}^{\infty}_F)$, we have Kudla's generating function

$$Z_{\phi,L}(g) := \sum_{T \in \mathrm{Herm}_r(F)^+} \sum_{\substack{x \in L\backslash V^r \otimes_F \mathbb{A}^{\infty}_F \\ T(x) = T}} (\omega_r(g)\phi)(x) Z(x)_L \cdot q^T$$

as a formal power series indexed by totally semi-positive definite hermitian matrices T over E/F of rank r, with coefficients that are special cycles $Z(x)_L \in \mathrm{CH}^r(X_L)$ indexed by elements $x \in L\backslash V^r \otimes_F \mathbb{A}^{\infty}_F$ with moment matrix T. Denote by $Z^{\pi}_{\phi,L}(g)$ its image under the composition of the absolute cycle class map $\mathrm{CH}^r(X_L) \to \mathrm{H}^{2r}(X_L, \overline{\mathbb{Q}}_p(r))$ and the canonical map $\wp_{\pi}\colon \mathrm{H}^{2r}(X_L, \overline{\mathbb{Q}}_p(r)) \to \mathrm{H}^1(E, \mathrm{V}_{\pi,L})$. We say that π satisfies the *Modularity Hypothesis* if:

There exists a (unique) holomorphic automorphic form $\mathcal{Z}^{\pi}_{\phi,L}$ *on* $G_r(\mathbb{A}_F)$ *valued in the Bloch–Kato Selmer group* $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L})$ [BK90] *such that the q-expansion of* $g \cdot \mathcal{Z}^{\pi}_{\phi,L}$ *coincides with* $Z^{\pi}_{\phi,L}(g)$ *for every* $g \in G_r(\mathbb{A}^{\infty}_F)$*.*

Our first result concerns the Modularity Hypothesis under certain assumptions.

Assumption 1.6. Suppose that $F \neq \mathbb{Q}$, that $\mathtt{V}_F^{\mathrm{spl}}$ contains all 2-adic (and p-adic) places, and that every prime in $\mathtt{V}_F^{\mathrm{ram}}$ is unramified over $\mathbb{Q}$. Suppose that the relevant $\mathbb{L}$-representation π of $G_r(\mathbb{A}_F^\infty)$ (with $\mathbb{L}/\mathbb{Q}_p$ a finite extension contained in $\overline{\mathbb{Q}}_p$) satisfies:

1. For every $v \in \mathtt{V}_F^{\mathrm{ram}}$, π_v is spherical with respect to $K_{r,v}$, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.
2. For every $v \in \mathtt{V}_F^{\mathrm{int}}$, π_v is either unramified or almost unramified (see [LL21, Remark 1.4(3)]) with respect to $K_{r,v}$; moreover, if π_v is almost unramified, then v is unramified over $\mathbb{Q}$.
3. We have $\mathtt{R}_\pi \cup \mathtt{S}_\pi \subseteq \mathtt{V}_F^{\heartsuit}$ (see below), where
    - $\mathtt{R}_\pi \subseteq \mathtt{V}_F^{\mathrm{spl}}$ denotes the (finite) subset for which π_v is ramified,
    - $\mathtt{S}_\pi \subseteq \mathtt{V}_F^{\mathrm{int}}$ denotes the (finite) subset for which π_v is almost unramified.
4. For every $v \in \mathtt{V}_F^{(p)}$, π_v is Panchishkin unramified.

Here, we recall from [LL22] (and refer to [LL22, Remark 1.2] for its technical nature) that $\mathtt{V}_F^{\heartsuit}$ is the subset of $\mathtt{V}_F^{\mathrm{spl}} \cup \mathtt{V}_F^{\mathrm{int}}$ consisting of v satisfying that for every $v' \in \mathtt{V}_F^{(p_v)} \cap \mathtt{V}_F^{\mathrm{ram}}$, the subfield of $\overline{F_v}$ generated by F_v and the Galois closure of $E_{v'}$ is unramified over F_v. In particular, $\mathtt{V}_F^{\heartsuit}$ contains $\mathtt{V}_F^{(p)}$.

Theorem 1.7 (Theorem 4.21)**.** *Suppose that we are in the situation of Assumption 1.6 and $n < p$. If the vanishing order of $\mathscr{L}_p^{\diamondsuit}(\pi)$ at $\mathbf{1}$ is one, then π satisfies the Modularity Hypothesis.*

1.3. **A p-adic arithmetic inner product formula.** In this subsection, we construct a Selmer group analogue of the (arithmetic) theta lift, and state a corresponding inner product formula for it, which we call the *p-adic arithmetic inner product formula*. The details can be found in §4.3 and §4.4. We keep the setup from the previous subsection.

Suppose that both π and $\tilde{\pi}$ satisfy the Modularity Hypothesis. For every $\varphi \in \tilde{\pi}$, we define $\Theta_\phi^{\mathrm{Sel}}(\varphi)_L$ to be the convolution of $\varphi^\dagger$ and $\mathcal{Z}_{\phi,L}^\pi$, which is an element of $\mathrm{H}_f^1(E, \mathrm{V}_{\pi,L})$. The element $\Theta_\phi^{\mathrm{Sel}}(\varphi)_L$ is the Selmer group analogue of the arithmetic theta lift constructed in [Liu11a, LL21], which we call a *Selmer theta lift*.

The Poincaré duality for X_L induces a pairing $\mathrm{V}_{\pi,L} \times \mathrm{V}_{\tilde{\pi},L} \to \overline{\mathbb{Q}}_p(1)$. By Nekovář's theory [Nek93], we have a p-adic height pairing

$$\langle\ ,\ \rangle_E \coloneqq \mathrm{H}_f^1(E, \mathrm{V}_{\pi,L}) \times \mathrm{H}_f^1(E, \mathrm{V}_{\tilde{\pi},L}) \to \Gamma_{E,p} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$$

using certain canonical Hodge splitting at p-adic places. For every $\varphi_1 \in \tilde{\pi}$, every $\varphi_2 \in \pi$ and every pair $\phi_1, \phi_2 \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$, the height

$$\mathrm{vol}^\natural(L) \cdot \langle \Theta_{\phi_1}^{\mathrm{Sel}}(\varphi_1)_L, \Theta_{\phi_2}^{\mathrm{Sel}}(\varphi_2)_L \rangle_E \in \Gamma_{E,p} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$$

is independent of L, where $\mathrm{vol}^\natural(L)$ denotes a certain canonical volume of L introduced in [LL21, Definition 3.8]. We will denote the above canonical value as $\langle \Theta_{\phi_1}^{\mathrm{Sel}}(\varphi_1), \Theta_{\phi_2}^{\mathrm{Sel}}(\varphi_2) \rangle_{\pi,E}^\natural$.

Theorem 1.8 (p-adic arithmetic inner product formula, Theorem 4.22)**.** *Suppose that we are in the situation of Assumption 1.6 and $n < p$.*

1. *If the vanishing order of $\mathscr{L}_p^{\diamondsuit}(\pi)$ at $\mathbf{1}$ is one (so that both π and $\tilde{\pi}$ satisfy the Modularity Hypothesis by Theorem 1.7 and Remark 1.5(3)), then for every choice of elements*
    - *$\varphi_1 = \otimes_v \varphi_{1,v} \in \tilde{\pi}$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \pi$ such that for every $v \notin \mathtt{V}_F^{(\diamondsuit)}$, $\varphi_{1,v}$ and $\varphi_{2,v}$ are fixed by $K_{r,v}$ such that $\langle \varphi_{1,v}, \varphi_{2,v} \rangle_{\pi_v} = 1$,*
    - *$\phi_1 = \otimes_v \phi_{1,v}, \phi_2 = \otimes_v \phi_{2,v} \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)$ with $\phi_1^\diamondsuit = \phi_2^\diamondsuit$ being the characteristic function of $(\Lambda^\diamondsuit)^r$ in which $\Lambda^\diamondsuit$ is a self-dual lattice of $V \otimes_F \mathbb{A}_F^\diamondsuit$,*

    the identity

    $$\mathrm{Nm}_{E/F} \langle \Theta_{\phi_1}^{\mathrm{Sel}}(\varphi_1), \Theta_{\phi_2}^{\mathrm{Sel}}(\varphi_2) \rangle_{\pi,E}^\natural = \partial \mathscr{L}_p^{\diamondsuit}(\pi)(\mathbf{1}) \cdot \prod_{v \in \mathtt{V}_F^{(p)}} \prod_{u \in \mathtt{P}_v} \gamma(\tfrac{1+r}{2}, \underline{\pi_u}, \psi_{F,v}) \cdot \prod_{v \in \mathtt{V}_F^{(\diamondsuit \setminus \{\infty\})}} Z(\varphi_{1,v}^\dagger \otimes \varphi_{2,v}, f_{\phi_{1,v} \otimes \phi_{2,v}}^{\mathrm{SW}})$$

    holds in $\Gamma_{F,p} \otimes_{\mathbb{Z}_p} \mathbb{C}$, where the term $Z(\varphi_{1,v}^\dagger \otimes \varphi_{2,v}, f_{\phi_{1,v} \otimes \phi_{2,v}}^{\mathrm{SW}})$ is the local doubling zeta integral with respect to the Siegel–Weil section $f_{\phi_{1,v} \otimes \phi_{2,v}}^{\mathrm{SW}}$ associated with $\phi_{1,v} \otimes \phi_{2,v}$.

(2) *If the vanishing order of $\mathscr{L}_p^\diamond(\pi)$ at* **1** *is not one, then assuming that both π and $\hat{\pi}$ satisfy the Modularity Hypothesis, we have*

$$\mathrm{Nm}_{E/F}\langle\Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1),\Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^{\natural}_{\pi,E}=0$$

for every $\varphi_1\in\hat{\pi}$, $\varphi_2\in\pi$, and $\phi_1,\phi_2\in\mathscr{S}(V^r\otimes_F\mathbb{A}_F^\infty)$. (See Theorem 4.22(2) for a version of this part that does not rely on the Modularity Hypothesis.)

The above theorem is only nontrivial when $r[F:\mathbb{Q}]+|\mathsf{S}_\pi|$ is odd (Remark 4.23(3)).

The theorem has applications to the *p-adic* Beilinson–Bloch–Kato conjecture. Associated with π, we have a semisimple continuous representation ρ_π of $\mathrm{Gal}(\overline{E}/E)$ of dimension n with coefficients in $\overline{\mathbb{Q}}_p$, satisfying $\rho_\pi^{\mathtt{c}}\simeq\rho_\pi^\vee(1-n)$ (Lemma 4.11). Then in the interpolation property of $\mathscr{L}_p^\diamond(\pi)$ in Theorem 1.4, we have

$$L(\tfrac{1}{2},\mathrm{BC}(\iota\pi^\diamond)\otimes(\iota\chi^\diamond\circ\mathrm{Nm}_{E/F}))=L^\diamond(0,\iota(\rho_\pi(r)\otimes\chi|_{\mathrm{Gal}(\overline{E}/E)})),$$

where on the right-hand side we view χ as a $\overline{\mathbb{Q}}_p$-valued character of $\mathrm{Gal}(\overline{E}/F)$ via the global class field theory. The following corollary provides evidence toward the p-adic Beilinson–Bloch–Kato conjecture for (genuinely) higher-dimensional motives, whose deduction is provided after Remark 4.23.

Corollary 1.9. *Suppose that we are in the situation of Assumption 1.6 and $n<p$. If the vanishing order of $\mathscr{L}_p^\diamond(\pi)$ at* **1** *is one, then*

$$\dim_{\overline{\mathbb{Q}}_p}\mathrm{H}^1_f(E,\rho_\pi(r))\geqslant 1.$$

Remark 1.10. When $n=2$, this result is a variant of the main application of the p-adic Gross–Zagier formula of [PR87], as generalized to totally real fields by one of us [Dis17] following the development of [GZ86] in [YZZ13]. In different directions, Perrin-Riou's results had been generalized to the case of higher-weight modular forms by Nekovář [Nek95] and further to the case with twists by higher-weight Hecke characters by Shnidman [Shn16], to the supersingular case by Kobayashi [Kob13], and to the case where p is not necessarily relative split by one of us [Dis].[2] A common generalization of [Nek95, Shn16, Dis17, Dis] was developed in [Dis22].

Remark 1.11. Strictly speaking, Theorem 1.8 (together with Corollary 1.9 and Corollary 1.12 below) relies on a hypothesis on the characterization of the tempered part of the cohomology of certain unitary Shimura varieties (see Hypothesis 4.12 and Remark 4.13), which is expected to be verified in a sequel of the work [KSZ].

1.4. **Application to symmetric power of elliptic curves.**

The above results have applications to the motives of symmetric power of elliptic curves. We consider a *modular* elliptic curve A over F without complex multiplication that has *ordinary good reduction* at every p-adic place of F. Denote by $\mathtt{V}_F^A\subseteq\mathtt{V}_F^{\mathrm{fin}}$ the subset consisting of places over which A has bad reduction.

By the very recent breakthrough on the automorphy of symmetric powers of Hilbert modular forms [NT], there exists a unique cuspidal automorphic representation $\Pi(\mathrm{Sym}^{n-1}A)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ satisfying

- for every $v\in\mathtt{V}_F^{(\infty)}$, the base change of $\Pi(\mathrm{Sym}^{n-1}A)_v$ to $\mathrm{GL}_n(\mathbb{C})$ is the principal series representation of characters $(\arg^{1-n},\arg^{3-n},\dots,\arg^{n-3},\arg^{n-1})$, where $\arg\colon\mathbb{C}^\times\to\mathbb{C}^\times$ is the character given by $\arg(z):=z/\sqrt{z\overline{z}}$;
- for every $v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^A$, $\Pi(\mathrm{Sym}^{n-1}A)_v$ is unramified with the Satake polynomial

$$\prod_{j=0}^{n-1}\left(T-\alpha_{v,1}^j\alpha_{v,2}^{n-1-j}\right)\in\mathbb{Q}[T],$$

  where $\alpha_{v,1}$ and $\alpha_{v,2}$ are the two roots of the polynomial $T^2-a_v(A)T+q_v$ (with q_v the residue cardinality of F_v).

Let $\Pi(\mathrm{Sym}^{n-1}A_E)$ be the (solvable) base change of $\Pi(\mathrm{Sym}^{n-1}A)$ to E, which is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. The representation $\Pi(\mathrm{Sym}^{n-1}A_E)$ satisfies $\Pi(\mathrm{Sym}^{n-1}A_E)^\vee\simeq\Pi(\mathrm{Sym}^{n-1}A_E)\simeq\Pi(\mathrm{Sym}^{n-1}A_E)^{\mathtt{c}}$, hence is a relevant representation in the sense of [LTX⁺22, Definition 1.1.3]. By [LTX⁺22, Remark 1.1.4] and the endoscopic classification for quasi-split unitary groups [Mok15], there exists a cuspidal automorphic representation $\pi(\mathrm{Sym}^{n-1}A_E)$ of $G_r(\mathbb{A}_F)$ satisfying

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[2] In fact, in [Kob13], a formula in the nonsplit case is deduced from the split case by making use of some special features of the setup under consideration.

- for every $v \in \mathtt{V}_F^{(\infty)}$, $\pi(\mathrm{Sym}^{n-1} A_E)_v$ is isomorphic to $\pi_v^{[r]}$;
- for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^A$, $\pi(\mathrm{Sym}^{n-1} A_E)_v$ is spherical with respect to $K_{r,v}$ and its base change to $\mathrm{GL}_n(E_v)$ is isomorphic to $\Pi(\mathrm{Sym}^{n-1} A_E)_v$.

In particular, there exists a relevant $\mathbb{Q}$-representation π in the sense of Definition 1.1 such that $\pi \otimes_{\mathbb{Q}} \mathbb{C} \simeq \pi(\mathrm{Sym}^{n-1} A_E)^\infty$. Moreover, for every $v \in \mathtt{V}_F^{(p)}$, $\pi_v \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is Panchishkin unramified. Applying Theorem 1.4 to π (or rather $\pi \otimes_{\mathbb{Q}} \mathbb{Q}_p$), we obtain a bounded analytic function $\mathscr{L}_p^\diamond(\pi)$ on $\mathscr{X}_{F,p}$ for every finite set $\diamond$ of places of $\mathbb{Q}$ containing $\{\infty, p\}$ and every prime number underlying $\mathtt{V}_F^{\mathrm{ram}} \cup \mathtt{V}_F^A$. For every $v \in \mathtt{V}_F^{(p)}$ and $u \in \mathtt{P}_v$, the unramified representation $\underline{\pi}_u$ of $\mathrm{GL}_r(F_v)$ is the one with the Satake polynomial

$$\prod_{j=r}^{n-1} \left(T - \alpha_{v,1}^j \alpha_{v,2}^{n-1-j}\right) \in \mathbb{Q}_p[T],$$

where we have ordered $\alpha_{v,1}, \alpha_{v,2} \in \mathbb{Q}_p^\times$ in the way that $\alpha_{v,i} \in q_v^{i-1}\mathbb{Z}_p^\times$. The following is an immediate consequence of Corollary 1.9 in which $\mathtt{S}_\pi = \emptyset$.

Corollary 1.12. *Under the above setup, we further assume that*

- $n < p$,
- $[F : \mathbb{Q}] > 1$,
- $r[F : \mathbb{Q}]$ *is odd,*
- *every prime in* $\mathtt{V}_F^{\mathrm{ram}}$ *is unramified over* $\mathbb{Q}$,
- $\mathtt{V}_F^A \cup \mathtt{V}_F^{(2)}$ *is contained in* $\mathtt{V}_F^{\mathrm{spl}}$.

Then $\mathscr{L}_p^\diamond(\pi)(\mathbf{1}) = 0$. *Moreover, if* $\partial\mathscr{L}_p^\diamond(\pi)(\mathbf{1}) \neq 0$, *then*

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(E, \mathrm{Sym}^{n-1} \mathrm{H}^1_{\text{ét}}(A_{\overline{E}}, \mathbb{Q}_p)(r)) \geqslant 1.$$

1.5. **Structure and strategy.** We explain the structure of the article and the strategy for the proofs. Before that, we point out that throughout the article, we have restricted ourselves to only use p-adic measures valued in finite products of finite extensions of $\mathbb{Q}_p$ to reduce the technical burden such as infinite dimensional p-adic Banach spaces.

In Section 2, we make preparation for proving the rationality property of our p-adic L-function. In §2.1, we collect two sets of more specialized notation that will be used throughout the main part of the article. In §2.2, we introduce the notion of Siegel hermitian varieties which are over $\mathbb{Q}_p$ and are the main stage to characterize the rationality of automorphic forms on the unitary group G_r. In §2.3, we review the construction of an auxiliary Shimura variety over $\mathbb{Q}$ that is of PEL type in the sense of Kottwitz, which is needed to prove the rationality of certain Eisenstein series used in the doubling method. The main reason we pass to this auxiliary one is that the theory of algebraic q-expansions is only available for such Shimura varieties. However, if the reader is satisfied with fixing an isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ from the beginning and does not care about the field of definition of the p-adic L-function, then there is no need to use those parts of §2.2 that are related to Shimura varieties and the entire §2.3.

In Section 3, we construct the p-adic L-function. The main strategy is to use the doubling method for an "analytic" family of sections in the degenerate principal series of the doubling unitary group G_{2r}, similar to [EHLS20]. However, it is worth pointing out that our computation makes no use of Weil representations (or their twisted versions). In particular, we do not need any explicit Schwartz functions on hermitian spaces. In fact, we do not even need an explicit formula for the sections in the degenerate principal series at p-adic places – what we need is just their Fourier transforms, which have very simple forms. The main reason we can simplify the computation is a formula obtained in the previous work [LL21] for computing the local doubling zeta integral (see Lemma 3.26). Using this formula, the gamma factor in Theorem 1.4 appears naturally and immediately. In §3.1, we review the doubling degenerate principal series and collect some facts on their Siegel–Fourier coefficients. In §3.2, we review the doubling Eisenstein series and prove a certain rationality property of their pullbacks to the diagonal block. In §3.3, we make all the representational-theoretical preparations; in particular, we study Panchishkin unramified representations. In §3.4, we prove several formulae for local doubling zeta integrals. In §3.5, we complete the construction of the p-adic L-function by defining it as an inner product of a specific element in $\tilde{\pi} \boxtimes \pi$ and the pullback of the family of doubling Eisenstein series with respect to a careful choice of sections in degenerate principal series. In §3.6, we collect some basic facts about p-adic measures that will be used later.

In Section 4, we construct the so-called Selmer theta lifts, which are Selmer group analogues of the classical theta lifts, and study their p-adic heights. In §4.1, we introduce further notation for the whole section and study the rationality of local theta liftings. In §4.2, we construct a canonical projection from the absolute p-adic cohomology to the Galois cohomology $\mathrm{H}^1(E, \mathrm{V}_{\pi,L})$ or $\mathrm{H}^1(E, \mathrm{V}_{\tilde{\pi},L})$, and define p-adic height pairings on the latter. In §4.3, we state a theorem (Theorem 4.21) on the modularity of Kudla's generating functions valued in the Selmer groups of the above Galois cohomology, and then construct Selmer theta lifts which belong to the Selmer groups. In §4.4, we state the precise version of our p-adic arithmetic inner formula (Theorem 4.22), which is slightly stronger than Theorem 1.8 by taking rationality into account. In §4.5, we present our strategy of reducing Theorem 4.21 to the problem of computing p-adic height itself, and explain that it makes sense and suffices to consider the p-adic height pairing $\langle Z_{T_1}(\phi_1), Z_{T_2}(\phi_2)\rangle_E$ between (weighted) special cycles for a certain pool of Schwartz functions, together with a formula decomposing the (global) p-adic height pairing into local ones. In §4.6, we compute local p-adic height pairings between special cycles at (nonarchimedean) places of E not above p, based on a result of Scholl that relates local p-adic heights to Beilinson's local indices and the formulae for the latter from previous works [LL21, LL22]. In §4.7, we study local p-adic height pairings between special cycles at p-adic places of E. With a crucial ingredient (Theorem A.8) on the crystalline property of the corresponding bi-extensions, we show that the local p-adic heights approach 0 p-adically when one repeatedly applies a certain operator U_p to the Schwartz functions. In §4.8, we finish the proof of Theorem 1.9 assuming a nonvanishing result (Proposition 4.46), by using the previous formulae on local p-adic heights together with certain limit processes. In §4.9, we prove Proposition 4.46 by a variant of the p-adic doubling formula from §3.5; then we complete the proof of Theorem 4.22.

The article has two appendices. In Appendix A, we develop further the theory of p-adic heights on *general varieties*, after Nekovář. In particular, we prove a comparison result between local p-adic heights and Beilinson's local indices. For local p-adic heights above p, we prove a key theorem (Theorem A.8) on the crystalline property for certain bi-extensions, whose proof occupies the entire Appendix B.

1.6. Notation and conventions.

- We denote $\mathbb{N} \coloneqq \{0, 1, 2, \dots\}$.
- We denote by $\overline{\mathbb{Z}}_p$ the ring of integers of $\overline{\mathbb{Q}}_p$.
- We write $\boldsymbol{\pi} = 3.1415926\dots$, to be distinguished from the representation π. We also write $\boldsymbol{i}$ for the imaginary unit in $\mathbb{C}$, to be distinguished from the commonly used index i.
- When we have a function f on a product set $A_1 \times \cdots \times A_s$, we will write $f(a_1, \dots, a_s)$ instead of $f((a_1, \dots, a_s))$ for its value at an element $(a_1, \dots, a_s) \in A_1 \times \cdots \times A_s$.
- For a set S, we denote by $\mathbf{1}_S$ the characteristic function of S.
- All rings are commutative and unital; and ring homomorphisms preserve units. However, we use the word *algebra* in the general sense, which is not necessarily commutative or unital.
- If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.
- For an abelian group A and a ring R, we put $A_R \coloneqq A \otimes R$ as an R-module.
- For an abelian group A, we denote by A^{fr} its free quotient.
- For a ring R, we denote by $\mathrm{Sch}'_{/R}$ the category of locally Noetherian schemes over R.
- We denote by $\mathbf{G}$ the multiplicative group scheme, that is, $\operatorname{Spec} \mathbb{Z}[X, X^{-1}]$.
- For an integer $m \geqslant 0$, we denote by 0_m and 1_m the null and identity matrices of rank m, respectively, and by $\mathtt{w}_m$ the matrix $\begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix}$.
- Let $\psi_{\mathbb{Q}} \colon \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^\times$ be the standard automorphic additive character that sends w^{-1} at a prime w to $\exp(-2\boldsymbol{\pi}\boldsymbol{i}/w)$, and put $\psi_K \coloneqq \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{K/\mathbb{Q}}$ for every number field K.
- For a subring $R \subseteq \mathbb{C}$ and a positive integer Δ, we denote by $R\langle\Delta\rangle \subseteq \mathbb{C}$ the subring generated by Δ^l-th roots of unity for all $l \geqslant 0$.
- For a locally compact totally disconnected space X and a ring R, we denote by $\mathscr{S}(X, R)$ the R-module of R-valued locally constant compactly supported functions on X. We omit R from the notation when $R = \mathbb{C}$.

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2. SIEGEL HERMITIAN VARIETIES

Recall that we have fixed the CM extension E/F of number fields with the Galois involution $\mathtt{c}$, such that every p-adic place of F splits in E.

2.1. **Running notation.** We introduce two sets of more specialized notation that will be used throughout the main part of the article.

- (F1) We denote by
    - $\mathtt{V}_F$ and $\mathtt{V}_F^{\mathrm{fin}}$ the set of all places and non-archimedean places of F, respectively;
    - $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$ and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert and ramified in E, respectively;
    - $\mathtt{V}_F^{(\diamond)}$ the subset of $\mathtt{V}_F$ of places above a finite set $\diamond$ of places of $\mathbb{Q}$.

  Moreover,
    - for every $v \in \mathtt{V}_F$, we put $E_v := E \otimes_F F_v$;
    - for every finite set $\diamond$ of places of $\mathbb{Q}$, we put $F_\diamond := \prod_{v \in \mathtt{V}_F^{(\diamond)}} F_v$;
    - for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we denote by p_v the underlying rational prime of v and by $\mathfrak{p}_v$ the maximal ideal of O_{F_v}, put $q_v := |O_{F_v}/\mathfrak{p}_v|$ which is a power of p_v, and let $d_v \geqslant 0$ be the integer such that $\mathfrak{p}_v^{d_v}$ generates the different ideal of $F_v/\mathbb{Q}_{p_v}$.
- (F2) For every $v \in \mathtt{V}_F^{(p)}$, let $\mathtt{P}_v$ be the set of places of E above v. Put $\mathtt{P} := \bigcup_{v \in \mathtt{V}_F^{(p)}} \mathtt{P}_v$. We fix a subset $\mathtt{P}_{\mathrm{CM}}$ of $\mathtt{P}$ satisfying that $\mathtt{P}_{\mathrm{CM}} \cap \mathtt{P}_v$ is a singleton for every $v \in \mathtt{V}_F^{(p)}$.
- (F3) Let $m \geqslant 0$ be an integer.
    - We denote by Herm_m the subscheme of $\mathrm{Res}_{O_E/O_F} \mathrm{Mat}_{m,m}$ of m-by-m matrices b satisfying ${}^{\mathrm{t}}b^{\mathtt{c}} = b$. Put $\mathrm{Herm}_m^\circ := \mathrm{Herm}_m \cap \mathrm{Res}_{O_E/O_F} \mathrm{GL}_m$.
    - For every (ordered) partition $m = m_1 + \cdots + m_s$ with m_i a positive integer, we denote by
    $$\partial_{m_1,\dots,m_s} \colon \mathrm{Herm}_m \to \mathrm{Herm}_{m_1} \times \cdots \times \mathrm{Herm}_{m_s}$$
    the morphism that extracts the diagonal blocks with corresponding ranks.
    - We denote by $\mathrm{Herm}_m(F)^+$ (resp. $\mathrm{Herm}_m^\circ(F)^+$) the subset of $\mathrm{Herm}_m(F)$ of elements that are totally semi-positive definite (resp. totally positive definite).
- (F4) Let $\eta_{E/F} \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character associated with E/F. For every finite character $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ and every integer $m \geqslant 1$, we put
    - for every $v \in \mathtt{V}_F$,
    $$b_{m,v}(\chi) := \prod_{i=1}^{m} L(i, \chi_v \eta_{E/F,v}^{m-i});$$
    - for a finite set $\diamond$ of places of $\mathbb{Q}$,
    $$b_{m,\diamond}(\chi) := \prod_{v \in \mathtt{V}_F^{(\diamond)}} b_{m,v}(\chi), \quad b_m^\diamond(\chi) := \prod_{v \in \mathtt{V}_F \backslash \mathtt{V}_F^{(\diamond)}} b_{m,v}(\chi),$$
    in which the latter product is absolutely convergent when m is even or $\chi \neq \mathbf{1}$.

Let $m \geqslant 1$ be an integer. We equip $W_m = E^{2m}$ and $\bar{W}_m = E^{2m}$ with the skew-hermitian forms (that are E-linear in the first variable) given by the matrices $\mathtt{w}_m$ and $-\mathtt{w}_m$, respectively.

- (G1) Let G_m be the unitary group of both W_m and $\bar{W}_m$. We write elements of W_m and $\bar{W}_m$ in the row form, on which G_m acts from the right. Denote by $\dagger$ the involution of G_m given by the conjugation by the element $\begin{pmatrix} 1_m & \\ & -1_m \end{pmatrix}$ inside $\mathrm{Res}_{E/F} \mathrm{GL}_{2m}$.
- (G2) We denote by $\{e_1, \dots, e_{2m}\}$ and $\{\bar{e}_1, \dots, \bar{e}_{2m}\}$ the natural bases of W_m and $\bar{W}_m$, respectively.
- (G3) Let $P_m \subseteq G_m$ be the parabolic subgroup stabilizing the subspace generated by $\{e_{m+1}, \dots, e_{2m}\}$, and $N_m \subseteq P_m$ its unipotent radical.
- (G4) We have
    - a homomorphism $m \colon \mathrm{Res}_{E/F} \mathrm{GL}_m \to P_m$ sending a to
    $$m(a) := \begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathtt{c},-1} \end{pmatrix},$$

which identifies $\mathrm{Res}_{E/F}\,\mathrm{GL}_m$ as a Levi factor of P_m, denoted by M_m.

- a homomorphism $n\colon \mathrm{Herm}_m \to N_m$ sending b to
$$n(b) := \begin{pmatrix} 1_m & b \\ & 1_m \end{pmatrix},$$
which is an isomorphism.

(G5) We define a maximal compact subgroup $K_m = \prod_{v\in\mathtt{V}_F} K_{m,v}$ of $G_m(\mathbb{A}_F)$ in the following way:
- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $K_{m,v}$ is the stabilizer of the lattice $O_{E_v}^{2m}$;
- for $v \in \mathtt{V}_F^{(\infty)}$, $K_{m,v}$ is the subgroup of the form
$$[k_1, k_2] := \frac{1}{2}\begin{pmatrix} k_1 + k_2 & -\boldsymbol{i}k_1 + \boldsymbol{i}k_2 \\ \boldsymbol{i}k_1 - \boldsymbol{i}k_2 & k_1 + k_2 \end{pmatrix},$$
in which $k_i \in \mathrm{GL}_m(\mathbb{C})$ satisfies $k_i\,{}^{\mathrm{t}}\overline{k_i} = 1_m$ for $i = 1, 2$.[3]

Moreover,
- for every place w of $\mathbb{Q}$, put $K_{m,w} := \prod_{v\in\mathtt{V}_F^{(w)}} K_{m,v}$;
- for a set $\Diamond$ of places of $\mathbb{Q}$, put $K_m^{\Diamond} := \prod_{w\notin\Diamond} K_{m,w}$.

(G6) For every $v \in \mathtt{V}_F^{(\infty)}$, we have a character $\kappa_{m,v}\colon K_{m,v} \to \mathbb{C}^\times$ that sends $[k_1, k_2]$ to $\det k_1/\det k_2$.

(G7) For every $v \in \mathtt{V}_F$, we define a Haar measure $\mathrm{d}g_v$ on $G_m(F_v)$ as follows:
- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathrm{d}g_v$ is the Haar measure under which $K_{m,v}$ has volume 1;
- for $v \in \mathtt{V}_F^{(\infty)}$, $\mathrm{d}g_v$ is the product of the Haar measure on $K_{m,v}$ under which $K_{m,v}$ has volume 1 and the standard hyperbolic measure on $G_m(F_v)/K_{m,v}$ (see, for example, [EL, Section 2.1]).

Put $\mathrm{d}g = \prod_v \mathrm{d}g_v$, which is a Haar measure on $G_m(\mathbb{A}_F)$.

(G8) Let $m_1, \dots, m_s$ be finitely many positive integers. Put
$$G_{m_1,\dots,m_s} := G_{m_1} \times \cdots \times G_{m_s}.$$
We denote by $\mathcal{A}_{m_1,\dots,m_s}$ the space of both $\mathcal{Z}(\mathfrak{g}_{m_1,\dots,m_s,\infty})$-finite and $K_{m_1,\infty} \times \cdots \times K_{m_s,\infty}$-finite automorphic forms (in the sense of [BJ79, §4.2]) on $G_{m_1,\dots,m_s}(\mathbb{A}_F)$, where $\mathcal{Z}(\mathfrak{g}_{m_1,\dots,m_s,\infty})$ denotes the center of the complexified universal enveloping algebra of the Lie algebra $\mathfrak{g}_{m_1,\dots,m_s,\infty}$ of $G_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{R}$. For every integer $w \geqslant 0$ (as weight), we denote by
- $\mathcal{A}^{[w]}_{m_1,\dots,m_s}$ the maximal subspace of $\mathcal{A}_{m_1,\dots,m_s}$ on which for every $v \in \mathtt{V}_F^{(\infty)}$ and every $1 \leqslant j \leqslant s$, $K_{m_j,v}$ acts by the character $\kappa^w_{m_j,v}$,
- $\mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \subseteq \mathcal{A}^{[w]}_{m_1,\dots,m_s}$ the subspace of holomorphic ones.

(G9) For every vector space $\mathcal{H}$ on which $G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)$ acts, we put $\mathcal{H}(K) := \mathcal{H}^K$ for every open compact subgroup $K \subseteq G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)$.

2.2. Siegel hermitian varieties and line bundles of automorphy.

We first recall the construction of a CM moduli problem following [LTX⁺22, Section 3.5]. Let T be the subtorus of $\mathrm{Res}_{E/\mathbb{Q}}\,\mathbf{G}$ that is the inverse image of $\mathbf{G}_{\mathbb{Q}}$ under the norm map $\mathrm{Nm}_{E/F}\colon \mathrm{Res}_{E/\mathbb{Q}}\,\mathbf{G} \to \mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{G}$.

For every nonzero element $\delta \in E^{\mathtt{c}=-1}$, we denote by W^δ the E-vector space E (itself) together with a pairing $\langle\ ,\ \rangle^\delta\colon E \times E \to \mathbb{Q}$ given by $\langle x, y\rangle^\delta = \mathrm{Tr}_{E/\mathbb{Q}}(\delta x y^{\mathtt{c}})$. For every $\mathbb{Q}$-ring R, we have $T(R) = \{t \in (E \otimes_{\mathbb{Q}} R)^\times \mid \langle tx, ty\rangle^\delta = c(t)\langle x, y\rangle^\delta$ for some $c(t) \in R^\times\}$.

For every neat open compact subgroup K_T of $T(\mathbb{A}^\infty)$, we define a moduli problem $\mathbf{\Sigma}^\delta(K_T)$ on $\mathrm{Sch}'_{/\mathbb{Q}_p}$ as follows: for every $S \in \mathrm{Sch}'_{/\mathbb{Q}_p}$, $\mathbf{\Sigma}^\delta(K_T)(S)$ is the set of equivalence classes of quadruples $(A_0, i_0, \lambda_0, \eta_0)$ in which

- A_0 is an abelian scheme over S of relative dimension $[F : \mathbb{Q}]$,
- $i_0\colon E \to \mathrm{End}_S(A_0) \otimes \mathbb{Q}$ is an E-action such that for every $x \in E$, $\mathrm{tr}(i_0(x) \mid \mathrm{Lie}_S(A_0)) = \sum_{u\in\mathtt{P}_{\mathrm{CM}}} \mathrm{Tr}_{E_u/\mathbb{Q}_p}(x)$, where $\mathtt{P}_{\mathrm{CM}}$ is the fixed subset of $\mathtt{P}$ (§2.1(F2)),
- $\lambda_0\colon A_0 \to A_0^\vee$ is a quasi-polarization under which the Rosati involution coincides with the complex conjugation on E under i_0,

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[3] Here, we choose a complex embedding of E above v to identify $G_m(F_v)$ as a subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$. However, neither $K_{m,v}$ nor the character $\kappa_{m,v}$ in (G6) depends on such a choice.

- $\eta_0\colon W^\delta\otimes_{\mathbb{Q}}\mathbb{A}^\infty\to \mathrm{H}_1^{\text{ét}}(A_0,\mathbb{A}^\infty)$ is a K_T-level structure (see, for example, [LTX+22, Definition 3.5.4]).[4]

It is known that $\mathbf{\Sigma}^\delta(K_T)$ is a nonempty scheme finite étale over $\mathbb{Q}_p$, which admits a natural action by the finite group $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$ such that each orbit is Galois over $\operatorname{Spec}\mathbb{Q}_p$ with the Galois group $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$. We fix such an orbit $\mathbf{\Sigma}^\delta_0(K_T)$.

For every neat open compact subgroup $K\subseteq G_m(\mathbb{A}_F^\infty)$, we consider the moduli problem $\mathbf{\Sigma}^\delta_m(K,K_T)$ on $\mathrm{Sch}'_{/\mathbb{Q}_p}$ as follows: for every $S\in\mathrm{Sch}'_{/\mathbb{Q}_p}$, $\mathbf{\Sigma}^\delta_m(K,K_T)(S)$ is the set of equivalent classes of octuples $(A_0,i_0,\lambda_0,\eta_0;A,i,\lambda,\eta)$ in which

- $(A_0,i_0,\lambda_0,\eta_0)$ is an element of $\mathbf{\Sigma}^\delta_0(K_T)(S)$,
- A is an abelian scheme over S of relative dimension $2m[F:\mathbb{Q}]$,
- $i\colon E\to \mathrm{End}_S(A)\otimes\mathbb{Q}$ is an E-action such that for every $x\in E$, $\mathrm{tr}(i(x)\mid \mathrm{Lie}_S(A)) = m\,\mathrm{Tr}_{E/\mathbb{Q}}(x)$,
- $\lambda\colon A\to A^\vee$ is a quasi-polarization under which the Rosati involution coincides with the complex conjugation on E under i,
- $\eta\colon W^\delta_m\otimes_E\mathbb{A}_E^\infty\to \mathrm{Hom}_{\mathbb{A}_E^\infty}(\mathrm{H}_1^{\text{ét}}(A_0,\mathbb{A}^\infty),\mathrm{H}_1^{\text{ét}}(A,\mathbb{A}^\infty))$ is a K-level structure, where W^δ_m denotes the space E^{2m} equipped with the *hermitian* form $\delta^{-1}\cdot\mathtt{w}_m$ (see, for example, [LTX+22, Definition 4.2.2]).

It is known that $\mathbf{\Sigma}^\delta_m(K,K_T)$ is a scheme finite type over $\mathbf{\Sigma}^\delta_0(K_T)$, which admits a natural lift of the action of $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$. We denote by $\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$ the quotient of $\mathbf{\Sigma}^\delta_m(K,K_T)$ by $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$, as a presheaf on $\mathrm{Sch}'_{/\mathbb{Q}_p}$.

Now we discuss the relation between $\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$ and usual Shimura varieties. For every CM type $\mathbf{\Phi}$, we have the Deligne homomorphism

$$\begin{aligned}\mathrm{h}^{\mathbf{\Phi}}_m\colon\ \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbf{G}&\to(\mathrm{Res}_{F/\mathbb{Q}}\,G_m)\otimes_{\mathbb{Q}}\mathbb{R}\\ z&\to([1_m,(\overline{z}/z)1_m],\cdots,[1_m,(\overline{z}/z)1_m])\in K_{m,\infty},\end{aligned}$$

in which for every archimedean place v of F, the notation $[1_m,(\overline{z}/z)1_m]$ is understood via the unique complex embedding of E in $\mathbf{\Phi}$ inducing v. Then we obtain a projective system of Shimura varieties $\{\mathbf{\Sigma}^{\mathbf{\Phi}}_m(K)\}_K$ associated with the Shimura data $(\mathrm{Res}_{F/\mathbb{Q}}\,G_m,\mathrm{h}^{\mathbf{\Phi}}_m)$ indexed by neat open compact subgroups $K\subseteq G_m(\mathbb{A}_F^\infty)$, which are smooth quasi-projective complex schemes of dimension $m^2[F:\mathbb{Q}]$, with the complex analytification

$$\mathbf{\Sigma}^{\mathbf{\Phi}}_m(K)^{\mathrm{an}}=G_m(F)\backslash G_m(\mathbb{A}_F)/K_{m,\infty}K.$$

For every embedding $\iota\colon\mathbb{Q}_p\to\mathbb{C}$, we denote by $\mathbf{\Phi}_\iota$ the set of complex embeddings $i\colon E\to\mathbb{C}$ such that the p-adic place induced by the embedding $i\colon E\hookrightarrow i(E).\iota(\mathbb{Q}_p)$ belongs to P_{CM} (§2.1(F2)). Then $\mathbf{\Phi}_\iota$ is a CM type of E.

Lemma 2.1. *The presheaf* $\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$ *is a scheme over* $\mathbb{Q}_p$ *independent of the choices of* K_T, δ, *and the orbit* $\mathbf{\Sigma}^\delta_0(K_T)$.[5] *Moreover, for every embedding* $\iota\colon\mathbb{Q}_p\to\mathbb{C}$, *we have a canonical isomorphism*

$$\mathbf{\Sigma}^\delta_m(K,K_T)^\flat\otimes_{\mathbb{Q}_p,\iota}\mathbb{C}\xrightarrow{\sim}\mathbf{\Sigma}^{\mathbf{\Phi}_\iota}_m(K).$$

Proof. By definition, the reflex field $E_{\mathbf{\Phi}_\iota}\subseteq\mathbb{C}$ of $\mathbf{\Phi}_\iota$ is contained in $\iota(\mathbb{Q}_p)$. Then there is a canonical isomorphism

$$\left(X_K\otimes_{E_{\mathbf{\Phi}_\iota}}Y_{K_T}\right)\otimes_{E_{\mathbf{\Phi}_\iota},\iota^{-1}}\mathbb{Q}_p\simeq\mathbf{\Sigma}^\delta_m(K,K_T)$$

of schemes over $\mathbb{Q}_p$, where X_K and Y_{K_T} are the usual Shimura varieties for G_m and T of level K and K_T, respectively, over their common reflex field $E_{\mathbf{\Phi}_\iota}$. Under such isomorphism, $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$ acts on the left side via the second factor Y_{K_T} whose quotient is nothing but $\operatorname{Spec}E_{\mathbf{\Phi}_\iota}$. Thus, we obtain a canonical isomorphism $X_K\otimes_{E_{\mathbf{\Phi}_\iota},\iota^{-1}}\mathbb{Q}_p\simeq\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$. The lemma follows. ☐

Definition 2.2. We define the Siegel hermitian variety (of genus m and level K) over $\mathbb{Q}_p$, denoted as $\mathbf{\Sigma}_m(K)$, to be $\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$, which makes sense by the lemma above.[6]

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[4] In this article, we have been vague in writing level structures: Strictly speaking, one should choose a geometric point s on every connected component of S and the level structure is a $\pi_1(S,s)$-invariant orbit (with respect to the level subgroup) of isometries concerning the fiber at s.

[5] But $\mathbf{\Sigma}^\delta_m(K,K_T)^\flat$ depends on the fixed subset P_{CM}.

[6] By construction, $\mathbf{\Sigma}_m(K)$ also depends on the choice of the subset P_{CM} of P (§2.1(F2)).

Now we define the *line bundle of automorphy* on $\mathbf{\Sigma}_m(K)$. Denote by $\boldsymbol{A}$ (the A part of) the universal object over $\mathbf{\Sigma}^\delta_m(K,K_T)$. Then $\mathrm{Lie}(\boldsymbol{A})$ is a projective $\mathscr{O}\otimes_{\mathbb{Q}}E$-module of rank m, where $\mathscr{O}=\mathscr{O}_{\mathbf{\Sigma}^\delta_m(K,K_T)}$ is the structure sheaf. Put

$$\boldsymbol{\omega}^\delta_m := \det{}_{\mathscr{O}}\left(\det{}_{\mathscr{O}\otimes_{\mathbb{Q}}E}\mathrm{Lie}(\boldsymbol{A})^\vee\right),$$

which is a line bundle on $\mathbf{\Sigma}^\delta_m(K,K_T)$. Since $T(\mathbb{A}^\infty)/T(\mathbb{Q})K_T$ acts trivially on $\boldsymbol{\omega}^\delta_m$, $\boldsymbol{\omega}^\delta_m$ descends to a line bundle $\boldsymbol{\omega}_m$ on $\mathbf{\Sigma}_m(K)$. It is easy too see that $\boldsymbol{\omega}_m$ does not depend on the choices of K_T, δ, and the orbit $\mathbf{\Sigma}^\delta_0(K_T)$.

Now suppose that we are given a partition $m=m_1+\cdots+m_s$ of m into positive integers. We have a natural isometry

$$W_{m_1}\oplus\cdots\oplus W_{m_s}\simeq W_m \tag{2.1}$$

such that if we write $\{e^j_1,\dots,e^j_{2m_j}\}$ as the standard bases for W_{m_j} for $1\leqslant j\leqslant s$, then the standard basis of W_m is identified with $\{e^1_1,\dots,e^1_{m_1},\dots,e^s_1,\dots,e^s_{m_s},e^1_{m_1+1},\dots,e^1_{2m_1},\dots,e^s_{m_s+1},\dots,e^s_{2m_s}\}$. In particular, we may regard $G_{m_1,\dots,m_s}=G_{m_1}\times\cdots\times G_{m_s}$ as a subgroup of G_m. We obtain a map

$$\rho_{m_1,\dots,m_s}\colon \mathscr{A}^{[w]}_{m,\mathrm{hol}}\to\mathscr{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \tag{2.2}$$

(see §2.1(G8)) given by the restriction to the subgroup $G_{m_1,\dots,m_s}(\mathbb{A}_F)$.

For neat open compact subgroups $K_j\subseteq G_{m_j}(\mathbb{A}^\infty_F)$ for $1\leqslant j\leqslant s$, we put

$$\begin{aligned}\mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1\times\cdots\times K_s,K_T) &:= \mathbf{\Sigma}^\delta_{m_1}(K_1,K_T)\times_{\mathbf{\Sigma}^\delta_0(K_T)}\cdots\times_{\mathbf{\Sigma}^\delta_0(K_T)}\mathbf{\Sigma}^\delta_{m_s}(K_s,K_T),\\ \boldsymbol{\omega}^\delta_{m_1,\dots,m_s} &:= \boldsymbol{\omega}^\delta_{m_1}\boxtimes\cdots\boxtimes\boldsymbol{\omega}^\delta_{m_s};\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Sigma}_{m_1,\dots,m_s}(K_1\times\cdots\times K_s) &:= \mathbf{\Sigma}_{m_1}(K_1)\times_{\mathbb{Q}_p}\cdots\times_{\mathbb{Q}_p}\mathbf{\Sigma}_{m_s}(K_s),\\ \boldsymbol{\omega}_{m_1,\dots,m_s} &:= \boldsymbol{\omega}_{m_1}\boxtimes\cdots\boxtimes\boldsymbol{\omega}_{m_s}.\end{aligned}$$

We have the natural quotient map

$$\xi_{m_1,\dots,m_s}\colon \mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1\times\cdots\times K_s,K_T)\to\mathbf{\Sigma}_{m_1,\dots,m_s}(K_1\times\cdots\times K_s)$$

under which $\xi^*_{m_1,\dots,m_s}\boldsymbol{\omega}_{m_1,\dots,m_s}\simeq\boldsymbol{\omega}^\delta_{m_1,\dots,m_s}$.

For a neat open compact subgroup $K\subseteq G_m(\mathbb{A}^\infty_F)$ containing $K_1\times\cdots\times K_s$, there is a natural morphism

$$\sigma^\delta_{m_1,\dots,m_s}\colon \mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1\times\cdots\times K_s,K_T)\to\mathbf{\Sigma}^\delta_m(K,K_T)$$

sending $((A_0,i_0,\lambda_0,\eta_0;A_j,i_j,\lambda_j,\eta_j))_{1\leqslant j\leqslant s}$ to

$$(A_0,i_0,\lambda_0,\eta_0;A_1\times\cdots\times A_s,(i_1,\dots,i_s),\lambda_1\times\cdots\times\lambda_s,(\eta_1,\dots,\eta_s)).$$

It is clear that $\sigma^\delta_{m_1,\dots,m_s}$ descends to a morphism

$$\sigma_{m_1,\dots,m_s}\colon \mathbf{\Sigma}_{m_1,\dots,m_s}(K_1\times\cdots\times K_s)\to\mathbf{\Sigma}_m(K)$$

rendering the following diagram

(2.3)
$$\begin{tikzcd}
\mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1\times\cdots\times K_s,K_T) \arrow[r, "\sigma^\delta_{m_1,\dots,m_s}"] \arrow[d, "\xi_{m_1,\dots,m_s}"'] & \mathbf{\Sigma}^\delta_m(K,K_T) \arrow[d, "\xi_m"] \\
\mathbf{\Sigma}_{m_1,\dots,m_s}(K_1\times\cdots\times K_s) \arrow[r, "\sigma_{m_1,\dots,m_s}"] & \mathbf{\Sigma}_m(K)
\end{tikzcd}$$

in $\mathrm{Sch}'_{/\mathbb{Q}_p}$ commutative. It is independent of the choices of K_T, δ, and the orbit $\mathbf{\Sigma}^\delta_0(K_T)$. For the line bundles of automorphy, we have $(\sigma^\delta_{m_1,\dots,m_s})^*\boldsymbol{\omega}^\delta_m\simeq\boldsymbol{\omega}^\delta_{m_1,\dots,m_s}$, and hence $\sigma^*_{m_1,\dots,m_s}\boldsymbol{\omega}_m\simeq\boldsymbol{\omega}_{m_1,\dots,m_s}$.

For every integer $w\geqslant 0$, put

$$\begin{aligned}\mathscr{H}^w_{m_1,\dots,m_s}(K_1\times\cdots\times K_s) &:= \mathrm{H}^0(\mathbf{\Sigma}_{m_1,\dots,m_s}(K_1\times\cdots\times K_s),\boldsymbol{\omega}^{\otimes w}_{m_1,\dots,m_s}),\\ \mathscr{H}^w_{m_1,\dots,m_s} &:= \varinjlim_{K_1,\dots,K_s}\mathscr{H}^w_{m_1,\dots,m_s}(K_1\times\cdots\times K_s).\end{aligned}$$

For every embedding $\iota\colon \mathbb{Q}_p \to \mathbb{C}$, we have an injective map

$$\boldsymbol{h}^\iota_{m_1,\dots,m_s}\colon \mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \to \mathcal{H}^w_{m_1,\dots,m_s} \otimes_{\mathbb{Q}_p,\iota} \mathbb{C}, \tag{2.4}$$

which fits into the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{[w]}_{m,\mathrm{hol}} \arrow[r, "\rho_{m_1,\dots,m_s}", "(2.2)"'] \arrow[d, "\boldsymbol{h}^\iota_m"'] & \mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \arrow[d, "\boldsymbol{h}^\iota_{m_1,\dots,m_s}"] \\
\mathcal{H}^w_m \otimes_{\mathbb{Q}_p,\iota} \mathbb{C} \arrow[r, "\sigma^*_{m_1,\dots,m_s}"] & \mathcal{H}^w_{m_1,\dots,m_s} \otimes_{\mathbb{Q}_p,\iota} \mathbb{C}
\end{tikzcd}$$

of complex vector spaces.

Definition 2.3. Let the notation be as above.

(1) We define $\mathcal{H}^{[w]}_{m_1,\dots,m_s}$ to be the maximal subspace of $\mathcal{H}^w_{m_1,\dots,m_s}$ such that for every embedding $\iota\colon \mathbb{Q}_p \to \mathbb{C}$, $\mathcal{H}^{[w]}_{m_1,\dots,m_s} \otimes_{\mathbb{Q}_p,\iota} \mathbb{C}$ is contained in the image of $\mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}}$ under $\boldsymbol{h}^\iota_{m_1,\dots,m_s}$.

(2) For every $\varphi \in \mathcal{H}^{[w]}_{m_1,\dots,m_s}$ and every embedding $\iota\colon \mathbb{Q}_p \to \mathbb{C}$, we denote by φ^ι the unique element in $\mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}}$ such that $\boldsymbol{h}^\iota_{m_1,\dots,m_s}(\varphi^\iota) = \iota\varphi$.

Remark 2.4. We have the following remarks concerning $\mathcal{H}^{[w]}_{m_1,\dots,m_s}$.

(1) The inclusion $\mathcal{H}^{[w]}_{m_1,\dots,m_s} \subseteq \mathcal{H}^w_{m_1,\dots,m_s}$ is proper in general since in the definition of $\mathcal{H}^w_{m_1,\dots,m_s}$, we do not impose any growth condition along the boundary.

(2) It is clear that the subspace $\mathcal{H}^{[w]}_{m_1,\dots,m_s}$ is closed under the action of $G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)$. Moreover, in its definition, it suffices to check for *some* embedding ι.

(3) The natural map $\mathcal{H}^{[w]}_{m_1} \otimes_{\mathbb{Q}_p} \cdots \otimes_{\mathbb{Q}_p} \mathcal{H}^{[w]}_{m_s} \to \mathcal{H}^{[w]}_{m_1,\dots,m_s}$ given by exterior product is an isomorphism. Indeed, it suffices to check it at every finite level, which is then an isomorphism of *finite-dimensional* $\mathbb{Q}_p$-vector spaces.

To end this subsection, we review the notion of analytic q-expansion (or Siegel–Fourier expansion).

Definition 2.5. For every ring R, we denote by $\mathrm{SF}_{m_1,\dots,m_s}(R)$ the R-module of formal power series

$$\sum_{(T_1,\dots,T_s)\in\mathrm{Herm}_{m_1}(F)^+\times\cdots\times\mathrm{Herm}_{m_s}(F)^+} a_{T_1,\dots,T_s} q^{T_1,\dots,T_s}, \quad a_{T_1,\dots,T_s} \in R$$

in which $a_{T_1,\dots,T_s}$ vanishes unless the entries of $T_1,\dots,T_s$ are in a certain fractional ideal of E. We have a restriction map

$$\varrho_{m_1,\dots,m_s}\colon \mathrm{SF}_m(R) \to \mathrm{SF}_{m_1,\dots,m_s}(R)$$

sending

$$\sum_{T\in\mathrm{Herm}_m(F)^+} a_T q^T$$

to

$$\sum_{(T_1,\dots,T_s)\in\mathrm{Herm}_{m_1}(F)^+\times\cdots\times\mathrm{Herm}_{m_s}(F)^+} \left(\sum_{\substack{T\in\mathrm{Herm}_m(F)^+ \\ \partial_{m_1,\dots,m_s}T=(T_1,\dots,T_s)}} a_T \right) q^{T_1,\dots,T_s},$$

where $\partial_{m_1,\dots,m_s}$ is the map from §2.1(F3). It is an easy exercise to show that the interior summation is always a finite sum.

For every integer $w \geqslant 0$, we have a map

$$\begin{aligned} \boldsymbol{q}^{\mathrm{an}}_{m_1,\dots,m_s}\colon \mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} &\to \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C}) \\ \varphi &\mapsto \sum_{(T_1,\dots,T_s)\in\mathrm{Herm}_{m_1}(F)^+\times\cdots\times\mathrm{Herm}_{m_s}(F)^+} a_{T_1,\dots,T_s}(\varphi) q^{T_1,\dots,T_s} \end{aligned} \tag{2.5}$$

in which $a_{T_1,\dots,T_s}(\varphi)$ equals

$$\int_{\mathrm{Herm}_{m_1}(F)\backslash\mathrm{Herm}_{m_1}(\mathbb{A}_F)}\cdots\int_{\mathrm{Herm}_{m_s}(F)\backslash\mathrm{Herm}_{m_s}(\mathbb{A}_F)}\varphi(n(b_1),\dots,n(b_s))\psi_F(\mathrm{tr}\,T_1b_1)^{-1}\cdots\psi_F(\mathrm{tr}\,T_sb_s)^{-1}\,\mathrm{d}b_1\cdots\mathrm{d}b_s$$

with $\mathrm{d}b_1,\dots,\mathrm{d}b_s$ being the Tamagawa measures.

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{[w]}_{m,\mathrm{hol}} \arrow[r, "\rho_{m_1,\dots,m_s}", "(2.2)"'] \arrow[d, "\boldsymbol{q}^{\mathrm{an}}_m"'] & \mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \arrow[d, "\boldsymbol{q}^{\mathrm{an}}_{m_1,\dots,m_s}"] \\
\mathrm{SF}_m(\mathbb{C}) \arrow[r, "\varrho_{m_1,\dots,m_s}"] & \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C})
\end{tikzcd}$$

under restrictions.

We also need an equivariant version of the above constructions for use in §4.

Definition 2.6. For every ring R, we denote by $\mathcal{SF}_{m_1,\dots,m_s}(R)$ the $R[G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)]$-module

$$\mathrm{Map}\left(G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty),\mathrm{SF}_{m_1,\dots,m_s}(R)\right)$$

in which $G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)$ acts via the right translation. We have an injective $\mathbb{C}[G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty)]$-equivariant map

$$\boldsymbol{q}^\infty_{m_1,\dots,m_s}\colon\mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}}\to\mathcal{SF}_{m_1,\dots,m_s}(C)$$

such that $\boldsymbol{q}^\infty_{m_1,\dots,m_s}(\varphi)$ sends g to $\boldsymbol{q}^{\mathrm{an}}_{m_1,\dots,m_s}(g\cdot\varphi)$.

2.3. **Relation with PEL type moduli spaces.** In order to show the rationality of some Eisenstein series later, we need the theory of algebraic q-expansions. However, since such theory was only developed for PEL type Shimura varieties (in the sense of Kottwitz), we need to study its relation with our Siegel hermitian varieties.

Let $\widetilde{W}_m$ be the space E^{2m} equipped with the pairing $\mathrm{Tr}_{E/\mathbb{Q}}\circ\langle\ ,\ \rangle_{W_{2m}}\colon E^{2m}\times E^{2m}\to\mathbb{Q}$. Let $\widetilde{G}_m$ be the similitude group of $\widetilde{W}_m$, which is a reductive group over $\mathbb{Q}$. Let $\widetilde{P}_m\subseteq\widetilde{G}_m$ be the parabolic subgroup stabilizing the subspace generated by $\{e_{m+1},\dots,e_{2m}\}$,

Consider a partition $m=m_1+\cdots+m_s$ of m into positive integers. We denote by $\widetilde{G}_{m_1,\dots,m_s}$ the subgroup of $\widetilde{G}_{m_1}\times\cdots\times\widetilde{G}_{m_s}$ of common similitudes; in other words, it fits into a Cartesian diagram

$$\begin{tikzcd}
\widetilde{G}_{m_1,\dots,m_s} \arrow[r] \arrow[d] & \widetilde{G}_{m_1}\times\cdots\times\widetilde{G}_{m_s} \arrow[d] \\
\mathbf{G}_{\mathbb{Q}} \arrow[r, "\text{diagonal}"] & \mathbf{G}_{\mathbb{Q}}^s
\end{tikzcd}$$

in which the vertical arrows are similitude maps. In particular, we may regard $\widetilde{G}_{m_1,\dots,m_s}$ as a subgroup of $\widetilde{G}_m$. Put $\widetilde{P}_{m_1,\dots,m_s}\coloneqq\widetilde{G}_{m_1,\dots,m_s}\cap\widetilde{P}_m$.

For every neat open compact subgroup $\widetilde{K}_{m_1,\dots,m_s}\subseteq\widetilde{G}_{m_1,\dots,m_s}(\mathbb{A}^\infty)$, we consider the PEL type moduli problem $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})$ on $\mathrm{Sch}'_{/\mathbb{Q}}$ as follows: for every $S\in\mathrm{Sch}'_{/\mathbb{Q}}$, $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})(S)$ is the set of equivalence classes of s-tuples of quadruples $((A_1,i_1,\lambda_1,\widetilde{\eta}_1),\dots,(A_s,i_s,\lambda_s,\widetilde{\eta}_s))$ in which

- for $1\leqslant j\leqslant s$, A_j is an abelian scheme over S of relative dimension $2m_j[F:\mathbb{Q}]$,
- for $1\leqslant j\leqslant s$, $i_j\colon E\to\mathrm{End}_S(A_j)\otimes\mathbb{Q}$ is an E-action such that for every $x\in E$, $\mathrm{tr}(i_j(x)\mid\mathrm{Lie}_S(A_j))=m_j\,\mathrm{Tr}_{E/\mathbb{Q}}(x)$,
- for $1\leqslant j\leqslant s$, $\lambda_j\colon A_j\to A_j^\vee$ is a quasi-polarization under which the Rosati involution coincides with the complex conjugation on E under i_j,
- $\{\widetilde{\eta}_j\colon\widetilde{W}_m\otimes_{\mathbb{Q}}\mathbb{A}^\infty\to\mathrm{H}_1^{\text{ét}}(A_j,\mathbb{A}^\infty)\}_{1\leqslant j\leqslant s}$ is a $\widetilde{K}_{m_1,\dots,m_s}$-orbit of skew-hermitian similitudes with similitude factors independent of j.

Then $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})$ is a scheme of finite type over $\mathbb{Q}$. Now for a neat open compact subgroup $\widetilde{K}\subseteq\widetilde{G}_m(\mathbb{A}^\infty)$ containing $\widetilde{K}_{m_1,\dots,m_s}$, we have an obvious morphism

$$\widetilde{\sigma}_{m_1,\dots,m_s}\colon\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})\to\widetilde{\mathbf{\Sigma}}_m(\widetilde{K})$$

over $\mathbb{Q}$ by "taking the product of all factors". For neat open compact subgroups $\widetilde{K}_j \subseteq \widetilde{G}_{m_j}(\mathbb{A}^\infty)$ containing the image of $\widetilde{K}_{m_1,\dots,m_s}$ under the natural projection map $\widetilde{G}_{m_1,\dots,m_s} \to \widetilde{G}_{m_j}$, we have another obvious map

$$\tau_{m_1,\dots,m_s} \colon \widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \to \widetilde{\mathbf{\Sigma}}_{m_1}(\widetilde{K}_1) \times_{\mathbb{Q}} \cdots \times_{\mathbb{Q}} \widetilde{\mathbf{\Sigma}}_{m_s}(\widetilde{K}_s)$$

over $\mathbb{Q}$. On $\widetilde{\mathbf{\Sigma}}_m(\widetilde{K})$, we have the line bundle of automorphy $\widetilde{\boldsymbol{\omega}}_m$ similar to $\boldsymbol{\omega}_m^\delta$, which satisfies

$$\widetilde{\sigma}^*_{m_1,\dots,m_s}\widetilde{\boldsymbol{\omega}}_m \simeq \tau^*_{m_1,\dots,m_s}\left(\widetilde{\boldsymbol{\omega}}_{m_1} \boxtimes \cdots \boxtimes \widetilde{\boldsymbol{\omega}}_{m_s}\right).$$

Put $\widetilde{\boldsymbol{\omega}}_{m_1,\dots,m_s} \coloneqq \widetilde{\sigma}^*_{m_1,\dots,m_s}\widetilde{\boldsymbol{\omega}}_m$ for future use.

Remark 2.7. For every $1 \leqslant j \leqslant s$, we have an isometry $W^\delta_{m_j} \otimes_E W^\delta \xrightarrow{\sim} \widetilde{W}_{m_j}$. These isometries induce a homomorphism

$$\zeta_{m_1,\dots,m_s} \colon \operatorname{Res}_{F/\mathbb{Q}} G_{m_1,\dots,m_s} \times T \to \widetilde{G}_{m_1,\dots,m_s}$$

sending $(g_1,\dots,g_s,t)$ to $(g_1t,\dots,g_st)$, which is independent of the choice of δ. Using this map, we regard $\operatorname{Res}_{F/\mathbb{Q}} G_{m_1,\dots,m_s}$ as a subgroup of $\widetilde{G}_{m_1,\dots,m_s}$ in what follows.

For neat open compact subgroups $K_j \subseteq G_{m_j}(\mathbb{A}_F^\infty)$ for $1 \leqslant j \leqslant s$ and $K_T \subseteq T(\mathbb{A}^\infty)$ such that $K_1 \times \cdots \times K_s \times K_T$ is contained in $\widetilde{K}_{m_1,\dots,m_s}$, we have a natural morphism

$$\zeta_{m_1,\dots,m_s} \colon \mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1 \times \cdots \times K_s, K_T) \to \widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

sending $((A_0,i_0,\lambda_0,\eta_0;A_j,i_j,\lambda_j,\eta_j))_{1\leqslant j\leqslant s}$ to $((A_j,i_j,\lambda_j,\widetilde{\eta}_j))_{1\leqslant j\leqslant s}$, where $\widetilde{\eta}_j$ sends $w \otimes v$ to $\eta_j(w)(\eta_0(v))$. The morphism $\zeta_{m_1,\dots,m_s}$ is finite étale.

In summary, for every neat open compact subgroup $K \subseteq G_m(\mathbb{A}_F^\infty)$ containing $K_1 \times \cdots \times K_s$ and such that $\zeta_m(K \times K_T)$ is contained in $\widetilde{K}$, we have a diagram

(2.6)

$$\begin{tikzcd}
\widetilde{\mathbf{\Sigma}}_{m_1}(\widetilde{K}_1)_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \cdots \times_{\mathbb{Q}_p} \widetilde{\mathbf{\Sigma}}_{m_s}(\widetilde{K}_s)_{\mathbb{Q}_p} & \widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})_{\mathbb{Q}_p} \arrow[l, "\tau_{m_1,\dots,m_s}"'] \arrow[r, "\widetilde{\sigma}_{m_1,\dots,m_s}"] & \widetilde{\mathbf{\Sigma}}_m(\widetilde{K})_{\mathbb{Q}_p} \\
\mathbf{\Sigma}^\delta_{m_1}(K_1,K_T) \times_{\mathbf{\Sigma}^\delta_0(K_T)} \cdots \times_{\mathbf{\Sigma}^\delta_0(K_T)} \mathbf{\Sigma}^\delta_{m_s}(K_s,K_T) \arrow[u, "\zeta_{m_1}\times\cdots\times\zeta_{m_s}"] \arrow[d, "\xi_{m_1}\times\cdots\times\xi_{m_s}"'] \arrow[r, equal, "\mathrm{def}"] & \mathbf{\Sigma}^\delta_{m_1,\dots,m_s}(K_1 \times \cdots \times K_s, K_T) \arrow[u, "\zeta_{m_1,\dots,m_s}"] \arrow[d, "\xi_{m_1,\dots,m_s}"'] \arrow[r, "\sigma^\delta_{m_1,\dots,m_s}"] & \mathbf{\Sigma}^\delta_m(K,K_T) \arrow[u, "\zeta_m"'] \arrow[d, "\xi_m"] \\
\mathbf{\Sigma}_{m_1}(K_1) \times_{\mathbb{Q}_p} \cdots \times_{\mathbb{Q}_p} \mathbf{\Sigma}_{m_s}(K_s) \arrow[r, equal, "\mathrm{def}"] & \mathbf{\Sigma}_{m_1,\dots,m_s}(K_1 \times \cdots \times K_s) \arrow[r, "\sigma_{m_1,\dots,m_s}"] & \mathbf{\Sigma}_m(K)
\end{tikzcd}$$

in $\mathrm{Sch}'_{/\mathbb{Q}_p}$ expanding (2.3) as the lower-right square, in which various line bundles of automorphy are compatible under pullbacks.

Similar to $\mathcal{A}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}}$ (§2.1(G8)), we define a space $\widetilde{\mathcal{A}}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}}$ of certain automorphic forms on $\widetilde{G}_{m_1,\dots,m_s}(\mathbb{A})$ with the additional requirement that $(t1_{m_1},\dots,t1_{m_s})$ acts trivially for every $t \in T(\mathbb{R})$. We have a map

$$\widetilde{\rho}_{m_1,\dots,m_s} \colon \widetilde{\mathcal{A}}^{[w]}_{m,\mathrm{hol}} \to \widetilde{\mathcal{A}}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \tag{2.7}$$

given by the restriction to the subgroup $\widetilde{G}_{m_1,\dots,m_s}(\mathbb{A})$.

For every integer $w \geqslant 0$, put

$$\widetilde{\mathcal{H}}^w_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \coloneqq \mathrm{H}^0(\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}), \widetilde{\boldsymbol{\omega}}^{\otimes w}_{m_1,\dots,m_s}),$$
$$\widetilde{\mathcal{H}}^w_{m_1,\dots,m_s} \coloneqq \varinjlim_{\widetilde{K}_{m_1,\dots,m_s}} \widetilde{\mathcal{H}}^w_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}).$$

Definition 2.8. Similar to (2.4), we have an injective map

$$\widetilde{\boldsymbol{h}}_{m_1,\dots,m_s} \colon \widetilde{\mathcal{A}}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \to \widetilde{\mathcal{H}}^w_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{C}$$

for $w \geqslant 0$. We define $\widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s}$ to be the subspace of $\widetilde{\mathcal{H}}^{w}_{m_1,\dots,m_s}$ such that the image of $\widetilde{\boldsymbol{h}}_{m_1,\dots,m_s}$ coincides with $\widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{C}$. Thus, we obtain an isomorphism

$$\widetilde{\boldsymbol{h}}_{m_1,\dots,m_s} \colon \widetilde{\mathcal{A}}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \xrightarrow{\sim} \widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.8}$$

Now we review the algebraic theory of q-expansion for $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}$ from [Lan12]. Take an open compact subgroup $\widetilde{K}_{m_1,\dots,m_s} \subseteq \widetilde{G}_{m_1,\dots,m_s}(\mathbb{A}^\infty)$. We choose a smooth projective toroidal compactification $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})^{\mathrm{tor}}$ of $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})$ over $\mathbb{Q}$, and let $\widetilde{\boldsymbol{\omega}}^{\mathrm{tor}}_{m_1,\dots,m_s}$ be the canonical extension of $\widetilde{\boldsymbol{\omega}}_{m_1,\dots,m_s}$ to $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})^{\mathrm{tor}}$. Then by [Lan12, Definition 5.3.4], for every $w \geqslant 0$, we have the *algebraic q-expansion map*

$$\mathrm{H}^0(\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})^{\mathrm{tor}}, (\widetilde{\boldsymbol{\omega}}^{\mathrm{tor}}_{m_1,\dots,m_s})^{\otimes w}) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C})$$

(Definition 2.5) at the cusp "at infinity". We remark that the map $\boldsymbol{q}_{m_1,\dots,m_s}$ is not necessarily injective, since we only expand the section on the connected component of $\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})^{\mathrm{tor}} \otimes_{\mathbb{Q}} \mathbb{C}$ that contains the cusp "at infinity". By [Lan12, Remark 5.2.14], the natural map $\widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \to \mathrm{H}^0(\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}), (\widetilde{\boldsymbol{\omega}}_{m_1,\dots,m_s})^{\otimes w})$ (here we adopt a similar notation as in §2.1(G9)) factors through a map

$$\widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \to \mathrm{H}^0(\widetilde{\mathbf{\Sigma}}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})^{\mathrm{tor}}, (\widetilde{\boldsymbol{\omega}}^{\mathrm{tor}}_{m_1,\dots,m_s})^{\otimes w}),$$

hence we obtain a map

$$\boldsymbol{q}_{m_1,\dots,m_s} \colon \widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s}) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C}), \tag{2.9}$$

which is independent of the choice of the toroidal compactification. Thus, by passing to the colimit, we obtain a map

$$\boldsymbol{q}_{m_1,\dots,m_s} \colon \widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C}), \tag{2.10}$$

which fits into the following commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{A}}^{[w]}_{m,\mathrm{hol}} \arrow[r, "\widetilde{\rho}_{m_1,\dots,m_s}", "(2.7)"'] \arrow[d, "\widetilde{\boldsymbol{h}}_m"'] & \widetilde{\mathcal{A}}^{[w]}_{m_1,\dots,m_s,\mathrm{hol}} \arrow[d, "\widetilde{\boldsymbol{h}}_{m_1,\dots,m_s}"] \\
\widetilde{\mathcal{H}}^{[w]}_{m} \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\widetilde{\sigma}^*_{m_1,\dots,m_s}"] \arrow[d, "\boldsymbol{q}_m"'] & \widetilde{\mathcal{H}}^{[w]}_{m_1,\dots,m_s} \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, "\boldsymbol{q}_{m_1,\dots,m_s}"] \\
\mathrm{SF}_m(\mathbb{C}) \arrow[r, "\varrho_{m_1,\dots,m_s}", "\text{Def. 2.5}"'] & \mathrm{SF}_{m_1,\dots,m_s}(\mathbb{C})
\end{tikzcd} \tag{2.11}$$

of complex vector spaces.

Definition 2.9. Denote by $\mathfrak{D}_E \subseteq O_E$ the different ideal of $E/\mathbb{Q}$. The (projective) O_E-lattice $\mathcal{W}_m := (O_E)^m \oplus (\mathfrak{D}_E^{-1})^m$ of W_m defines an integral model $\mathcal{G}_m$ (resp. $\widetilde{\mathcal{G}}_m$) of G_m (resp. $\widetilde{G}_m$) over O_F (resp. $\mathbb{Z}$).[7] Similarly, we have $\mathcal{G}_{m_1,\dots,m_s}$ and $\widetilde{\mathcal{G}}_{m_1,\dots,m_s}$ and their parabolic subgroups $\mathcal{P}_{m_1,\dots,m_s}$ and $\widetilde{\mathcal{P}}_{m_1,\dots,m_s}$, respectively.

Notation 2.10. For future use, we introduce some standard open compact subgroups. Take two positive integers Δ and Δ' that are coprime to each other. We put

$$\widetilde{K}_{m_1,\dots,m_s}(\Delta,\Delta') := \widetilde{\mathcal{G}}_{m_1,\dots,m_s}(\widehat{\mathbb{Z}}) \times_{\widetilde{\mathcal{G}}_{m_1,\dots,m_s}(\mathbb{Z}/\Delta\Delta')} \widetilde{\mathcal{P}}_{m_1,\dots,m_s}(\mathbb{Z}/\Delta),$$

$$K_{m_1,\dots,m_s}(\Delta,\Delta') := G_{m_1,\dots,m_s}(\mathbb{A}_F^\infty) \cap \widetilde{K}_{m_1,\dots,m_s}(\Delta,\Delta')$$

in view of Remark 2.7.

Lemma 2.11. *When $\widetilde{K}_{m_1,\dots,m_s} = \widetilde{K}_{m_1,\dots,m_s}(\Delta,\Delta')$, the map (2.9) is equivariant under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta'\rangle)$, where we recall that $\mathbb{Q}\langle\Delta'\rangle \subseteq \mathbb{C}$ is the subfield generated by Δ'^l-th roots of unity for all $l \geqslant 1$.*

Proof. This follows from the fact that the cusp "at infinity" is defined over the subfield $\mathbb{Q}\langle\Delta'\rangle$ at this level structure. See [Lan12] for more details. □

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[7] For $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathcal{G}_m(O_{F_v}) = K_{m,v}$ if and only if $d_v = 0$ and $v \notin \mathtt{V}_F^{\mathrm{ram}}$.

Remark 2.12. Denote by $\widetilde{G}^{\mathrm{der}}_{m_1,\dots,m_s}$ the derived subgroup of $\widetilde{G}_{m_1,\dots,m_s}$ and consider the maximal abelian quotient $\widetilde{G}^{\mathrm{ab}}_{m_1,\dots,m_s} := \widetilde{G}_{m_1,\dots,m_s}/\widetilde{G}^{\mathrm{der}}_{m_1,\dots,m_s}$. Since $\widetilde{G}^{\mathrm{der}}_{m_1,\dots,m_s}$ is simply connected, for every open compact subgroup $\widetilde{K}_{m_1,\dots,m_s} \subseteq \widetilde{G}_{m_1,\dots,m_s}(\mathbb{A}^\infty)$, the natural map

$$\mathbf{\Sigma}_{m_1,\dots,m_s}(\widetilde{K}_{m_1,\dots,m_s})(\mathbb{C}) \to \widetilde{G}^{\mathrm{ab}}_{m_1,\dots,m_s}(\mathbb{Q})\backslash \widetilde{G}^{\mathrm{ab}}_{m_1,\dots,m_s}(\mathbb{A}^\infty)/\widetilde{K}^{\mathrm{ab}}_{m_1,\dots,m_s}$$

has connected fibers, where $\widetilde{K}^{\mathrm{ab}}_{m_1,\dots,m_s}$ denotes the image of $\widetilde{K}_{m_1,\dots,m_s}$ in $\widetilde{G}^{\mathrm{ab}}_{m_1,\dots,m_s}(\mathbb{A}^\infty)$. It is clear that $\widetilde{K}_{m_1,\dots,m_s}(\Delta,\Delta')^{\mathrm{ab}}$ depends only on Δ', which we denote by $\widetilde{K}^{\mathrm{ab}}_{m_1,\dots,m_s}(\Delta')$.

3. Cyclotomic p-adic L-function

In this section, we construct the p-adic L-function. We fix an even positive integer $n = 2r$.

3.1. **Doubling space and degenerate principal series.** We have the doubling skew-hermitian space $W_r^\square := W_r \oplus \bar{W}_r$. Let $G_r^\square$ be the unitary group of $W_r^\square$, which admits a canonical embedding $\imath\colon G_r \times G_r \hookrightarrow G_r^\square$. We now take a basis $\{e_1^\square,\dots,e_{4r}^\square\}$ of $W_r^\square$ by the formula

$$e_i^\square = e_i, \quad e_{r+i}^\square = -\bar{e}_i, \quad e_{2r+i}^\square = e_{r+i}, \quad e_{3r+i}^\square = \bar{e}_{r+i}$$

for $1 \leqslant i \leqslant r$, under which we may identify $W_r^\square$ with W_{2r} and $G_r^\square$ with G_{2r}. Put $\mathtt{w}_r^\square := \mathtt{w}_{2r}$, $P_r^\square := P_{2r}$ and $N_r^\square := N_{2r}$. We denote by

$$\delta_r^\square\colon P_r^\square \to \mathbf{G}_F$$

the composition of the Levi quotient map $P_r^\square = P_{2r} \to M_{2r}$, the isomorphism $m^{-1}\colon M_{2r} \to \mathrm{Res}_{E/F}\mathrm{GL}_{2r}$, the determinant $\mathrm{Res}_{E/F}\mathrm{GL}_{2r} \to \mathrm{Res}_{E/F}\mathbf{G}$ and the norm $\mathrm{Nm}_{E/F}\colon \mathrm{Res}_{E/F}\mathbf{G} \to \mathbf{G}_F$. Put

$$\mathbf{w}_r := \begin{pmatrix} & & 1_r & \\ & 1_r & & \\ -1_r & 1_r & & \\ & & 1_r & 1_r \end{pmatrix} \in G_r^\square(F). \tag{3.1}$$

Then $P_r^\square \cdot \mathbf{w}_r \cdot \imath(G_r \times G_r)$ is Zariski open in $G_r^\square$.

In what follows, we will regard $G_r \times G_r$ as a subgroup of $G_{2r} = G_r^\square$ via the isometry (2.1), which is precisely the embedding

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}. \tag{3.2}$$

Remark 3.1. The embedding $\imath\colon G_r \times G_r \hookrightarrow G_r^\square = G_{2r}$ coincides with the embedding (3.2) twisted by the involution $\mathrm{id} \times \dagger$ on $G_r \times G_r$.

Let $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$ be a finite character, regarded as an automorphic character of $\mathbb{A}_F^\times$. For every place v of F, we have the degenerate principal series of $G_r^\square(F_v)$, which is defined as the normalized induced representation

$$\mathrm{I}_{r,v}^\square(\chi_v) := \mathrm{Ind}_{P_r^\square(F_v)}^{G_r^\square(F_v)}(\chi_v \circ \delta_{r,v}^\square)$$

of $G_r^\square(F_v)$ with complex coefficients. For every $f \in \mathrm{I}_{r,v}^\square(\chi_v)$ and every $T^\square \in \mathrm{Herm}_{2r}^\circ(F_v)$, we can regularize the following integral

$$W_{T^\square}(f) := \int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b))\psi_{F,v}(\mathrm{tr}\, T^\square b)^{-1}\,\mathrm{d}b, \tag{3.3}$$

where $\mathrm{d}b$ is the self-dual measure on $\mathrm{Herm}_{2r}(F_v)$ with respect to $\psi_{F,v}$. Indeed, one has a family of integrals $W_{T^\square}(f_s)$ for $s \in \mathbb{C}$, where $f_s \in \mathrm{I}_{r,v}^\square(\chi_v|\,|_{F_v}^s)$ is the standard section induced by f; it is absolutely convergent when $\mathrm{Re}\, s$ is large enough and has an analytic continuation to $\mathbb{C}$. Then $W_{T^\square}(f)$ is defined as the value at 0 of this analytic continuation. See [Wal88, Theorem 8.1] and [Kar79, Corollary 3.6.1] for more details.

In order to show the rationality of our p-adic L-function, we need to extend the degenerate principal series to $\widetilde{G}_{2r}$. Recall that we have a natural inclusion $\mathrm{Res}_{F/\mathbb{Q}} G_r^\square = \mathrm{Res}_{F/\mathbb{Q}} G_{2r} \hookrightarrow \widetilde{G}_{2r}$. We have a map

$$s\colon \mathbf{G}_\mathbb{Q} \to \widetilde{G}_{2r}$$

sending c to $\begin{pmatrix} c1_{2r} & \\ & 1_{2r} \end{pmatrix}$. Then the natural map $\mathrm{Res}_{F/\mathbb{Q}} P_{2r} \times s(\mathbf{G}_{\mathbb{Q}}) \to \widetilde{P}_{2r}$ is an isomorphism.

Take a place w of $\mathbb{Q}$. Put

$$\psi_{F,w} := \prod_{v \in \mathtt{V}_F^{(w)}} \psi_{F,v}, \quad \chi_w := \prod_{v \in \mathtt{V}_F^{(w)}} \chi_v, \quad \mathrm{I}^{\Box}_{r,w}(\chi_w) := \bigotimes_{v \in \mathtt{V}_F^{(w)}} \mathrm{I}^{\Box}_{r,v}(\chi_v),$$

and

$$\delta^{\Box}_{r,w} := \prod_{v \in \mathtt{V}_F^{(w)}} \delta^{\Box}_{r,v} \colon \prod_{v \in \mathtt{V}_F^{(w)}} P^{\Box}_r(F_v) = (\mathrm{Res}_{F/\mathbb{Q}} P^{\Box}_r)(\mathbb{Q}_w) \to (F_w)^{\times}.$$

The map $\delta^{\Box}_{r,w}$ extends uniquely to a map $\widetilde{\delta}^{\Box}_{r,w}$ along the inclusion $(\mathrm{Res}_{F/\mathbb{Q}} P^{\Box}_r)(\mathbb{Q}_w) = (\mathrm{Res}_{F/\mathbb{Q}} P_{2r})(\mathbb{Q}_w) \subseteq \widetilde{P}_{2r}(\mathbb{Q}_w)$ that sends $s(c)$ to c^{2r} for $c \in \mathbb{Q}_w^{\times}$. Then we have a canonical isomorphism

$$\mathrm{I}^{\Box}_{r,w}(\chi_w) \simeq \mathrm{Ind}_{\widetilde{P}_{2r}(\mathbb{Q}_w)}^{\widetilde{G}_{2r}(\mathbb{Q}_w)}(\chi_w \circ \widetilde{\delta}^{\Box}_{r,w})$$

so that $\mathrm{I}^{\Box}_{r,w}(\chi_w)$ becomes a representation of $\widetilde{G}_{2r}(\mathbb{Q}_w)$. For every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F_w)$, we define the functional $W_{T^{\Box}}(-)$ on $\mathrm{I}^{\Box}_{r,w}(\chi_w)$ to be the product of the corresponding ones over $v \in \mathtt{V}_F^{(w)}$.

Lemma 3.2. *For every $v \in \mathtt{V}_F^{(\infty)}$, denote by $f_v^{[r]} \in \mathrm{I}^{\Box}_{r,v}(\chi_v) = \mathrm{I}^{\Box}_{r,v}(\mathbf{1})$ the unique section whose restriction to $K_{2r,v}$ is the character $\kappa^r_{2r,v}$. Put $f_\infty^{[r]} := \otimes_{v \in \mathtt{V}_F^{(\infty)}} f_v^{[r]}$. Then there exists $W_{2r} \in \mathbb{Q}_{>0}$ such that*

$$W_{T^{\Box}}(f_\infty^{[r]}) = W_{2r} \cdot b_{2r}^{\infty}(\mathbf{1}) \cdot \exp(-2\boldsymbol{\pi}\, \mathrm{Tr}_{F/\mathbb{Q}} \operatorname{tr} T^{\Box})$$

for every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F)^{+}$.

Proof. For two elements $x, y \in \mathbb{C}^{\times}$, we write $x \sim y$ if their quotient is rational.

By [Liu11a, Proposition 4.5(2)], we have

$$W_{T^{\Box}}(f_\infty^{[r]}) = \left(\frac{(2\boldsymbol{\pi})^{r(2r+1)}}{\Gamma(1)\Gamma(2)\cdots\Gamma(2r)}\right)^{[F:\mathbb{Q}]} \exp(-2\boldsymbol{\pi}\, \mathrm{Tr}_{F/\mathbb{Q}} \operatorname{tr} T^{\Box})$$

for every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F)^{+}$. The positivity of W_{2r} then follows. Thus, it remains to show that $b_{2r}^{\infty}(\mathbf{1}) \sim \boldsymbol{\pi}^{r(2r+1)[F:\mathbb{Q}]}$.

Write $L(s, \eta^i_{E/F})$ for the complete L-function for the self-dual character $\eta^i_{E/F}$. Then by the functional equation, we have

$$\prod_{i=1}^{2r} L(i, \eta^i_{E/F}) \sim \prod_{i=1}^{2r} L(1-i, \eta^i_{E/F}).$$

By a well-known result of Siegel, $\prod_{i=1}^{2r} L^{\infty}(1-i, \eta^i_{E/F})$ is rational. It follows that

$$b_{2r}^{\infty}(\mathbf{1}) \sim \frac{\prod_{i=1}^{2r} L_\infty(1-i, \eta^i_{E/F})}{\prod_{i=1}^{2r} L_\infty(i, \eta^i_{E/F})} = \left(\frac{\prod_{i=1}^{2r} L_{\mathbb{R}}(1-i, \mathrm{sgn}^i)}{\prod_{i=1}^{2r} L_{\mathbb{R}}(i, \mathrm{sgn}^i)}\right)^{[F:\mathbb{Q}]} \sim \left(\frac{\boldsymbol{\pi}^{r^2}}{\boldsymbol{\pi}^{-r(r+1)}}\right)^{[F:\mathbb{Q}]} = \boldsymbol{\pi}^{r(2r+1)[F:\mathbb{Q}]}.$$

The lemma follows. □

From now to the end of this subsection, we assume $w \neq \infty$.

Lemma 3.3. *We have*

(1) For $v \in \mathtt{V}_F^{(w)}$ and $b \in \mathrm{Herm}_{2r}(F_v)$, the relation

$$W_{T^{\Box}}(n(b)f) = \psi_{F,v}(\operatorname{tr} T^{\Box} b) \cdot W_{T^{\Box}}(f)$$

holds for every $f \in \mathrm{I}^{\Box}_{r,v}(\chi_v)$ and every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F_v)$.

(2) For $v \in \mathtt{V}_F^{(w)}$ and $a \in \mathrm{GL}_{2r}(E_v)$, the relation

$$W_{T^{\Box}}(m(a)f) = \chi_v(\mathrm{Nm}_{E_v/F_v} \det a)^{-1} |\det a|^{r}_{E_v} \cdot W_{{}^{t}a^{c} T^{\Box} a}(f)$$

holds for every $f \in \mathrm{I}^{\Box}_{r,v}(\chi_v)$ and every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F_v)$.

(3) For $c \in \mathbb{Q}_w^{\times}$, the relation

$$W_{T^{\Box}}(s(c)f) = \chi_w(c)^{-2r} |c|^{2r^2}_{F_v} \cdot W_{cT^{\Box}}(f)$$

holds for every $f \in \mathrm{I}^{\Box}_{r,w}(\chi_w)$ and every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F_w)$.

Proof. This is well-known. For readers' convenience, we give a (formal) proof.

For (1), we have

$$\begin{aligned} W_{T^\square}(n(b)f) &= \int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b')n(b))\psi_{F,v}(\operatorname{tr} T^\square b')^{-1}\,\mathrm{d}b' \\ &= \psi_{F,v}(\operatorname{tr} T^\square b)\int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b'+b))\psi_{F,v}(\operatorname{tr} T^\square(b'+b))^{-1}\,\mathrm{d}b' = \psi_{F,v}(\operatorname{tr} T^\square b)\cdot W_{T^\square}(f). \end{aligned}$$

For (2), we have

$$\begin{aligned} W_{T^\square}(m(a)f) &= \int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b)m(a))\psi_{F,v}(\operatorname{tr} T^\square b)^{-1}\,\mathrm{d}b \\ &= \int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square m(a)n(a^{-1}b\,{}^{\mathrm{t}}a^{\mathtt{c},-1}))\psi_{F,v}(\operatorname{tr} T^\square b)^{-1}\,\mathrm{d}b \\ &= \int_{\mathrm{Herm}_{2r}(F_v)} f(m({}^{\mathrm{t}}a^{\mathtt{c},-1})\mathtt{w}_r^\square n(a^{-1}b\,{}^{\mathrm{t}}a^{\mathtt{c},-1}))\psi_{F,v}(\operatorname{tr}({}^{\mathrm{t}}a^{\mathtt{c}}T^\square a)(a^{-1}b\,{}^{\mathrm{t}}a^{\mathtt{c},-1}))^{-1}\,\mathrm{d}b \\ &= \chi_v(\mathrm{Nm}_{E_v/F_v}\det a)^{-1}|\det a|_{E_v}^{r}\int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b))\psi_{F,v}(\operatorname{tr}({}^{\mathrm{t}}a^{\mathtt{c}}T^\square a)b)^{-1}\,\mathrm{d}b \\ &= \chi_v(\mathrm{Nm}_{E_v/F_v}\det a)^{-1}|\det a|_{E_v}^{r}\cdot W_{{}^{\mathrm{t}}a^{\mathtt{c}}T^\square a}(f). \end{aligned}$$

The proof for (3) is similar to (2) and we omit it. The lemma is proved. □

Notation 3.4. Let $v \in \mathtt{V}_F^{\mathrm{fin}}$ be a finite place.

(1) We denote by $\mathrm{I}_{r,v}^\square(\chi_v)^\circ$ the subspace of $\mathrm{I}_{r,v}^\square(\chi_v)$ consisting of sections that are supported on the big Bruhat cell $P_r^\square(F_v)\cdot\mathtt{w}_r^\square\cdot N_r^\square(F_v)$.

(2) When $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^{(p)}$, we denote by $f_{\chi_v}^{\mathrm{sph}} \in \mathrm{I}_{r,v}^\square(\chi_v)$ the unique section that takes value 1 on $K_{2r,v}$.

It is clear that $\mathrm{I}_{r,v}^\square(\chi_v)^\circ$ is stable under the action of $P_r^\square(F_v)$. For $f \in \mathrm{I}_{r,v}^\square(\chi_v)^\circ$ and $T^\square \in \mathrm{Herm}_{2r}(F_v)$, we put

$$W_{T^\square}(f) := \int_{\mathrm{Herm}_{2r}(F_v)} f(\mathtt{w}_r^\square n(b))\psi_{F,v}(\operatorname{tr} T^\square b)^{-1}\,\mathrm{d}b,$$

which is in fact a finite sum and coincides with (3.3) for $T^\square \in \mathrm{Herm}_{2r}^\circ(F_v)$. It is clear that the assignment $T^\square \mapsto W_{T^\square}(f)$ is a Schwartz function on $\mathrm{Herm}_{2r}(F_v)$. Conversely, using the Fourier inversion formula, we know that for every $\mathtt{f} \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$, there exists a unique section $\mathtt{f}^{\chi_v} \in \mathrm{I}_{r,v}^\square(\chi_v)^\circ$ such that $W_{T^\square}(\mathtt{f}^{\chi_v}) = \mathtt{f}(T^\square)$ holds for every $T^\square \in \mathrm{Herm}_{2r}(F_v)$. In other words, we obtain a bijection

$$-^{\chi_v}\colon \mathscr{S}(\mathrm{Herm}_{2r}(F_v)) \xrightarrow{\sim} \mathrm{I}_{r,v}^\square(\chi_v)^\circ. \tag{3.4}$$

Put $\mathrm{I}_{r,w}^\square(\chi_w)^\circ := \bigotimes_{v\in\mathtt{V}_F^{(w)}} \mathrm{I}_{r,v}^\square(\chi_v)^\circ$ and we obtain an isomorphism

$$-^{\chi_w}\colon \mathscr{S}(\mathrm{Herm}_{2r}(F_w)) \xrightarrow{\sim} \mathrm{I}_{r,w}^\square(\chi_w)^\circ$$

by taking product over $v \in \mathtt{V}_F^{(w)}$.

Lemma 3.5. *Suppose that (the rational prime) $w \neq p$.*

(1) For every $v \in \mathtt{V}_F^{(w)} \setminus \mathtt{V}_F^{\mathrm{ram}}$ and every $g \in G_{2r}(F_v)$, there exists a finitely generated ring $\mathbb{O}_g$ contained in $\mathbb{Z}_{(p)}\langle w\rangle$ such that for every $T^\square \in \mathrm{Herm}_{2r}^\circ(F_v)$, there exists a unique element ${}^{g}\mathsf{W}_{T^\square,v}^{\mathrm{sph}} \in \mathbb{O}_g[X, X^{-1}]$ such that

$${}^{g}\mathsf{W}_{T^\square,v}^{\mathrm{sph}}(\chi_v(\varpi_v)) = b_{2r,v}(\chi)\cdot W_{T^\square}(g\cdot f_{\chi_v}^{\mathrm{sph}})$$

holds for every finite character $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$, where ϖ_v is an arbitrary uniformizer of F_v. Moreover, $\mathsf{W}_{T^\square,v}^{\mathrm{sph}} := {}^{1_{4r}}\mathsf{W}_{T^\square,v}^{\mathrm{sph}} \in \mathbb{Z}[X]$.

(2) For every $f \in \mathrm{I}_{r,w}^\square(\chi_w)$ and every $T^\square \in \mathrm{Herm}_{2r}^\circ(F_w)$, we have

$$W_{T^\square}(\sigma f) = \sigma W_{T^\square}(f)$$

for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle w\rangle)$.

(3) For every $f \in \mathrm{I}^{\Box}_{r,w}(\chi_w)^{\circ}$ that is fixed by $\widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w)$ and every $T^{\Box} \in \mathrm{Herm}^{\circ}_{2r}(F_w)$, we have

$$W_{T^{\Box}}(\sigma f) = \sigma W_{T^{\Box}}(f)$$

for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Proof. For (1), by Lemma 3.3(1,2) and the Iwasawa decomposition $G_{2r}(F_v) = P_{2r}(F_v)K_{2r,v}$, it suffices to consider the case where $g = 1_{4r}$. Then the statement follows from [LZ22, Theorem 3.5.1], together with the discussion in [LZ22, Sections 3.2 & 3.3].[8]

Part (2) follows from the proof of [Kar79, Corollary 3.6.1] and the fact that $\psi_{F,w}$ takes values in $\mathbb{Q}\langle w\rangle$.

For (3), put $\mathfrak{D}_b := \{c\,{}^{\mathrm{t}}a^{\mathtt{c}}ba \mid a \in \mathrm{GL}_{2r}(O_{E_w}), c \in \mathbb{Z}_w^{\times}\}$ for every $b \in \mathrm{Herm}_{2r}(F_w)$, which is an open compact subset of $\mathrm{Herm}_{2r}(F_w)$. It follows easily that

$$c_{T^{\Box},\mathfrak{D}_b} := \int_{\mathfrak{D}_b} \psi_{F,v}(\mathrm{tr}\, T^{\Box}b')^{-1}\,\mathrm{d}b' \in \mathbb{Q}.$$

Since χ_w is unramified, the assignment $b' \mapsto f(\mathtt{w}_r^{\Box}n(b'))$ is constant on each subset $\mathfrak{D}_b$, which we denote as $f_{\mathfrak{D}_b}$. Then $(\sigma f)_{\mathfrak{D}_b} = \sigma f_{\mathfrak{D}_b}$. It follows that

$$W_{T^{\Box}}(\sigma f) = \sum_{\mathfrak{D}} c_{T^{\Box},\mathfrak{D}} \cdot (\sigma f)_{\mathfrak{D}} = \sum_{\mathfrak{D}} c_{T^{\Box},\mathfrak{D}} \cdot \sigma f_{\mathfrak{D}} = \sigma \sum_{\mathfrak{D}} c_{T^{\Box},\mathfrak{D}} \cdot f_{\mathfrak{D}} = \sigma W_{T^{\Box}}(f)$$

in which the sum is taken over a finite set of disjoint open compact subset of $\mathrm{Herm}_{2r}(F_w)$ of the form $\mathfrak{D}_b$. Thus, (3) follows. ☐

Lemma 3.6. *The representation $\mathrm{I}^{\Box}_{r,w}(\chi_w)$ is semisimple and of finite length as a representation of $\widetilde{G}_{2r}(\mathbb{Q}_w)$. When $w \neq p$, every irreducible summand of $\mathrm{I}^{\Box}_{r,w}(\chi_w)$ contains a nonzero element f in $\mathrm{I}^{\Box}_{r,w}(\chi_w)^{\circ}$ that is fixed by $\widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w)$.*

Proof. The first statement follows since it is the parabolic induction of a unitary character.

Now we show the second statement. For every $v \in \mathtt{V}_F^{(v)}$, by [KS97, Theorem 1.2 & Theorem 1.3], $\mathrm{I}^{\Box}_{r,v}(\chi_v)$ is an irreducible representation of $G_{2r}(F_v)$ unless $\chi_v^2 = \mathbf{1}$. Moreover, when $\chi_v^2 = \mathbf{1}$, each direct summand of $\mathrm{I}^{\Box}_{r,v}(\chi_v)$ is of the form $\mathrm{I}(V_v)$ for some (nondegenerate) hermitian space V_v over E_v of rank $2r$. Here, $\mathrm{I}(V_v)$ is the image of the Siegel–Weil section map $\mathscr{S}(V_v^{2r}) \to \mathrm{I}^{\Box}_{r,v}(\chi_v)$ under the Weil representation with respect to (the standard additive character $\psi_{F,v}$ and) the splitting character $\chi_v \circ \mathrm{Nm}_{E_v/F_v}$ (again see [KS97]). Put $\mathtt{V} := \{v \in \mathtt{V}_F^{(w)} \mid \chi_v^2 = \mathbf{1}\}$.

Now let I be an irreducible summand of $\mathrm{I}^{\Box}_{r,w}(\chi_w)$ as a representation of $\widetilde{G}_{2r}(\mathbb{Q}_w)$. One can find a collection of hermitian spaces V_v over E_v of rank $2r$ for $v \in \mathtt{V}$ such that I contains

$$\left(\bigotimes_{v\in\mathtt{V}} \mathrm{I}(V_v)\right) \otimes \left(\bigotimes_{v\in\mathtt{V}_F^{(v)}\setminus\mathtt{V}} \mathrm{I}^{\Box}_{r,v}(\chi_v)\right).$$

For every $v \in \mathtt{V}_F^{(w)}$, we define a subset $\mathfrak{T}_v$ of $\mathrm{Herm}^{\circ}_{2r}(F_v)$ as follows. If $v \in \mathtt{V}$, then we define $\mathfrak{T}_v$ to be the intersection of $\mathrm{Herm}^{\circ}_{2r}(F_v)$ and the image of the moment map $V_v^{2r} \to \mathrm{Herm}_{2r}(F_v)$ (see §4.1(H1) if one needs recall). If $v \notin \mathtt{V}$, then we define $\mathfrak{T}_v$ to be $\mathrm{Herm}^{\circ}_{2r}(F_v)$. Take any open compact subset $\mathfrak{T}$ of $\mathrm{Herm}_{2r}(F_w) = \prod_{v\in\mathtt{V}_F^{(w)}} \mathrm{Herm}_{2r}(F_v)$ that is contained in $\prod_{v\in\mathtt{V}_F^{(w)}} \mathfrak{T}_v \cap \mathrm{Herm}_{2r}(O_{F_v})$ satisfying that $c\,{}^{\mathrm{t}}a^{\mathtt{c}}\mathfrak{T}a = \mathfrak{T}$ for every $a \in \mathrm{GL}_{2r}(O_{E,w})$ and every $c \in \mathbb{Z}_w^{\times}$. Then $(\mathbf{1}_{\mathfrak{T}})^{\chi_w} \in \mathrm{I}^{\Box}_{r,w}(\chi_w)^{\circ}$ is a nonzero element of I. Moreover, by Lemma 3.3, it is fixed by $\widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w)$.

The lemma is proved. ☐

In the rest of this subsection, we construct some explicit sections in $\mathrm{I}^{\Box}_{r,p}(\chi_p)^{\circ}$.

Notation 3.7. For every place $v \in \mathtt{V}_F^{(p)}$, we

- fix a uniformizer ϖ_v of F_v,
- for every element $e = (e_u)_u \in \mathbb{Z}^{\mathtt{P}_v}$, put $|e| := \sum_{u\in\mathtt{P}_v} e_u$ and denote by ϖ_v^e the element in $E_v = \prod_{u\in\mathtt{P}_v} E_u$ whose component in E_u is $\varpi_v^{e_u}$,
- for $u \in \mathtt{P}_v$, denote by $1_u \in \mathbb{Z}^{\mathtt{P}_v}$ the element that takes values 1 at u and 0 at $u^{\mathtt{c}}$,

---

[8] Though [LZ22] only treats the case where v is inert in E, the same argument works in the case where v splits in E as well.

- for every $u \in \mathtt{P}_v$, introduce an element
$$\mathrm{U}_u := \sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \left[\begin{pmatrix} 1_r & \varpi_v^{-d_v} b^\sharp \\ & 1_r \end{pmatrix} \begin{pmatrix} \varpi_v^{1_u} \cdot 1_r & \\ & \varpi_v^{-1_{u^\mathtt{c}}} \cdot 1_r \end{pmatrix} \right] \in \mathbb{Z}[G_r(F_v)],$$
where $b^\sharp \in \mathrm{Herm}_r(O_{F_v})$ denotes the Teichmüller lift of b,
- for every $e = (e_u)_u \in \mathbb{N}^{\mathtt{P}_v}$, define
$$\mathrm{U}_v^e := \prod_{u \in \mathtt{P}_v} \mathrm{U}_u^{e_u} \in \mathbb{Z}[G_r(F_v)],$$
where we note that the subalgebra of $\mathbb{Z}[G_r(F_v)]$ generated by U_u for $u \in \mathtt{P}_v$ is commutative.

Construction 3.8. For $v \in \mathtt{V}_F^{(p)}$ and every element $e \in \mathbb{Z}^{\mathtt{P}_v}$, put
$$\mathfrak{T}_v^{[e]} := \left\{ T^\square = \begin{pmatrix} T_{11}^\square & T_{12}^\square \\ T_{21}^\square & T_{22}^\square \end{pmatrix} \in \mathrm{Herm}_{2r}(F_v) \,\middle|\, T_{11}^\square, T_{22}^\square \in \mathrm{Herm}_r(O_{F_v}), T_{12}^\square \in \varpi_v^{-e} \cdot \mathrm{GL}_r(O_{E_v}) \right\}$$
as a subset of $\mathrm{Herm}_{2r}(F_v)$. Define a function $\mathfrak{f}_{\chi_v}^{[e]} \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$ by the formula
$$\mathfrak{f}_{\chi_v}^{[e]}(T^\square) := \chi_v(\mathrm{Nm}_{E_v/F_v} \det T_{12}^\square) \cdot \mathbf{1}_{\mathfrak{T}_v^{[e]}}(T^\square).$$
In particular, we obtain a section $(\mathfrak{f}_{\chi_v}^{[e]})^{\chi_v} \in \mathrm{I}_{r,v}^\square(\chi_v)^\circ$ by (3.4).

In what follows, we will identity $\mathbb{Z}^\mathtt{P}$ and $\mathbb{N}^\mathtt{P}$ with $\prod_{v \in \mathtt{V}_F^{(p)}} \mathbb{Z}^{\mathtt{P}_v}$ and $\prod_{v \in \mathtt{V}_F^{(p)}} \mathbb{N}^{\mathtt{P}_v}$, respectively. For $e \in \mathbb{Z}^\mathtt{P}$, we put
$$\|e\| := \max_{v \in \mathtt{V}_F^{(p)}} |e_v|, \quad \mathfrak{T}_p^{[e]} := \prod_{v \in \mathtt{V}_F^{(p)}} \mathfrak{T}_v^{[e_v]}, \quad \mathfrak{f}_{\chi_p}^{[e]} := \bigotimes_{v \in \mathtt{V}_F^{(p)}} \mathfrak{f}_{\chi_v}^{[e_v]}.$$
For $e \in \mathbb{N}^\mathtt{P}$, we put
$$\mathrm{U}_p^e := \bigotimes_{v \in \mathtt{V}_F^{(p)}} \mathrm{U}_v^{e_v} \in \bigotimes_{v \in \mathtt{V}_F^{(p)}} \mathbb{Z}[G_r(F_v)] = \mathbb{Z}[G_r(F \otimes \mathbb{Z}_p)].$$
For two elements $e_1, e_2 \in \mathbb{N}^\mathtt{P}$, we have the element $\mathrm{U}_p^{e_1} \times \mathrm{U}_p^{e_2}$ as the image of $\mathrm{U}_p^{e_1} \otimes \mathrm{U}_p^{e_2}$ under the natural map $\mathbb{Z}[G_r(F \otimes \mathbb{Z}_p)] \otimes \mathbb{Z}[G_r(F \otimes \mathbb{Z}_p)] \to \mathbb{Z}[G_{2r}(F \otimes \mathbb{Z}_p)]$ induced by the embedding (3.2).

Example 3.9. Suppose that $F = \mathbb{Q}$ and write $\mathtt{P} = \{u, u^\mathtt{c}\}$. If we take $\varpi_p = p$ and identify $G_{2r}(\mathbb{Q}_p)$ with $\mathrm{GL}_{4r}(\mathbb{Q}_p)$ via u, then
$$\mathrm{U}_p^{1_u} \times \mathrm{U}_p^0 = \sum_{b \in \mathrm{Herm}_r(\mathbb{F}_p)} \left[\begin{pmatrix} 1_r & & b^\sharp & \\ & 1_r & & \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \begin{pmatrix} p1_r & & & \\ & 1_r & & \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \right]$$
$$\mathrm{U}_p^{1_{u^\mathtt{c}}} \times \mathrm{U}_p^0 = \sum_{b \in \mathrm{Herm}_r(\mathbb{F}_p)} \left[\begin{pmatrix} 1_r & & b^\sharp & \\ & 1_r & & \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \begin{pmatrix} 1_r & & & \\ & 1_r & & \\ & & p^{-1}1_r & \\ & & & 1_r \end{pmatrix} \right]$$
$$\mathrm{U}_p^0 \times \mathrm{U}_p^{1_u} = \sum_{b \in \mathrm{Herm}_r(\mathbb{F}_p)} \left[\begin{pmatrix} 1_r & & & \\ & 1_r & & b^\sharp \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \begin{pmatrix} 1_r & & & \\ & p1_r & & \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \right]$$
$$\mathrm{U}_p^0 \times \mathrm{U}_p^{1_{u^\mathtt{c}}} = \sum_{b \in \mathrm{Herm}_r(\mathbb{F}_p)} \left[\begin{pmatrix} 1_r & & & \\ & 1_r & & b^\sharp \\ & & 1_r & \\ & & & 1_r \end{pmatrix} \begin{pmatrix} 1_r & & & \\ & 1_r & & \\ & & 1_r & \\ & & & p^{-1}1_r \end{pmatrix} \right]$$
and the general ones $\mathrm{U}_p^{e_1} \times \mathrm{U}_p^{e_2}$ are the composition of the above four.

Lemma 3.10. *For every element $e \in \mathbb{Z}^{\mathtt{P}}$, the section $(\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p} \in \mathrm{I}_{r,p}^{\Box}(\chi_p)^{\circ}$ is invariant under the subgroup $\widetilde{\mathcal{P}}_{r,r}(\mathbb{Z}_p)$ (Definition 2.9) of $\widetilde{G}_{2r}(\mathbb{Q}_p)$.*

Proof. This follows immediately from the construction of $\mathfrak{f}_{\chi_p}^{[e]}$. ☐

Lemma 3.11. *For every element $e \in \mathbb{Z}^{\mathtt{P}}$ and every $e_1, e_2 \in \mathbb{N}^{\mathtt{P}}$, we have*

$$(\mathrm{U}_p^{e_1} \times \mathrm{U}_p^{e_2})(\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p} = (\mathfrak{f}_{\chi_p}^{[e+e_1^{\mathtt{c}}+e_2]})^{\chi_p},$$

where $e_1^{\mathtt{c}} := e_1 \circ \mathtt{c}$.

Proof. By induction, we may assume either $e_1 = 0$ or $e_2 = 0$. We consider the case where $e_2 = 0$ and leave the other similar case to the reader. Again by induction, we may assume $e_1 = 1_u$ for some $u \in \mathtt{P}$, with $v \in \mathtt{V}_F^{(p)}$ its underlying place.

For two square matrices a and b, we write $[a, b]$ for the block diagonal matrix. As an element in $\mathbb{Z}[\widetilde{G}_{2r}(\mathbb{Q}_p)]$, we have

$$\mathrm{U}_p^{e_1} \times \mathrm{U}_p^{e_2} = \sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \left[n([\varpi_v^{-d_v} \cdot b^{\sharp}, 0_r]) \cdot m([\varpi_v^{1_u} \cdot 1_r, 1_r]) \right]$$

in which all components away from v are 1_{4r}. By Lemma 3.3, we have

$$\begin{aligned}
& W_{T^{\Box}}((\mathrm{U}_p^{e_1} \times \mathrm{U}_p^{e_2})(\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p}) \\
&= \sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} W_{T^{\Box}}(n([\varpi_v^{-d_v} \cdot b^{\sharp}, 0_r]) \cdot m([\varpi_v^{1_u} \cdot 1_r, 1_r]) \cdot (\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p}) \\
&= \left(\sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\varpi_v^{-d_v} \operatorname{tr} T_{11,v}^{\Box} b^{\sharp}) \right) W_{T^{\Box}}(m([\varpi_v^{1_u} \cdot 1_r, 1_r]) \cdot (\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p}) \\
&= \left(\sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\varpi_v^{-d_v} \operatorname{tr} T_{11,v}^{\Box} b^{\sharp}) \right) \chi_v(\varpi_v^r)^{-1} q_v^{-r^2} \cdot \mathfrak{f}_{\chi_p}^{[e]}([\varpi_v^{1_{u^{\mathtt{c}}}} \cdot 1_r, 1_r] T^{\Box} [\varpi_v^{1_u} \cdot 1_r, 1_r]) \\
&= \left(\sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\varpi_v^{-d_v} \operatorname{tr} T_{11,v}^{\Box} b^{\sharp}) \right) \chi_v(\varpi_v^r)^{-1} q_v^{-r^2} \cdot \mathfrak{f}_{\chi_p}^{[e]} \left(\begin{pmatrix} \varpi_v \cdot T_{11}^{\Box} & \varpi_v^{1_{u^{\mathtt{c}}}} \cdot T_{12}^{\Box} \\ \varpi_v^{1_u} \cdot T_{21}^{\Box} & T_{22}^{\Box} \end{pmatrix} \right). \qquad (3.5)
\end{aligned}$$

Since

$$\sum_{b \in \mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\varpi_v^{-d_v} \operatorname{tr} T_{11,v}^{\Box} b^{\sharp}) = \begin{cases} q_v^{r^2} & \text{if } T_{11,v}^{\Box} \in \mathrm{Herm}_{2r}(O_{F_v}), \\ 0 & \text{if } T_{11,v}^{\Box} \in \varpi_v^{-1}\mathrm{Herm}_{2r}(O_{F_v}) \setminus \mathrm{Herm}_{2r}(O_{F_v}), \end{cases}$$

we have

$$(3.5) = \chi_v(\varpi_v^r)^{-1} \chi_p(\mathrm{Nm}_{E_v/F_v} \det \varpi_v^{1_{u^{\mathtt{c}}}} T_{12}^{\Box}) \cdot \mathbf{1}_{\mathfrak{X}_p^{[e+e_1^{\mathtt{c}}]}}(T^{\Box}) = \chi_p(\mathrm{Nm}_{E_p/F_p} \det T_{12}^{\Box}) \cdot \mathbf{1}_{\mathfrak{X}_p^{[e+e_1^{\mathtt{c}}]}}(T^{\Box}) = \mathfrak{f}_{\chi_p}^{[e+e_1^{\mathtt{c}}]}(T^{\Box}).$$

The lemma follows. ☐

3.2. **Siegel hermitian Eisenstein series.** Let $\chi \colon \Gamma_{F,p} \to \mathbb{C}^{\times}$ be a finite character, regarded as an automorphic character of $\mathbb{A}_F^{\times}$. We define $\mathrm{I}_r^{\Box}(\chi)$ to be the restricted tensor product of $\mathrm{I}_{r,v}^{\Box}(\chi_v)$ over all places v of F, which is a smooth representation of $\widetilde{G}_{2r}(\mathbb{A})$. For $f_\chi \in \mathrm{I}_r^{\Box}(\chi)$, we have the Siegel hermitian Eisenstein series[9]

$$E(g, f_\chi) := \sum_{\gamma \in P_{2r}(F) \backslash G_{2r}(F)} f_\chi(\gamma g), \quad g \in G_{2r}(\mathbb{A}_F),$$

$$\widetilde{E}(g, f_\chi) := \sum_{\gamma \in \widetilde{P}_{2r}(\mathbb{Q}) \backslash \widetilde{G}_{2r}(\mathbb{Q})} f_\chi(\gamma g), \quad g \in \widetilde{G}_{2r}(\mathbb{A}).$$

---

[9] We remind the reader that the sums in the following expressions are not absolutely convergent in general; they are rather defined by analytic continuation. For example, one has a family of Eisenstein series $E(g, f_{\chi,s})$ for $s \in \mathbb{C}$, where $f_{\chi,s} \in \mathrm{I}_r^{\Box}(\chi|\ |_{\mathbb{A}_F}^s)$ is the standard section induced by f_χ; it is absolutely convergent when $\mathrm{Re}\, s$ is large enough and has a meromorphic continuation to $\mathbb{C}$. Then $E(g, f_\chi)$ is defined as the value at 0 of this continuation, known to be analytic.

For a finite set $\Diamond$ of places of $\mathbb{Q}$ containing $\{\infty, p\}$, an element $e \in \mathbb{Z}^{\mathsf{P}}$ and a section $f \in \mathrm{I}_r^{\Box}(\chi)^{\infty p}$, we put

$$\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f) := b_{2r}^{\Diamond}(\mathbf{1})^{-1} \cdot b_{2r}^{\Diamond}(\chi) \cdot \widetilde{E}(-, f_{\infty}^{[r]} \otimes (\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p} \otimes f), \tag{3.6}$$

where $b_{2r}^{\Diamond}(\mathbf{1})$ is defined in §2.1(F4); $f_{\infty}^{[r]}$ is introduced in Lemma 3.2; and $(\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p}$ is introduced in Construction 3.8. It is clear that $\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f)$ belongs to $\widetilde{\mathscr{A}}_{2r,\mathrm{hol}}^{[r]}$. Put

$$W_{2r}^{\Diamond} := W_{2r} \cdot b_{2r,\Diamond\setminus\{\infty\}}(\mathbf{1}) \in \mathbb{Q}^{\times}, \tag{3.7}$$

where W_{2r} is the constant in Lemma 3.2.

Lemma 3.12. *Suppose that $\|e\| > 0$. Then for $f = \otimes_{w\nmid\infty p} f_w$ that is a pure tensor,*

$$\boldsymbol{q}_{2r}\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f)\right) = W_{2r}^{\Diamond} \sum_{T^{\Box}\in\mathrm{Herm}_{2r}^{\circ}(F)^{+}} \left(\chi_p(\mathrm{Nm}_{E_p/F_p} \det T_{12}^{\Box})\mathbf{1}_{\mathfrak{T}_p^{[e]}}(T^{\Box}) \cdot \prod_{w\nmid\infty p} W_{T^{\Box}}^{\Diamond}(f_w)\right) q^{T}$$

in which the product is finite. Here, $\widetilde{\boldsymbol{h}}_{2r}$ is the map (2.8); $\boldsymbol{q}_{2r}$ is the map (2.10); and

$$W_{T^{\Box}}^{\Diamond}(f_w) := \begin{cases} W_{T^{\Box}}(f_w) & \text{if } w \in \Diamond, \\ b_{2r,w}(\chi) \cdot W_{T^{\Box}}(f_w) & \text{if } w \notin \Diamond. \end{cases}$$

Proof. First, note that when $\|e\| > 0$, we have $\mathfrak{f}_{\chi_p}^{[e]}(T^{\Box}) = 0$ for $T^{\Box} \in \mathrm{Herm}_{2r}(F) \setminus \mathrm{Herm}_{2r}^{\circ}(F)$. By the discussion in [Liu11b, Section 2B] and Lemma 3.2, the analytic q-expansion (2.5) of $\widetilde{E}(-, f_{\infty}^{[r]} \otimes (\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p} \otimes f)$ equals

$$W_{2r} \cdot b_{2r}^{\infty}(\mathbf{1}) \sum_{T^{\Box}\in\mathrm{Herm}_{2r}^{\circ}(F)^{+}} \left(\chi_p(\mathrm{Nm}_{E_p/F_p} \det T_{12}^{\Box})\mathbf{1}_{\mathfrak{T}_p^{[e]}}(T^{\Box}) \cdot \prod_{w\nmid\infty p} W_{T^{\Box}}(f_w)\right) q^{T^{\Box}},$$

in which we recall that $b_{2r}^{\infty}(\mathbf{1})$ is absolutely convergent as in §2.1(F4). It follows that the analytic q-expansion of $\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f)$ equals

$$W_{2r}^{\Diamond} \sum_{T^{\Box}\in\mathrm{Herm}_{2r}^{\circ}(F)^{+}} \left(\chi_p(\mathrm{Nm}_{E_p/F_p} \det T_{12}^{\Box})\mathbf{1}_{\mathfrak{T}_p^{[e]}}(T^{\Box}) \cdot \prod_{w\nmid\infty p} W_{T^{\Box}}^{\Diamond}(f_w)\right) q^{T^{\Box}}$$

in which the product is actually finite by [Tan99, Proposition 3.2]. The lemma follows by the coincidence of the analytic and the algebraic q-expansions [Lan12, Theorem 5.3.5]. □

Put

$$\widetilde{D}_{\Diamond}^{[e]}(-,\chi,f) := \widetilde{\rho}_{r,r}\left(\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f)\right) \in \widetilde{\mathscr{A}}_{r,r,\mathrm{hol}}^{[r]} \tag{3.8}$$

(see (2.7) for the map $\widetilde{\rho}_{r,r}$).[10] The following proposition concerns the rationality of $\widetilde{D}_{\Diamond}^{[e]}(-,\chi,f)$, which is the main result of this subsection.

Proposition 3.13. *Suppose that $\|e\| > 0$ and let $f \in \mathrm{I}_r^{\Box}(\chi)^{\infty p}$ be a section. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have*

$$\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}_{\Diamond}^{[e]}(-,\sigma\chi,\sigma f)\right) = \sigma\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}_{\Diamond}^{[e]}(-,\chi,f)\right),$$

where $\widetilde{\boldsymbol{h}}_{r,r}$ is the map (2.8).

Note that for $f \in \mathrm{I}_r^{\Box}(\chi)^{\infty p}$, $\sigma f \in \mathrm{I}_r^{\Box}(\sigma\chi)^{\infty p}$. Thus, the statement of the proposition makes sense.

Proof. Take an integer $d \geqslant 1$ such that $(\mathfrak{f}_{\chi_p}^{[e]})^{\chi_p}$ is fixed by the kernel of the map $\widetilde{\mathscr{G}}_{2r}(\mathbb{Z}_p) \to \widetilde{\mathscr{G}}_{2r}(\mathbb{Z}/p^d)$. The proof consists of two steps.

Step 1. We first show that for every $f \in \mathrm{I}_r^{\Box}(\chi)^{\infty p}$ and every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\rangle)$, the relation

$$\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}_{\Diamond}^{[e]}(-,\sigma\chi,\sigma f)\right) = \sigma\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}_{\Diamond}^{[e]}(-,\chi,f)\right) \tag{3.9}$$

holds.

Take an irreducible summand I of $\mathrm{I}_r^{\Box}(\chi)^{\infty p}$ (as a representation of $\widetilde{G}_{2r}(\mathbb{A}^{\infty p})$). Choose a positive integer $\Delta = \Delta_{\mathrm{I}} > 1$ that is coprime to p such that

---

[10] The letter D stands for pullback along the *diagonal* block.

1. for every rational prime w not dividing $p\Delta$, I_w has nonzero invariants under $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w)$;
2. one can write $\Delta = \Delta_1 \cdot \Delta_2$ with $(\Delta_1, \Delta_2) = 1$ such that for $i = 1, 2$, $\prod_{w|\Delta_i} \widetilde{P}_{2r}(\mathbb{Q}_w)$ maps surjectively to $\widetilde{G}^{\mathrm{ab}}_{2r}(\mathbb{Q})\backslash\widetilde{G}^{\mathrm{ab}}_{2r}(\mathbb{A}^\infty)/\widetilde{K}^{\mathrm{ab}}_{2r}(p^d)$ (Remark 2.12).

For every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\rangle)$, since the map

$$f \mapsto \widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\sigma\chi,\sigma f)\right) - \sigma\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\chi,f)\right)$$

is $\widetilde{G}_{2r}(\mathbb{A}^{\infty p})$-equivariant, it suffices to show that there exists a nonzero element $f = f_\sigma \in \mathrm{I}$ such that

$$\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\sigma\chi,\sigma f)\right) - \sigma\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\chi,f)\right) = 0. \tag{3.10}$$

In practice below, we will first check (3.10) for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta_1\rangle)$ and then for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta_2\rangle)$.

Choose a nonzero element $f = \otimes_{w\nmid\infty p} f_w \in \mathrm{I}$ such that f_w satisfies the condition in Lemma 3.6 (that is, it belongs to $\mathrm{I}^\square_{r,w}(\chi_w)^\circ$ and is fixed by $\widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w)$) for $w \mid \Delta$ and that f_w is the unique section that is fixed by $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w)$ and satisfies $f_w(1_{4r}) = 1$ for $w \nmid \Delta$. Replacing Δ by a power of Δ, we may assume that f_w is invariant under $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w) \times_{\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w/\Delta)} \widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w/\Delta)$ for every $w \nmid \infty p$. In particular, we have

$$\widetilde{E}^{[e]}_{\diamond}(-,\chi,f) \in \widetilde{\mathcal{A}}^{[r]}_{2r,\mathrm{hol}}(\widetilde{K}_{2r}(\Delta,p^d))$$

(Notation 2.10). For such f, we first show that (3.10) holds for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\Delta_1\rangle)$. By property (2) for Δ and Remark 2.12, the q-expansions of $\boldsymbol{h}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\chi,g\cdot f)\right)$ for all $g \in \prod_{w|\Delta_1}\widetilde{P}_{2r}(\mathbb{Q}_w)$ determines $\widetilde{E}^{[e]}_{\diamond}(-,\chi,f)$. For every $g \in \prod_{w|\Delta_1}\widetilde{P}_{2r}(\mathbb{Q}_w)$, there exists an integer $d_g \geqslant 1$ such that $\widetilde{E}^{[e]}_{\diamond}(-,\chi,g\cdot f)$ belongs to $\widetilde{\mathcal{A}}^{[r]}_{2r,\mathrm{hol}}(\widetilde{K}_{2r}(\Delta_2,p^d\Delta_1^{d_g}))$. Then by Lemma 2.11, to check (3.10) for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\Delta_1\rangle)$, it suffices to check that

$$\boldsymbol{q}_{2r}\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\sigma\chi,g\cdot\sigma f)\right) - \sigma\boldsymbol{q}_{2r}\widetilde{\boldsymbol{h}}_{2r}\left(\widetilde{E}^{[e]}_{\diamond}(-,\chi,g\cdot f)\right) = 0$$

for every $g \in \prod_{w|\Delta_1}\widetilde{P}_{2r}(\mathbb{Q}_w)$. However, this follows from Lemma 3.12 and Lemma 3.5(2,3). Since the roles of Δ_1 and Δ_2 are symmetric, (3.10) also holds for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\Delta_2\rangle)$. Together, (3.10) holds for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle p\rangle)$. Thus, (3.9) holds.

Step 2. By Step 1 and the upper square of the functorial diagram (2.11), for the proposition, it suffices to show that for every $f \in \mathrm{I}^\square_r(\chi)^{\infty p}$, there exists a positive integer Δ that is coprime to p such that

$$\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\sigma\chi,\sigma f)\right) - \sigma\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\chi,f)\right) = 0 \tag{3.11}$$

holds for every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta\rangle)$.

By linearity, we may assume that $f = \otimes_{w\nmid\infty p} f_w$ is a pure tensor. Let Δ be a positive integer that is coprime to p such that

3. for every prime w not dividing $p\Delta$, f_w is the unique section that is fixed by $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w)$ and satisfies $f_w(1_{4r}) = 1$;
4. for every prime w dividing Δ, f_w is fixed by the kernel of the map $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w) \to \widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w/\Delta)$.

Combining with Lemma 3.10, we see that $\widetilde{D}^{[e]}_{\diamond}(-,\chi,f)$ belongs to $\widetilde{\mathcal{A}}^{[r]}_{r,r,\mathrm{hol}}(\widetilde{K}_{r,r}(p^d,\Delta))$. Thus, by Lemma 2.11 (with $\Delta = p$ and $\Delta' = \Delta$), for (3.11), it suffices to show that

$$\boldsymbol{q}_{r,r}\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\sigma\chi,\sigma f)\right) - \sigma\boldsymbol{q}_{r,r}\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\chi,f)\right) = 0 \tag{3.12}$$

holds for every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta\rangle)$ (this time, we only need to consider the q-expansion on one connected component since we argue for all f). By Lemma 3.12 and Lemma 3.5(2), (3.12) holds for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}\langle\Delta\rangle)$.

The proposition is proved. □

3.3. Relevant representations.

Lemma 3.14. *Let $\mathbb{L}/\mathbb{Q}_p$ be a finite extension and let π be a relevant $\mathbb{L}$-representation of $G_r(\mathbb{A}^\infty_F)$ (Definition 1.1).*

1. *The representation $\tilde{\pi} := (\pi^\vee)^\dagger$ is also a relevant $\mathbb{L}$-representation of $G_r(\mathbb{A}^\infty_F)$, where $\dagger$ is the involution introduced at the beginning of §1.1.*
2. *The $\mathbb{L}$-vector space $\mathrm{Hom}_{G_r(\mathbb{A}^\infty_F)}(\pi, \mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}_p} \mathbb{L})$ has dimension 1.*

See Definition 2.3 for the notation $\mathcal{H}^{[r]}_r$.

Proof. Part (1) follows from the fact that for every $v \in \mathtt{V}_F^{(\infty)}$, $((\pi_v^{[r]})^\vee)^\dagger$ is isomorphic to $\pi_v^{[r]}$.

For (2), it suffices to show that for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, the complex vector space $\mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\iota\pi, \mathcal{A}^{[r]}_{r,\mathrm{hol}})$ has dimension 1. However, this follows from Arthur's multiplicity one property proved in [Mok15]. □

Now we fix a relevant $\mathbb{L}$-representation π of $G_r(\mathbb{A}_F^\infty)$ for some finite extension $\mathbb{L}/\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ such that π_v is *unramified* for every $v \in \mathtt{V}_F^{(p)}$. After Lemma 3.14, we let $\mathcal{V}_\pi$ and $\mathcal{V}_{\hat\pi}$ be the unique subspaces of $\mathcal{H}_r^{[r]} \otimes_{\mathbb{Q}_p} \mathbb{L}$ that are isomorphic to π and $\hat\pi$, respectively.

Notation 3.15. We fix a $G_r(\mathbb{A}_F^\infty)$-invariant pairing $\langle\ ,\ \rangle_\pi\colon \mathcal{V}_{\hat\pi}^\dagger \times \mathcal{V}_\pi \to \mathbb{L}$. For every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, since π is absolutely irreducible, there is a unique element $\mathrm{P}_\pi^\iota \in \mathbb{C}^\times$ such that

$$\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_1^\iota(g^\dagger)\varphi_2^\iota(g)\,\mathrm{d}g = \mathrm{P}_\pi^\iota \cdot \langle \varphi_1^\dagger, \varphi_2\rangle_\pi$$

for every $\varphi_1 \in \mathcal{V}_{\hat\pi}$ and $\varphi_2 \in \mathcal{V}_\pi$. See Definition 2.3 for the notation φ_i^ι.

Remark 3.16. Since $\hat{\hat\pi} = \pi$, the pairing $\langle\ ,\ \rangle_\pi$ is equivalent to a similar pairing $\langle\ ,\ \rangle_{\hat\pi}\colon \mathcal{V}_\pi^\dagger \times \mathcal{V}_{\hat\pi} \to \mathbb{L}$ for $\hat\pi$, for which we have $\mathrm{P}_{\hat\pi}^\iota = \mathrm{P}_\pi^\iota$ for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$.

Lemma 3.17. *There is a unique* $\mathbb{L}$*-linear map*

$$\mathrm{pr}_\pi\colon \mathcal{H}_r^{[r]} \otimes_{\mathbb{Q}_p} \mathbb{L} \to \mathcal{V}_\pi$$

satisfying that for every $\mathcal{Z} \in \mathcal{H}_r^{[r]} \otimes_{\mathbb{Q}_p} \mathbb{L}$*, every* $\varphi \in \mathcal{V}_{\hat\pi}$ *and every embedding* $\iota\colon \mathbb{L} \to \mathbb{C}$*,*

$$\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi^\iota(g^\dagger)\mathcal{Z}^\iota(g)\,\mathrm{d}g = \mathrm{P}_\pi^\iota \cdot \iota\langle \varphi^\dagger, \mathrm{pr}_\pi(\mathcal{Z})\rangle_\pi$$

holds.

Proof. Take an open compact subgroup K of $G_r(\mathbb{A}_F^\infty)$. The $\mathbb{L}$-vector space $\mathcal{H}_r^{[r]}(K) \otimes_{\mathbb{Q}_p} \mathbb{L}$ is a semisimple module over $\mathbb{L}[K\backslash G_r(\mathbb{A}_F^\infty)/K]$, in which $\mathcal{V}_\pi(K)$ is the unique summand that is isomorphic to π^K. We denote by $\mathcal{H}_r^{[r]}(K)^\pi \subseteq \mathcal{H}_r^{[r]}(K) \otimes_{\mathbb{Q}_p} \mathbb{L}$ the direct sum of the remaining summands. Then we have a direct sum decomposition $\mathcal{H}_r^{[r]}(K) \otimes_{\mathbb{Q}_p} \mathbb{L} = \mathcal{V}_\pi(K) \oplus \mathcal{H}_r^{[r]}(K)^\pi$ of $\mathbb{L}[K\backslash G_r(\mathbb{A}_F^\infty)/K]$-modules. Denote by $\mathrm{pr}_\pi^K\colon \mathcal{H}_r^{[r]}(K) \otimes_{\mathbb{Q}_p} \mathbb{L} \to \mathcal{V}_\pi(K) \subseteq \mathcal{V}_\pi$ the corresponding projection map. It is clear that the maps pr_π^K are compatible with each other for different K, hence defining a map $\mathrm{pr}_\pi\colon \mathcal{H}_r^{[r]} \otimes_{\mathbb{Q}_p} \mathbb{L} \to \mathcal{V}_\pi$ which satisfies the property of the lemma by construction. The lemma is proved as the uniqueness is obvious. □

In the rest of this subsection, we take an element $v \in \mathtt{V}_F^{(p)}$. For every $u \in \mathtt{P}_v$, we have the representation π_u of $\mathrm{GL}_n(F_v)$ as a local component of π via the isomorphism $G_r(F_v) \simeq \mathrm{GL}_n(E_u) = \mathrm{GL}_n(F_v)$ (recall that $n = 2r$). In particular, $\pi_u^\vee \simeq \pi_{u^\mathtt{c}} \simeq (\pi^\vee)_u$. Note that we will also speak of π_v, a representation of $G_r(F_v)$ without any identification with $\mathrm{GL}_n(F_v)$.

Definition 3.18. We let $\{\alpha_{u,1}, \dots, \alpha_{u,n}\} \subseteq \overline{\mathbb{Q}}_p^\times$ (as a multi-subset) be the Satake parameter of π_u.

1. Define the *Satake polynomial* of π_u to be
$$\mathsf{P}_{\pi_u}(T) := \prod_{j=1}^{n}\left(T - \alpha_{u,j}\sqrt{q_v}^{\,n-1}\right).$$
2. For every integer $1 \leqslant m \leqslant n$, put
$$\mathsf{A}(\pi_u, m) := \left\{ \left(\prod_{j\in J}\alpha_{u,j}\right)\sqrt{q_v}^{\,m(n-m)} \,\middle|\, J \subseteq \{1,\dots,n\}, |J| = m \right\}$$
as a multi-subset of $\overline{\mathbb{Q}}_p$.

Note that to define the Satake parameter, one needs to choose a square root of q_v in $\overline{\mathbb{Q}}_p$. However, both $\mathsf{P}_{\pi_u}(T)$ and $\mathsf{A}(\pi_u, m)$ are independent of such choice.

Lemma 3.19. *We have*

(1) *There exist* $\beta_{u,1},\dots,\beta_{u,n}\in O_{\mathbb{L}}$ *such that*

$$\mathsf{P}_{\pi_u}(T)=T^n+\beta_{u,1}\cdot T^{n-1}+\beta_{u,2}\cdot q_v\cdot T^{n-2}+\cdots+\beta_{u,r}\cdot q_v^{\frac{r(r-1)}{2}}\cdot T^r+\cdots+\beta_{u,n}\cdot q_v^{\frac{n(n-1)}{2}}.$$

(2) *For every integer* $1\leqslant m\leqslant n$, $\mathsf{A}(\pi_u,m)$ *is contained in* $\overline{\mathbb{Z}}_p$ *and contains at most one element (with multiplicity one) in* $\overline{\mathbb{Z}}_p^\times$. *Moreover,* $\mathsf{A}(\pi_u,m)\cap\overline{\mathbb{Z}}_p^\times\neq\emptyset$ *if and only if* $\beta_{u,m}\in O_{\mathbb{L}}^\times$.

(3) *We have that* $\mathsf{A}(\pi_u,m)\cap\overline{\mathbb{Z}}_p^\times\neq\emptyset$ *is equivalent to that* $\mathsf{A}(\pi_{u^{\mathtt{c}}},n-m)\cap\overline{\mathbb{Z}}_p^\times\neq\emptyset$.

Proof. Part (1) follows from Definition 1.1 and [Hid98, Theorem 8.1(3)], that is, the Newton polygon is above the Hodge polygon.

For (2), we may order the multi-set $\{\alpha_{u,1},\dots,\alpha_{u,n}\}$ in the way that $\alpha_{u,j+1}/\alpha_{u,j}\in\overline{\mathbb{Z}}_p$ for $1\leqslant j<n$. Then it follows by (1) and induction that for every $1\leqslant m\leqslant n$, $\prod_{j=1}^m(\alpha_{u,j}\sqrt{q_v}^{n-1})\in q_v^{\frac{m(m-1)}{2}}\overline{\mathbb{Z}}_p$. Thus, (2) follows.

Part (3) follows from the fact that $\prod_{j=1}^n\alpha_{u,j}$ is a root of unity and the fact that $\{\alpha_{u^{\mathtt{c}},1},\dots,\alpha_{u^{\mathtt{c}},n}\}=\{\alpha_{u,1}^{-1},\dots,\alpha_{u,n}^{-1}\}$. □

Put

$$I_v:=\mathcal{G}_r(O_{F_v})\times_{\mathcal{G}_r(O_{F_v}/\varpi_v)}\mathcal{P}_r(O_{F_v}/\varpi_v) \tag{3.13}$$

which is an open compact subgroup of $G_r(F_v)$. For every $u\in\mathsf{P}_v$, define two Hecke operators

$$\mathrm{T}_u^{\pm}:=I_v\begin{pmatrix}\varpi_v^{\pm 1_u}\cdot 1_r & \\ & \varpi_v^{\mp 1_{u^{\mathtt{c}}}}\cdot 1_r\end{pmatrix}I_v$$

(in which the volume of I_v is regarded as 1). In particular, $\mathrm{T}_u^+=\mathrm{U}_u\cdot I_v$ (Notation 3.7).

Lemma 3.20. *For every* $u\in\mathsf{P}_v$, *the multisets of generalized eigenvalues of the actions of* T_u^+ *and* T_u^- *on* $\pi_v^{I_v}$ *are* $\mathsf{A}(\pi_u,r)$ *and* $\mathsf{A}(\pi_{u^{\mathtt{c}}},r)$, *respectively.*

The proof of this lemma will be given at the end of this subsection.

Definition 3.21. We say that the (unramified) representation π_v of $G_r(F_v)$ is *Panchishkin* if $\beta_{u,r}\in O_{\mathbb{L}}^\times$ for every $u\in\mathsf{P}_v$ under the notation in Lemma 3.19.

Lemma 3.22. *The following statements are equivalent:*

(1) π_v *is Panchishkin unramified;*

(2) $\hat{\pi}_v$ *is Panchishkin unramified;*

(3) $\mathsf{A}(\pi_u,r)$ *contains a unique element in* $O_{\mathbb{L}}^\times$ *for some* $u\in\mathsf{P}_v$.

Proof. This is an immediate consequence of Lemma 3.19. The fact that the unique element in $\mathsf{A}(\pi_u,r)\cap\overline{\mathbb{Z}}_p^\times$ belongs to $\mathbb{L}$ follows from the Galois action and the uniqueness. □

Lemma 3.23. *Suppose that* π_v *is Panchishkin unramified.*

(1) *The one-dimensional subspace of* $\pi_v^{I_v}$ *that is the eigenspace of the operator* T_u^+ *for the eigenvalue that is the unique element in* $\mathsf{A}(\pi_u,r)\cap O_{\mathbb{L}}^\times$ *is independent of* $u\in\mathsf{P}_v$.

(2) *The one-dimensional subspace of* $\pi_v^{I_v}$ *that is the eigenspace of the operator* T_u^- *for the eigenvalue that is the unique element in* $\mathsf{A}(\pi_{u^{\mathtt{c}}},r)\cap O_{\mathbb{L}}^\times$ *is independent of* $u\in\mathsf{P}_v$.

The proof of this lemma will be given at the end of this subsection.

Notation 3.24. Suppose that π_v is Panchishkin unramified.

(1) For every $u\in\mathsf{P}_v$, we denote by $\alpha(\pi_u)\in O_{\mathbb{L}}^\times$ the unique element in Lemma 3.23(3).

(2) In view of Lemma 3.20 and Lemma 3.23, we denote by π_v^+ and π_v^- the one-dimensional subspaces of $\pi_v^{I_v}$ that are the eigenspaces of the operators T_u^+ and T_u^- for the eigenvalues $\alpha(\pi_u)$ and $\alpha(\pi_{u^{\mathtt{c}}})$ for every $u\in\mathsf{P}_v$, respectively.

Proposition 3.25. *Suppose that* π_v *is Panchishkin unramified.*

(1) *For every $u \in \mathtt{P}_v$, there is a unique polynomial $\mathsf{Q}_{\pi_u}(T) \in \mathbb{L}[T]$ that divides $\mathsf{P}_{\pi_u}(T)$ and has the form*

$$\mathsf{Q}_{\pi_u}(T) = T^r + \gamma_{u,1} \cdot T^{r-1} + \gamma_{u,2} \cdot q_v \cdot T^{r-2} + \cdots + \gamma_{u,r} \cdot q_v^{\frac{r(r-1)}{2}}$$

with $\gamma_{u,r} \in O_{\mathbb{L}}^{\times}$.

(2) *There is a unique $\mathbb{L}$-valued locally constant function ξ_{π_v} on $G_r(F_v)$ such that*
   (a) *there exist $\varphi_v^\vee \in (\pi_v^\vee)^-$ and $\varphi_v \in \pi_v^-$ such that $\xi_{\pi_v} = \langle \pi_v^\vee(g)\varphi_v^\vee, \varphi_v\rangle_{\pi_v}$ for $g \in G_r(F_v)$;*
   (b) *$\xi_{\pi_v}(\mathtt{w}_r) = 1$.*

   In particular, ξ_{π_v} is bi-I_v-invariant.

(3) *For $u \in \mathtt{P}_v$, denote by $\underline{\pi_u}$ the unramified principal series of $\mathrm{GL}_r(F_v)$ with $\mathsf{Q}_{\pi_u}(T)$ as its Satake polynomial, which is defined over $\mathbb{L}$. Then there exist $\mathrm{GL}_r(O_{F_v})$-invariant vectors $\phi_u \in \underline{\pi_u}$ and $\phi_u^\vee \in (\underline{\pi_u})^\vee$ for every $u \in \mathtt{P}_v$ such that*

$$\xi_{\pi_v}(m(a)\mathtt{w}_r) = \prod_{u \in \mathtt{P}_v} \langle \underline{\pi_u}(a_{u^c})\phi_u, \phi_u^\vee\rangle_{(\underline{\pi_u})^\vee}$$

holds for every $a = (a_u)_u \in \mathrm{GL}_r(E_v) = \prod_{u\in\mathtt{P}_v} \mathrm{GL}_r(E_u)$.

In this rest of this subsection, we prove Lemma 3.20, Lemma 3.23 and Proposition 3.25. To ease the notation, we will suppress the subscript v hence $F = F_v$, $\mathtt{P} = \mathtt{P}_v$, and $\pi = \pi_v$ temporarily. It is easy to see that for these three statements, we may replace $\mathbb{L}$ by a finite extension (inside $\overline{\mathbb{Q}}_p$). Thus, without loss of generality, we may assume that $\mathbb{L}$ contains both $\sqrt{q}$ and $\alpha_{u,j}$ for $u \in \mathtt{P}$ and $1 \leqslant j \leqslant n$. We need some preparation on Jacquet modules.

For every subset $J \subseteq \{1, \dots, n\}$, put $\overline{J} := \{1, \dots, n\} \setminus J$. For every subset $J \subseteq \{1, \dots, n\}$ of cardinality r, every $u \in \mathtt{P}$ and every sign $\epsilon \in \{+, -\}$, we denote by $\mathrm{I}(\alpha_{u,j}\sqrt{q}^{\epsilon r} \mid j \in J)$ the unramified principal series of $\mathrm{GL}_r(F)$ with the Satake parameter $\{\alpha_{u,j}\sqrt{q}^{\epsilon r} \mid j \in J\}$, with coefficients in $\mathbb{L}$.

Put $\overline{P}_r := \mathtt{w}_r^{-1} P_r \mathtt{w}_r$ and let $\overline{N}_r$ be its unipotent radical. We identify both Levi quotients P_r/N_r and $\overline{P}_r/\overline{N}_r$ with $\mathrm{Res}_{E/F}\,\mathrm{GL}_r$ via the map m in §2.1(G4). We define the Jacquet modules

$$\pi_{N_r} := \pi/\{\varphi - \pi(n)\varphi \mid n \in N_r(F), \varphi \in \pi\},$$
$$\pi_{\overline{N}_r} := \pi/\{\varphi - \pi(n)\varphi \mid n \in \overline{N}_r(F), \varphi \in \pi\},$$

which are admissible representations of $\mathrm{GL}_r(E)$ of finite length. Fix an order $\{u_1, u_2\}$ of $\mathtt{P}$. Recall that $\{\alpha_{u_1,1}, \dots, \alpha_{u_1,n}\} = \{\alpha_{u_2,1}^{-1}, \dots, \alpha_{u_2,n}^{-1}\}$. Without loss of generality, we may assume $\alpha_{u_1,j}\alpha_{u_2,j} = 1$ for $1 \leqslant j \leqslant n$. It is well-known that

$$\pi_{N_r}^{\mathrm{ss}} \simeq \bigoplus_{\substack{J \subseteq \{1,\dots,n\} \\ |J| = r}} \mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid j \in J) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid j \in \overline{J}),$$
$$\pi_{\overline{N}_r}^{\mathrm{ss}} \simeq \bigoplus_{\substack{J \subseteq \{1,\dots,n\} \\ |J| = r}} \mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{r} \mid j \in \overline{J}) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{r} \mid j \in J),$$

as representations of $\mathrm{GL}_r(E) = \mathrm{GL}_r(E_{u_1}) \times \mathrm{GL}_r(E_{u_2})$. Since $\mathtt{w}_r$ conjugates $m(a_1, a_2) \in G_{2r}(F)$ to $m({}^{\mathrm{t}}a_2^{-1}, {}^{\mathrm{t}}a_1^{-1})$, the isomorphism $\mathtt{w}_r \colon \pi \xrightarrow{\sim} \pi$ descends to an isomorphism $\pi_{N_r}^{\mathrm{ss}} \to \pi_{\overline{N}_r}^{\mathrm{ss}}$ that sends $\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid j \in J) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid j \in \overline{J})$ to $\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{r} \mid j \in \overline{J}) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{r} \mid j \in J)$.

Proofs of Lemma 3.20 and Lemma 3.23. The element

$$\begin{pmatrix} & \varpi^{-1_{u^c}} \cdot 1_r \\ -\varpi^{1_u} \cdot 1_r & \end{pmatrix} \in G_r(F)$$

normalizes I and induces an operator on π^I that switches T_u^+ and $\varpi^{1_u - 1_{u^c}} \cdot \mathrm{T}_u^-$. In particular, if the multiset of generalized eigenvalues of T_u^+ on π^I is $\mathsf{A}(\pi_u, r)$, then the multiset for T_u^- is

$$\left\{ \left(\prod_{j \in J} \alpha_{u,j}^{-1} \right) \sqrt{q}^{r^2} \,\middle|\, J \subseteq \{1, \dots, n\}, |J| = r \right\},$$

which is nothing but $\mathsf{A}(\pi_{u^c}, r)$. Thus, it suffices to study T_u^+ in both lemmas.

The quotient map $\pi \to \pi_{N_r}$ induces an isomorphism

$$\pi^I \xrightarrow{\sim} \pi_{N_r}^{\mathrm{GL}_r(O_E)}$$

under which the operator T_u^+ (which is nothing but the operator U_u in Notation 3.7) corresponds to the operator $q^{r^2} \cdot \left(\begin{smallmatrix} \varpi^{1_u} \cdot 1_r & \\ & 1_r \end{smallmatrix}\right)$.

Since the operator $\left(\begin{smallmatrix} \varpi^{1_u} \cdot 1_r & \\ & 1_r \end{smallmatrix}\right)$ acts on $\mathrm{I}(\alpha_{u,j}\sqrt{q}^{-r} \mid j \in J) \boxtimes \mathrm{I}(\alpha_{u^c,j}\sqrt{q}^{-r} \mid j \in \overline{J})$ by the scalar $\prod_{j\in J}\alpha_{u,j}\sqrt{q}^{-r^2}$, the multiset of (generalized) eigenvalues of T_u^+ on π^I is $\mathsf{A}(\pi_u, r)$. Lemma 3.20 is proved.

Now we consider Lemma 3.23, for which it suffices to show (1). For $i = 1, 2$, let J_i be the unique subset of $\{1, \dots, n\}$ of cardinality r such that $\prod_{j\in J_i}\alpha_{u_i,j}\sqrt{q}^{-r^2} \in \overline{\mathbb{Z}}_p^\times$. Then $J_1 \cup J_2 = \{1, \dots, n\}$. Thus, for both $i = 1, 2$, the one-dimensional subspace of π^I that is the eigenspace of the operator $\mathrm{T}_{u_i}^+$ for the eigenvalue that is the unique element in $\mathsf{A}(\pi_{u_i}, r) \cap O_{\mathbb{L}}^\times$ is the $\mathrm{GL}_r(O_E)$-invariant subspace of $\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid j \in J_1) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid j \in J_2)$. Lemma 3.23 is proved. □

Proof of Proposition 3.25. Without loss of generality, by Lemma 3.22, we may assume that the unique subset J of $\{1, \dots, n\}$ with $|J| = r$ such that $\sqrt{q}^{r^2}\prod_{j\in J}\alpha_{u_1,j} \in O_{\mathbb{L}}^\times$ is $\{1, \dots, r\}$. It follows that the unique subset J of $\{1, \dots, n\}$ with $|J| = r$ such that $\sqrt{q}^{r^2}\prod_{j\in J}\alpha_{u_2,j} \in O_{\mathbb{L}}^\times$ is $\{r + 1, \dots, n\}$.

For (1), note that every factor of $\mathsf{P}_{\pi_u}(T)$ in $\mathbb{L}[T]$ that is monic of degree r has the form

$$\prod_{j\in J}\left(T - \alpha_{u,j}\sqrt{q}^{n-1}\right)$$

for some $J \subseteq \{1, \dots, n\}$ with $|J| = r$. In particular, the corresponding term $\gamma_{u,r}$ equals $\sqrt{q}^{r^2}\prod_{j\in J}\alpha_{u,j}$. Thus, we must have

$$\mathsf{Q}_{\pi_{u_1}}(T) = \prod_{j=1}^{r}\left(T - \alpha_{u_1,j}\sqrt{q}^{n-1}\right), \quad \mathsf{Q}_{\pi_{u_2}}(T) = \prod_{j=r+1}^{n}\left(T - \alpha_{u_2,j}\sqrt{q}^{n-1}\right).$$

For (2) and (3), it suffices to show the following claim: For nonzero vectors $\varphi^\vee \in (\pi^\vee)^-$ and $\varphi \in \pi^-$, there exist nonzero $\mathrm{GL}_r(O_F)$-invariant vectors $\phi_1 \in \underline{\pi_{u_1}}$, $\phi_1^\vee \in (\underline{\pi_{u_1}})^\vee$, $\phi_2 \in \underline{\pi_{u_2}}$, $\phi_2^\vee \in (\underline{\pi_{u_2}})^\vee$ such that

$$\langle \pi^\vee(m(a_1, a_2)\mathtt{w}_r)\varphi^\vee, \varphi\rangle_\pi = \prod_{i=1}^{2}\langle \underline{\pi_{u_i}}(a_{3-i})\phi_i, \phi_i^\vee\rangle_{(\underline{\pi_{u_i}})^\vee}$$

holds for every $(a_1, a_2) \in \mathrm{GL}_r(E) = \mathrm{GL}_r(E_{u_1}) \times \mathrm{GL}_r(E_{u_2})$.

Again by Lemma 3.22, the two factors

$$\begin{gathered}\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid 1 \leqslant j \leqslant r) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid r + 1 \leqslant j \leqslant n), \\ \mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid r + 1 \leqslant r \leqslant n) \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid 1 \leqslant j \leqslant r)\end{gathered}$$

are direct summands of π_{N_r}. We see from the proof of Lemma 3.23 that under the projection $\pi \to \pi_{N_r}$, the one-dimensional subspaces $\pi^+, \pi^- \subseteq \pi^I$ map to

$$\begin{gathered}\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid 1 \leqslant j \leqslant r)^{\mathrm{GL}_r(O_F)} \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid r + 1 \leqslant j \leqslant n)^{\mathrm{GL}_r(O_F)}, \\ \mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid r + 1 \leqslant r \leqslant n)^{\mathrm{GL}_r(O_F)} \boxtimes \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid 1 \leqslant j \leqslant r)^{\mathrm{GL}_r(O_F)},\end{gathered}$$

respectively. However, we observe that

$$\mathrm{I}(\alpha_{u_1,j}\sqrt{q}^{-r} \mid r + 1 \leqslant r \leqslant n) \simeq (\underline{\pi_{u_2}})^\vee, \quad \mathrm{I}(\alpha_{u_2,j}\sqrt{q}^{-r} \mid 1 \leqslant j \leqslant r) \simeq (\underline{\pi_{u_1}})^\vee.$$

The claim follows.

The proposition is proved. □

3.4. **Local doubling zeta integral.** Let π be as in §3.3. Let $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$ be a finite character, regarded as an automorphic character of $\mathbb{A}_F^\times$.

Take a finite place $v \in \mathtt{V}_F^{\mathrm{fin}}$. For every $\varphi_v^\vee \in \pi_v^\vee$, $\varphi_v \in \pi_v$ and $f \in \mathrm{I}_{r,v}^\square(\chi_v)$, we have the local doubling zeta integral

$$Z^\iota(\varphi_v^\vee \otimes \varphi_v, f) := \int_{G_r(F_v)} \iota\langle \pi_v^\vee(g)\varphi_v^\vee, \varphi_v\rangle_{\pi_v} \cdot f(\mathbf{w}_r(g, 1_{2r}))\,\mathrm{d}g$$

for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$. Here, $\mathbf{w}_r$ is defined in (3.1) and $(g, 1_{2r})$ is an element of $G_{2r}(F_v)$ via the embedding (3.2). Since $\iota\pi_v$ is tempered, the above integral is absolutely convergent by [Yam14, Lemma 7.2].

Lemma 3.26. *Define a map $\varsigma\colon (\mathrm{Res}_{E/F}\mathrm{GL}_r) \times \mathrm{Herm}_F \times \mathrm{Herm}_F \to G_r$ by the formula*

$$\varsigma(a, u_1, u_2) := \begin{pmatrix} 1_r & u_2 \\ 0 & 1_r \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & -{}^{\mathrm{t}}a^{\mathrm{c},-1} \end{pmatrix}\mathtt{w}_r\begin{pmatrix} 1_r & u_1 \\ 0 & 1_r \end{pmatrix}$$

whose image is contained in the big Bruhat cell $P_r\mathtt{w}_rN_r$. Then $Z^\iota(\varphi_v^\vee \otimes \varphi_v, f)$ equals

$$\int_{P_r(F_v)\mathtt{w}_rN_r(F_v)} \iota\langle \pi_v^\vee(\varsigma(a, u_1, u_2))\varphi_v^\vee, \varphi_v\rangle_{\pi_v} \cdot \chi(\mathrm{Nm}_{E_v/F_v}\det a)|\det a|_{E_v}^r \cdot f\left(\mathtt{w}_r^\square \cdot n\begin{pmatrix} u & {}^{\mathrm{t}}a^{\mathrm{c}} \\ a & v \end{pmatrix}\right)\mathrm{d}\varsigma(a, u_1, u_2),$$

where the integral is absolutely convergent. Here, we recall that $\mathtt{w}_r^\square = \mathtt{w}_{2r} = \begin{pmatrix} & 1_{2r} \\ -1_{2r} & \end{pmatrix}$ from §1.6.

Proof. This formula is deduced in the proof of [LL21, Proposition 3.13]. □

Definition 3.27. For a pair $\varphi_v^\vee \in \pi_v^\vee$ and $\varphi_v \in \pi_v$, we say that an element $\mathtt{f} \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$ is $(\varphi_v^\vee, \varphi_v)$-*typical* if its Fourier transform $\widehat{\mathtt{f}} \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$ with respect to $\psi_{F,v}$ (recall from §1.6) satisfies

(1) $\widehat{\mathtt{f}}$ takes values in $\mathbb{Q}$;
(2) $\widehat{\mathtt{f}}$ is supported on the subset

$$\left\{\begin{pmatrix} u_1 & {}^{\mathrm{t}}a^{\mathrm{c}} \\ a & u_2 \end{pmatrix}\middle| a \in \mathrm{GL}_r(O_{E_v}), u_1, u_2 \in \mathrm{Herm}_r(O_{F_v})\right\} \subseteq \mathrm{Herm}_{2r}(F_v);$$

(3) $\widehat{\mathtt{f}}$ satisfies

$$\int_{G_r(F_v)} \langle \pi_v^\vee(\varsigma(a, u_1, u_2))\varphi_v^\vee, \varphi_v\rangle_{\pi_v} \cdot \widehat{\mathtt{f}}\left(\begin{pmatrix} u_1 & {}^{\mathrm{t}}a^{\mathrm{c}} \\ a & u_2 \end{pmatrix}\right) \cdot \mathrm{d}\varsigma(a, u_1, u_2) = 1,$$

where the integration is in fact a finite sum by (2) and ς is the map in Lemma 3.26.

Remark 3.28. It is easy to see that $(\varphi_v^\vee, \varphi_v)$-typical element exists if $\langle \pi_v^\vee(\mathtt{w}_r)\varphi_v^\vee, \varphi_v\rangle_{\pi_v} \in \mathbb{Q}^\times$.

Lemma 3.29. *Consider $\varphi_v^\vee \in \pi_v^\vee$, $\varphi_v \in \pi_v$ and a $(\varphi_v^\vee, \varphi_v)$-typical element $\mathtt{f} \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$. If χ_v is unramified, then*

$$Z^\iota(\varphi_v^\vee \otimes \varphi_v, \mathtt{f}^{\chi_v}) = 1$$

(see Notation 3.4 for $\mathtt{f}^{\chi_v}$) holds for every $\iota\colon \mathbb{L} \to \mathbb{C}$.

Proof. This is immediate from Lemma 3.26 and Definition 3.27. □

This following lemma will not be used until Section 4.

Lemma 3.30. *For every $\varphi_v^\vee \in \pi_v^\vee$, $\varphi_v \in \pi_v$ and a $\mathbb{Q}$-valued section $f \in \mathrm{I}_{r,v}^\square(\mathbf{1})$, there exists a unique element*

$$Z(\varphi_v^\vee \otimes \varphi_v, f) \in \mathbb{L}$$

such that for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, $\iota Z(\varphi_v^\vee \otimes \varphi_v, f)$ coincides with $Z^\iota(\varphi_v^\vee \otimes \varphi_v, f)$.

Proof. We may regard $\mathrm{I}_{r,v}^\square(\mathbf{1})$ as a representation with coefficients in $\mathbb{Q}$. Let Ω be the set of all embeddings $\iota\colon \mathbb{L} \to \mathbb{C}$. The assignment

$$(\varphi_v^\vee \otimes \varphi_v, f) \mapsto \{Z^\iota(\varphi_v^\vee \otimes \varphi_v, f)\}_{\iota\in\Omega}$$

defines an element

$$\mathfrak{Z} \in \mathrm{Hom}_{G_r(F_v)\times G_r(F_v)}\left((\pi_v^\vee \boxtimes \pi_v) \otimes \mathrm{I}_{r,v}^\square(\mathbf{1}), \mathbb{C}^\Omega\right).$$

We need to show that $\mathfrak{Z}$ takes values in $\mathbb{L}$, which is tautologically a subring of $\mathbb{C}^\Omega$. By [LL21, Proposition 3.6(1)], it suffices to find one pair of elements $(\varphi_v^\vee \otimes \varphi_v, f)$ such that $\mathfrak{Z}(\varphi_v^\vee \otimes \varphi_v, f) \in \mathbb{L}^\times$. Indeed, choose $\varphi_v^\vee \in \pi_v^\vee$, $\varphi_v \in \pi_v$

such that $\langle\pi_v^\vee(\mathtt{w}_r)\varphi_v^\vee,\varphi_v\rangle_{\pi_v}=1$, and a $(\varphi_v^\vee,\varphi_v)$-typical element $\mathtt{f}\in\mathscr{S}(\mathrm{Herm}_{2r}(F_v))$ (which exists by Remark 3.28). Then $\mathtt{f}^{\mathbf{1}}$ is $\mathbb{Q}$-valued. By Lemma 3.29, $\mathfrak{Z}(\varphi_v^\vee\otimes\varphi_v,\mathtt{f}^{\mathbf{1}})=1\in\mathbb{L}^\times$. The lemma is proved. □

Lemma 3.31. *Suppose that $v\notin\mathtt{V}_F^{(p)}$. If π_v is unramified with respect to $K_{r,v}$ and $\varphi_v^\vee,\varphi_v$ are both $K_{r,v}$-invariant such that $\langle\varphi_v^\vee,\varphi_v\rangle_{\pi_v}=1$, then for every $\iota\colon\mathbb{L}\to\mathbb{C}$,*

$$Z^\iota(\varphi_v^\vee\otimes\varphi_v,f_{\chi_v}^{\mathrm{sph}})=\frac{L(\frac{1}{2},\mathrm{BC}(\iota\pi_v)\otimes(\chi_v\circ\mathrm{Nm}_{E_v/F_v}))}{b_{2r,v}(\chi)},$$

where $f_{\chi_v}^{\mathrm{sph}}$ is defined in Notation 3.4(2).

Proof. This is a well-known calculation of Piatetski-Shapiro and Rallis. See [Li92, Theorem 3.1] for a full account including our case. □

Proposition 3.32. *Suppose that $v\in\mathtt{V}_F^{(p)}$ and that π_v is Panchishkin unramified. For every embedding $\iota\colon\mathbb{L}\to\mathbb{C}$, we have*

$$\int_{G_r(F_v)}\iota\xi_{\pi_v}(g)\cdot(\mathtt{f}_{\chi_v}^{[0]})^{\chi_v}(\mathbf{w}_r(g,1_{2r}))\,\mathrm{d}g=q_v^{d_vr^2}\prod_{u\in\mathtt{P}_v}\gamma(\tfrac{1+r}{2},\iota\underline{\pi_u}\otimes\chi_v,\psi_{F,v})^{-1}$$

where ξ_{π_v} and $\underline{\pi_u}$ are introduced in Proposition 3.25.

Note that the left-hand side is a local doubling zeta integral.

Proof. To ease notation, we omit v and ι in the proof. In particular, ϖ^d generates the different ideal of $F/\mathbb{Q}_p$, and ξ_π is $\mathbb{C}$-valued.

By Lemma 3.26, we have

$$\begin{aligned}&\int_{G_r(F)}\xi_\pi(g)\cdot(\mathtt{f}_\chi^{[0]})^\chi(\mathbf{w}_r(g,1_{2r}))\,\mathrm{d}g\\ \text{(3.14)}\qquad&=\int_{G_r(F)}\xi_\pi(\varsigma(a,u_1,u_2))\cdot\chi(\mathrm{Nm}_{E/F}\det a)\cdot|\mathrm{Nm}_{E/F}\det a|_{F_v}^r\cdot\widehat{\mathtt{f}_\chi^{[0]}}(a,u_1,u_2)\cdot\,\mathrm{d}\varsigma(a,u_1,u_2),\end{aligned}$$

where

$$\widehat{\mathtt{f}_\chi^{[0]}}(a,u_1,u_2):=\int_{\mathrm{Herm}_{2r}(F)}\mathtt{f}_\chi^{[0]}(T^\Box)\psi_F\left(\mathrm{tr}\begin{pmatrix}u_1&{}^\mathrm{t}a^c\\a&u_2\end{pmatrix}\begin{pmatrix}T_{11}^\Box&T_{12}^\Box\\T_{21}^\Box&T_{22}^\Box\end{pmatrix}\right)\mathrm{d}T^\Box$$

in which $\mathrm{d}T^\Box$ is the self-dual measure with respect to ψ_F.

It follows easily that

$$\text{(3.15)}\qquad\widehat{\mathtt{f}_\chi^{[0]}}(a,u_1,u_2)=\begin{cases}q^{-dr^2}\displaystyle\int_{\mathrm{GL}_r(O_E)}\chi(\mathrm{Nm}_{E/F}\det T)\psi_F\left(\mathrm{Tr}_{E/F}\mathrm{tr}\,aT\right)\mathrm{d}T&\text{if }u_1,u_2\in\varpi^{-d}\mathrm{Herm}_r(O_F),\\0&\text{otherwise,}\end{cases}$$

in which $\mathrm{d}T$ is the self-dual measure on $\mathrm{Mat}_{r,r}(E)$ with respect to ψ_F.

Since ξ_π is bi-I-invariant, (3.15) implies that

$$\begin{aligned}(3.14)=q^{dr^2}\int_{\mathrm{GL}_r(E)}&\xi_\pi(m(a)\mathtt{w}_r)\cdot\chi(\mathrm{Nm}_{E/F}\det a)\cdot|\mathrm{Nm}_{E/F}\det a|_F^r\\&\times\left(\int_{\mathrm{GL}_r(O_E)}\chi(\mathrm{Nm}_{E/F}\det T)\psi_F\left(\mathrm{Tr}_{E/F}\mathrm{tr}\,aT\right)\mathrm{d}T\right)\mathrm{d}a,\end{aligned}$$

which, by Proposition 3.25(3), equals

$$\begin{aligned}&=q^{dr^2}\prod_{u\in\mathtt{P}}\left(\int_{\mathrm{GL}_r(F)}\langle\underline{\pi_u}(a)\phi_u,\phi_u^\vee\rangle_{(\underline{\pi_u})^\vee}\cdot\chi(\det a)\cdot|\det a|_F^r\left(\int_{\mathrm{GL}_r(O_F)}\chi(\det T)\psi_F\left(\mathrm{tr}\,aT\right)\mathrm{d}T\right)\mathrm{d}a\right)\\&=q^{dr^2}\prod_{u\in\mathtt{P}}\left(\int_{\mathrm{GL}_r(F)}\langle(\underline{\pi_u}\otimes\chi)(a)\phi_u,\phi_u^\vee\rangle_{(\underline{\pi_u}\otimes\chi)^\vee}\cdot|\det a|_F^r\left(\int_{\mathrm{GL}_r(O_F)}\chi(\det T)\psi_F\left(\mathrm{tr}\,aT\right)\mathrm{d}T\right)\mathrm{d}a\right).\end{aligned}$$

Applying [Jac79, Proposition 1.2(3)] with $\Phi = (\chi \circ \det) \cdot \mathbf{1}_{\mathrm{GL}_r(O_F)}$, we have

$$\begin{aligned}
&\int_{\mathrm{GL}_r(F)} \langle (\underline{\pi_u} \otimes \chi)(a)\phi_u, \phi_u^\vee \rangle_{(\underline{\pi_u}\otimes\chi)^\vee} \cdot |\det a|_F^r \left(\int_{\mathrm{GL}_r(O_F)} \chi(\det T)\psi_F\,(\mathrm{tr}\, aT)\ \mathrm{d}T \right) \mathrm{d}a \\
&= \gamma(\tfrac{1-r}{2}, (\underline{\pi_u} \otimes \chi)^\vee, \psi_F) \int_{\mathrm{GL}_r(F)} \langle \phi_u, (\underline{\pi_u} \otimes \chi)^\vee(a)\phi_u^\vee \rangle_{(\underline{\pi_u}\otimes\chi)^\vee} \cdot |\det a|_F^{\frac{1-r}{2}} \cdot \chi(\det a) \cdot \mathbf{1}_{\mathrm{GL}_r(O_F)}(a)\,\mathrm{d}a \\
&= \gamma(\tfrac{1-r}{2}, (\underline{\pi_u} \otimes \chi)^\vee, \psi_F) \cdot \langle \phi_u^\vee, \phi_u \rangle_{\underline{\pi_u}} \\
&= \gamma(\tfrac{1+r}{2}, \underline{\pi_u} \otimes \chi, \psi_F)^{-1} \cdot \langle \phi_u^\vee, \phi_u \rangle_{\underline{\pi_u}}.
\end{aligned}$$

Together, we have

$$\begin{aligned}
(3.14) &= q^{dr^2} \prod_{u\in\mathrm{P}} \gamma(\tfrac{1+r}{2}, \underline{\pi_u} \otimes \chi, \psi_F)^{-1} \cdot \langle \phi_u^\vee, \phi_u \rangle_{\underline{\pi_u}} \\
&= q^{dr^2} \xi_\pi(\mathtt{w}_r) \prod_{u\in\mathrm{P}} \gamma(\tfrac{1+r}{2}, \underline{\pi_u} \otimes \chi, \psi_F)^{-1} = q^{dr^2} \prod_{u\in\mathrm{P}} \gamma(\tfrac{1+r}{2}, \underline{\pi_u} \otimes \chi, \psi_F)^{-1}.
\end{aligned}$$

The proposition is proved. □

3.5. **Construction of the p-adic L-function.** Let π be a relevant $\mathbb{L}$-representation of $G_r(\mathbb{A}_F^\infty)$ for some finite extension $\mathbb{L}/\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ such that π_v is *Panchishkin unramified* for every $v \in \mathtt{V}_F^{(p)}$.

Choose a finite set $\diamond$ of places of $\mathbb{Q}$ containing $\{\infty, p\}$ such that π_v is unramified (hence $v \notin \mathtt{V}_F^{\mathrm{ram}}$) for every $v \in \mathtt{V}_F \setminus \mathtt{V}_F^{(\diamond)}$.

We choose decomposable elements $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathcal{V}_{\hat{\pi}}$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \mathcal{V}_\pi$ satisfying

(T1) $\langle \pi_v^\vee(\mathtt{w}_r)\varphi_{1,v}^\dagger, \varphi_{2,v} \rangle_{\pi_v} = 1$ for $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}$,

(T2) $\varphi_{1,v}^\dagger \in (\pi_v^\vee)^-$, $\varphi_{2,v} \in \pi_v^-$ and $\langle \pi_v^\vee(\mathtt{w}_r)\varphi_{1,v}^\dagger, \varphi_{2,v} \rangle_{\pi_v} = q_v^{-d_v r^2}$ for $v \in \mathtt{V}_F^{(p)}$,

(T3) $\varphi_{1,v}^\dagger \in (\pi_v^\vee)^{K_{r,v}}$, $\varphi_{2,v} \in \pi_v^{K_{r,v}}$ and $\langle \varphi_{1,v}^\dagger, \varphi_{2,v} \rangle_{\pi_v} = 1$ for $v \in \mathtt{V}_F \setminus \mathtt{V}_F^{(\diamond)}$.

Note that (T2) is possible by Proposition 3.25(2). We also choose a $(\varphi_{1,v}^\dagger, \varphi_{2,v})$-typical element $\mathtt{f}_v \in \mathscr{S}(\mathrm{Herm}_{2r}(F_v))$ (Definition 3.21, which exists by (T1) and Remark 3.28) for $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}$.

For every finite character $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$, put

$$f_{\chi^{\infty p}} := \bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} f_{\chi_v} \in \mathrm{I}_r^\square(\chi)^{\infty p},$$

where $f_{\chi_v} \in \mathrm{I}_{r,v}^\square(\chi_v)$ is the section $\mathtt{f}_v^{\chi_v}$ (resp. $f_{\chi_v}^{\mathrm{sph}}$) for $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}$ (resp. $v \in \mathtt{V}_F \setminus \mathtt{V}_F^{(\diamond)}$).

Consider an open compact subset $\Omega \subseteq \Gamma_{F,p}$. By the linear independence of characters, one can write

$$\mathbf{1}_\Omega = \sum_i c_i \cdot \chi_i$$

as a finite sum in a unique way with $c_i \in \mathbb{C}$ and finite characters $\chi_i\colon \Gamma_{F,p} \to \mathbb{C}^\times$. For an element $e \in \mathbb{Z}^{\mathrm{P}}$, we put

$$\widetilde{D}_\diamond^{[e]}(-, \Omega) := \sum_i c_i \widetilde{D}_\diamond^{[e]}(-, \chi_i, f_{\chi_i^{\infty p}}),$$

where $\widetilde{D}_\diamond^{[e]}(-, \chi_i, f_{\chi_i^{\infty p}})$ is defined in (3.8).

For every $w \notin \diamond$, we choose a nonnegative power Δ_w of w such that $G_{2r}(F_w) \bigcap \widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w) \times_{\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w/\Delta_w)} \widetilde{\mathcal{P}}_{2r}(\mathbb{Z}_w/\Delta_w)$ is contained in $K_{2r,w}$ (and we may take $\Delta_w = 1$ when w is unramified in E). For every $w \in \diamond \setminus \{\infty, p\}$, we may choose a nonnegative power Δ_w of w such that $\otimes_{v\in\mathtt{V}_F^{(w)}} f_{\chi_v}$ is fixed by the kernel of the map $\widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w) \to \widetilde{\mathcal{G}}_{2r}(\mathbb{Z}_w/\Delta_w)$ for every finite character $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$. Indeed, by Definition 3.27(2), the restriction of f_{χ_v} to $K_{2r,v}$ is independent of χ_v, which implies the existence of Δ_w. Finally, put

$$\Delta := \prod_{w\notin\diamond} \Delta_w, \quad \Delta' := \prod_{w\in\diamond\setminus\{\infty,p\}} \Delta_w.$$

Lemma 3.33. *For every open compact subset* $\Omega \subseteq \Gamma_{F,p}$*, if* $\|e\| > 0$*, then*

$$\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\Omega)\right) \in \varinjlim_{d\in\mathbb{N}} \widetilde{\mathcal{H}}^{[r]}_{r,r}(\widetilde{K}_{r,r}(p^d\Delta,\Delta'))$$

(Notation 2.10).

Proof. By construction and Lemma 3.10, it is clear that

$$\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\Omega)\right) \in \varinjlim_{d\in\mathbb{N}} \widetilde{\mathcal{H}}^{[r]}_{r,r}(\widetilde{K}_{r,r}(p^d\Delta,\Delta')) \otimes_{\mathbb{Q}} \mathbb{C}.$$

It remains to show the rationality when $\|e\| > 0$.

Take an arbitrary element $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have $\sigma f_{\chi_v} = f_{\sigma\chi_v}$ for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^{(p)}$ and every finite character $\chi\colon \Gamma_{F,p} \to \mathbb{C}^\times$ by construction. By Proposition 3.13, we have

$$\sigma\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\Omega)\right) = \widetilde{\boldsymbol{h}}_{r,r}\left(\sum_i \sigma(c_i)\widetilde{D}^{[e]}_{\diamond}(-,\sigma\chi_i, f_{\sigma\chi_i^{\infty p}})\right).$$

On the other hand, we have $\mathbf{1}_\Omega = \sigma\mathbf{1}_\Omega = \sum_i \sigma(c_i)\cdot\sigma\chi_i$, which implies that

$$\widetilde{\boldsymbol{h}}_{r,r}\left(\sum_i \sigma(c_i)\widetilde{D}^{[e]}_{\diamond}(-,\sigma\chi_i, f_{\sigma\chi_i^{\infty p}})\right) = \widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\Omega)\right).$$

The lemma is proved. □

Lemma 3.34. *For every open compact subset* $\Omega \subseteq \Gamma_{F,p}$*, if* $\|e\| > 0$*, then there is a unique element*

$$\mathcal{D}^{[e]}_{\diamond}(-,\Omega) \in \varinjlim_{d\in\mathbb{N}} \mathcal{H}^{[r]}_{r,r}(K_{r,r}(p^d\Delta,\Delta'))$$

(Notation 2.10) such that

$$\xi^*_{r,r}\mathcal{D}^{[e]}_{\diamond}(-,\Omega) = \zeta^*_{r,r}\widetilde{\boldsymbol{h}}_{r,r}\left(\widetilde{D}^{[e]}_{\diamond}(-,\Omega)\right)$$

in terms of the diagram (2.6).

Proof. Since the center of $\widetilde{G}_{2r}(\mathbb{A}^\infty)$ (as a subgroup of $\widetilde{G}_{r,r}(\mathbb{A}^\infty)$) acts trivially on $\widetilde{D}^{[e]}_{\diamond}(-,\Omega)$, the element $\zeta^*_{r,r}\widetilde{D}^{[e]}_{\diamond}(-,\Omega)$ descends to the desired element $\mathcal{D}^{[e]}_{\diamond}(-,\Omega)$. □

Notation 3.35. By Remark 2.4(2), we have a map

$$\mathrm{pr}_{\pi,\hat{\pi}} \coloneqq \mathrm{pr}_\pi \otimes \mathrm{pr}_{\hat{\pi}}\colon \mathcal{H}^{[r]}_{r,r} = \mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}_p} \mathcal{H}^{[r]}_r \to \mathcal{V}_\pi \otimes_{\mathbb{L}} \mathcal{V}_{\hat{\pi}}$$

that is the tensor product of pr_π and $\mathrm{pr}_{\hat{\pi}}$ from Lemma 3.17. In what follows, for every $\Psi \in \mathcal{H}^{[r]}_{r,r}$, $\varphi_1 \in \mathcal{V}_{\hat{\pi}}$ and $\varphi_2 \in \mathcal{V}_\pi$, we put

$$\langle\varphi_1\otimes\varphi_2,\Psi\rangle_{\pi,\hat{\pi}} \coloneqq \left\langle\varphi_2^\dagger, \left\langle\varphi_1^\dagger, \mathrm{pr}_{\pi,\hat{\pi}}\Psi\right\rangle_\pi\right\rangle_{\hat{\pi}}.$$

Definition 3.36. We define an $\mathbb{L}$-valued distribution $\mathrm{d}\mathscr{L}_p^\diamond(\pi)$ on $\Gamma_{F,p}$ to be the following assignment

$$\Omega \subseteq \Gamma_{F,p} \mapsto \left(\prod_{u\in\mathrm{P}} \alpha(\pi_u)\right)^{-1} \left\langle\varphi_1\otimes\varphi_2, \mathcal{D}^{[1]}_{\diamond}(-,\Omega)\right\rangle_{\pi,\hat{\pi}},$$

which is additive from the construction. Here, 1 is regarded as a constant tuple in $\mathbb{N}^{\mathrm{P}}$.

Theorem 3.37. *The distribution* $\mathrm{d}\mathscr{L}_p^\diamond(\pi)$ *on* $\Gamma_{F,p}$ *in Definition 3.36 is a p-adic measure. Moreover, if we denote by* $\mathscr{L}_p^\diamond(\pi)$ *the induced (bounded) analytic function on* $\mathscr{X}_{F,p} \otimes_{\mathbb{Q}_p} \mathbb{L}$*, then for every finite (continuous) character* $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ *and every embedding* $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$*, we have*

$$\iota\mathscr{L}_p^\diamond(\pi)(\chi) = \frac{1}{\mathrm{P}^\iota_\pi}\cdot\frac{Z_r^{[F:\mathbb{Q}]}}{b^\diamond_{2r}(\mathbf{1})}\cdot\prod_{v\in\mathtt{V}_F^{(p)}}\prod_{u\in\mathrm{P}_v}\gamma(\tfrac{1+r}{2},\iota(\underline{\pi_u}\otimes\chi_v),\psi_{F,v})^{-1}\cdot L(\tfrac{1}{2},\mathrm{BC}(\iota\pi^\diamond)\otimes(\iota\chi^\diamond\circ\mathrm{Nm}_{E/F})),$$

where

$$Z_r \coloneqq (-1)^r 2^{-2r^2}\cdot 2^{r^2-r}\boldsymbol{\pi}^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}$$

and $\underline{\pi_u}$ is introduced in Proposition 3.25. In particular, in terms of the data chosen from this subsection, $\mathscr{L}_p^\diamond(\pi)$ depends on $\diamond$ only, justifying its notation.

Proof. For the first statement, it amounts to showing that the map

$$\Omega \mapsto \int_\Omega \mathrm{d}\mathscr{L}_p^\diamond(\pi) := \left(\prod_{u\in\mathtt{P}} \alpha(\pi_u)\right)^{-1} \left\langle \varphi_1 \otimes \varphi_2, \mathcal{D}_\diamond^{[1]}(-,\Omega)\right\rangle_{\pi,\hat{\pi}} \in \mathbb{L}$$

is uniformly bounded.

Now we show the uniform boundedness. By Lemma 3.38 below, for every Ω, there is an integer $e = e_\Omega \geqslant 1$, regarded as a constant tuple in $\mathbb{N}^{\mathtt{P}}$, such that

$$D_\diamond^{[e]}(-,\Omega) \in \mathcal{H}_{r,r}^{[r]}(K_{r,r}(p\Delta,\Delta')).$$

By (T2) and Lemma 3.11, we have

$$\begin{aligned}
\int_\Omega \mathrm{d}\mathscr{L}_p^\diamond(\pi) &= \left(\prod_{u\in\mathtt{P}} \alpha(\pi_u)\right)^{-e} \left\langle \varphi_1 \otimes (\mathrm{T}_p^-)^{e-1}\varphi_2, \mathcal{D}_\diamond^{[1]}(-,\Omega)\right\rangle_{\pi,\hat{\pi}} \\
&= \left(\prod_{u\in\mathtt{P}} \alpha(\pi_u)\right)^{-e} \left\langle \varphi_1 \otimes \varphi_2, (1\times \mathrm{U}_p^{e-1})\mathcal{D}_\diamond^{[1]}(-,\Omega)\right\rangle_{\pi,\hat{\pi}} \\
&= \left(\prod_{u\in\mathtt{P}} \alpha(\pi_u)\right)^{-e} \left\langle \varphi_1 \otimes \varphi_2, \mathcal{D}_\diamond^{[e]}(-,\Omega)\right\rangle_{\pi,\hat{\pi}}
\end{aligned}$$

where $\mathrm{T}_p^- := \prod_{u\in\mathtt{P}} \mathrm{T}_u^-$. Since $\alpha(\pi_u) \in O_{\mathbb{L}}^\times$ for every $u \in \mathtt{P}$ and $\mathcal{H}_{r,r}^{[r]}(K(p\Delta,\Delta'))$ is a finite-dimensional $\mathbb{Q}_p$-vector space, it suffices to show that there exists an integer $M \geqslant 0$ such that

$$p^M \boldsymbol{q}_{r,r}(g\cdot D_\diamond^{[e]}(-,\Omega)) \in \mathrm{SF}_{r,r}(\overline{\mathbb{Z}}_p)$$

holds for every $g \in G_{r,r}(\mathbb{A}^\diamond)$, every $e \geqslant 1$ and every Ω. By (2.11), it suffices to study $\boldsymbol{q}_{2r}(g\cdot E_\diamond^{[e]}(-,\Omega))$. We may choose M such that $p^M W_{2r}^\diamond \cdot \prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} \widehat{\mathfrak{f}_v}(T^\square) \in \mathbb{Z}_{(p)}$ for every $T^\square \in \mathrm{Herm}_{2r}^\circ(F)$. Then by Lemma 3.12 and Lemma 3.5(1), $p^M \boldsymbol{q}_{2r}(g\cdot E_\diamond^{[e]}(-,\Omega)) \in \mathrm{SF}_{2r}(\overline{\mathbb{Z}}_p)$ holds for every $1 \leqslant j \leqslant s$, every $e \geqslant 1$ and every Ω. Thus, we have shown that $\mathrm{d}\mathscr{L}_p^\diamond(\pi)$ is a p-adic measure.

Next, we show the second statement, that is, the interpolation property. By construction, Remark 3.16 and Lemma 3.11, for every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ and embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\begin{aligned}
\iota\mathscr{L}_p^\diamond(\pi)(\chi) &= \left(\iota\prod_{u\in\mathtt{P}} \alpha(\pi_u)\right)^{-1} \frac{1}{(\mathrm{P}_\pi^\iota)^2} \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} \varphi_1^\iota(g_1^\dagger)\varphi_2^\iota(g_2^\dagger)E_\diamond^{[1]}((g_1,g_2),\iota\chi,f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2 \\
&= \frac{1}{(\mathrm{P}_\pi^\iota)^2} \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} \varphi_1^\iota(g_1^\dagger)\varphi_2^\iota(g_2^\dagger)E_\diamond^{[0]}((g_1,g_2),\iota\chi,f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2
\end{aligned}$$

$$= \frac{1}{(\mathrm{P}_\pi^\iota)^2} \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} (\varphi_1^\dagger)^\iota(g_1)\varphi_2^\iota(g_2)E_\diamond^{[0]}((g_1,g_2^\dagger),\iota\chi,f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2. \tag{3.16}$$

By (3.6),

$$(3.16) = \frac{1}{(\mathrm{P}_\pi^\iota)^2}\cdot b_{2r}^\diamond(\mathbf{1})^{-1}\cdot b_{2r}^\diamond(\iota\chi) \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} (\varphi_1^\dagger)^\iota(g_1)\varphi_2^\iota(g_2)E((g_1,g_2^\dagger),f_\infty^{[r]}\otimes(\mathfrak{f}_{\iota\chi_p}^{[0]})^{\iota\chi_p}\otimes f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2$$

$$= \frac{1}{(\mathrm{P}_\pi^\iota)^2}\cdot b_{2r}^\diamond(\mathbf{1})^{-1}\cdot b_{2r}^\diamond(\iota\chi) \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} (\varphi_1^\dagger)^\iota(g_1)\varphi_2^\iota(g_2)E(\imath(g_1,g_2),f_\infty^{[r]}\otimes(\mathfrak{f}_{\iota\chi_p}^{[0]})^{\iota\chi_p}\otimes f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2, \tag{3.17}$$

where we have used $(g_1, g_2^\dagger) = \imath(g_1, g_2)$ as in Remark 3.1. By the well-known doubling integral expansion (see [Ral82] or [Liu11a, Section 2B] in the case of unitary groups) and Lemma 3.31, we have

$$\iint_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} (\varphi_1^\dagger)^\iota(g_1)\varphi_2^\iota(g_2)E(\imath(g_1,g_2), f_\infty^{[r]} \otimes (\mathfrak{f}^{[0]}_{\iota\chi_p})^{\iota\chi_p} f_{\iota\chi^{\infty p}})\,\mathrm{d}g_1\,\mathrm{d}g_2 = \frac{L(\frac{1}{2}, \mathrm{BC}(\iota\pi^\Diamond)\otimes(\iota\chi^\Diamond\circ\mathrm{Nm}_{E/F}))}{b_{2r}^\Diamond(\iota\chi)}$$
$$\times Z((\varphi_1^\dagger)^\iota_\infty\otimes(\varphi_2^\iota)_\infty, f_\infty^{[r]})\cdot\prod_{v\in\mathtt{V}_F^{(p)}} Z^\iota(\varphi_{1,v}^\dagger\otimes\varphi_{2,v}, (\mathfrak{f}^{[0]}_{\iota\chi_v})^{\iota\chi_v})\cdot\prod_{v\in\mathtt{V}_F^{(\Diamond\backslash\{\infty,p\})}} Z^\iota(\varphi_{1,v}^\dagger\otimes\varphi_{2,v}, f_{\iota\chi_v}).$$

There are three cases:

- By [EL, Theorem 1.3 & Proposition 3.3.2] (with $n = k = 2r$, $a = b = r$, $\tau_1 = \cdots = \tau_r = r$, $\nu_1 = \cdots = \nu_r = -r$, and $\chi_{\mathrm{ac}}^r = 1$), we have (see the proof of [LL21, Proposition 3.7] for more details)
$$Z((\varphi_1^\dagger)^\iota_\infty\otimes(\varphi_2^\iota)_\infty, f_\infty^{[r]}) = \mathrm{P}_\pi^\iota\cdot Z_r^{[F:\mathbb{Q}]}.$$
- By (T2) and Proposition 3.32, for $v\in\mathtt{V}_F^{(p)}$, we have
$$Z^\iota(\varphi_{1,v}^\dagger\otimes\varphi_{2,v}, (\mathfrak{f}^{[0]}_{\iota\chi_v})^{\iota\chi_v}) = \prod_{u\in\mathtt{P}_v}\gamma(\tfrac{1+r}{2}, \iota(\underline{\pi_u}\otimes\chi_v), \psi_{F,v})^{-1}.$$
- By Lemma 3.29, for $v\in\mathtt{V}_F^{(\Diamond\backslash\{\infty,p\})}$, we have $Z^\iota(\varphi_{1,v}^\dagger\otimes\varphi_{2,v}, f_{\iota\chi_v}) = 1$.

Putting together, we have

$$(3.17) = \frac{1}{\mathrm{P}_\pi^\iota}\cdot\frac{Z_r^{[F:\mathbb{Q}]}}{b_{2r}^\Diamond(\mathbf{1})}\cdot\prod_{v\in\mathtt{V}_F^{(p)}}\prod_{u\in\mathtt{P}_v}\gamma(\tfrac{1+r}{2}, \iota(\underline{\pi_u}\otimes\chi_v), \psi_{F,v})^{-1}\cdot L(\tfrac{1}{2}, \mathrm{BC}(\iota\pi^\Diamond)\otimes(\iota\chi^\Diamond\circ\mathrm{Nm}_{E/F})).$$

The theorem is proved. □

Lemma 3.38. *For every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$, there exists $e_\chi\in\mathbb{N}$ such that for every $e\in\mathbb{N}^{\mathtt{P}}$ satisfying $e_u\geqslant e_\chi$ for every $u\in\mathtt{P}$, the section $(\mathfrak{f}^{[e]}_{\chi_p})^{\chi_p}$ is invariant under $\prod_{v\in\mathtt{V}_F^{(p)}}\mathscr{G}_{r,r}(O_{F_v})\times_{\mathscr{G}_{r,r}(O_{F_v}/\varpi_v)}\mathscr{P}_{r,r}(O_{F_v}/\varpi_v)$ (Definition 2.9).*

Proof. It is well-known that for every $v\in\mathtt{V}_F^{(p)}$ and $u\in\mathtt{P}_u$, we have $I_v^d\mathrm{U}_uI_v^d = I_v^{d-1}\mathrm{U}_uI_v^d$ for every integer $d\geqslant 2$, where $I_v^d \coloneqq \mathscr{G}_r(O_{F_v})\times_{\mathscr{G}_r(O_{F_v}/\varpi_v)}\mathscr{P}_r(O_{F_v}/\varpi_v^d)$. Since $(\mathfrak{f}^{[0]}_{\chi_p})^{\chi_p}$ is fixed by $\prod_{v\in\mathtt{V}_F^{(p)}}\mathscr{P}_{r,r}(O_{F_v})$, it follows that there exists a pair $(e_{\chi1}, e_{\chi2})\in\mathbb{N}\times\mathbb{N}$ such that for every $(e_1,e_2)\in\mathbb{N}^{\mathtt{P}}\times\mathbb{N}^{\mathtt{P}}$ satisfying $e_{1,u}\geqslant e_{\chi1}$ and $e_{2,u}\geqslant e_{\chi2}$ for every $u\in\mathtt{P}$, $(\mathrm{U}_p^{e_1}\times\mathrm{U}_p^{e_2})(\mathfrak{f}^{[0]}_{\chi_p})^{\chi_p}$ is invariant under $\prod_{v\in\mathtt{V}_F^{(p)}}\mathscr{G}_{r,r}(O_{F_v})\times_{\mathscr{G}_{r,r}(O_{F_v}/\varpi_v)}\mathscr{P}_{r,r}(O_{F_v}/\varpi_v)$. By Lemma 3.11, $(\mathrm{U}_p^{e_1}\times\mathrm{U}_p^{e_2})(\mathfrak{f}^{[0]}_{\chi_p})^{\chi_p} = (\mathfrak{f}^{[e_1^{\mathtt{c}}+e_2]}_{\chi_p})^{\chi_p}$. Thus, the lemma follows by taking $e_\chi = e_{\chi1}+e_{\chi2}$. □

To end this subsection, we discuss the parity of the vanishing order of $\mathscr{L}_p^\Diamond(\pi)$ at $\mathbf{1}$. For every $v\in\mathtt{V}_F^{\mathrm{fin}}$, the root number $\epsilon(\mathrm{BC}(\pi_v)\otimes\iota\chi_v\circ\mathrm{Nm}_{E_v/F_v})$ does not depend on the finite character $\chi\colon\Gamma_{F,p}\to\overline{\mathbb{Q}}_p^\times$ and the embedding $\iota\colon\overline{\mathbb{Q}}_p\to\mathbb{C}$, which we denote by $\epsilon(\pi_v)$. Put $\epsilon(\pi)\coloneqq\prod_{v\in\mathtt{V}_F^{\mathrm{fin}}}\epsilon(\pi_v)$, which is indeed a finite product.

Proposition 3.39. *The vanishing order of $\mathscr{L}_p^\Diamond(\pi)$ at $\mathbf{1}$ has the same parity as $r[F:\mathbb{Q}]+\frac{1-\epsilon(\pi)}{2}$.*

Proof. Denote by $\Gamma_{F,p}^\Diamond$ the subgroup of $\Gamma_{F,p}$ generated by uniformizers above $\Diamond\setminus\{\infty,p\}$. For every $v\in\mathtt{V}_F^{(\Diamond\backslash\{\infty,p\})}$, there is a unique element $\mathscr{L}_p(\pi_v)\in\mathbb{L}(\Gamma_{F,p}^\Diamond)$ such that for every finite character $\chi\colon\Gamma_{F,p}\to\overline{\mathbb{Q}}_p^\times$ and every embedding $\iota\colon\overline{\mathbb{Q}}_p\to\mathbb{C}$, $\iota\mathscr{L}_p(\pi_v)(\chi) = L(\frac{1}{2}, \mathrm{BC}(\iota\pi_v)\otimes(\iota\chi_v\circ\mathrm{Nm}_{E_v/F_v}))$. In particular, $\mathscr{L}_p(\pi_v)$ has neither poles nor zeros at points corresponding to finite characters.

Put

$$\mathscr{L}_p(\pi)\coloneqq\mathscr{L}_p^\Diamond(\pi)\prod_{v\in\mathtt{V}_F^{(\Diamond\backslash\{\infty,p\})}} b_{2r,v}(\mathbf{1})^{-1}\cdot\mathscr{L}_p(\pi_v),$$

regarded as an element of $\mathbb{Z}_p[[\Gamma_{F,p}]] \otimes_{\mathbb{Z}_p[\Gamma^{\diamond}_{F,p}]} \mathbb{L}(\Gamma^{\diamond}_{F,p})$. Then $\mathscr{L}_p(\pi)$ is the unique element such that for every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^{\times}$ and every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\iota\mathscr{L}_p(\pi)(\chi) = \frac{1}{\mathrm{P}^{\iota}_{\pi}} \cdot \frac{Z_r^{[F:\mathbb{Q}]}}{b_{2r}^{\infty p}(\mathbf{1})} \cdot \prod_{v\in\mathtt{V}_F^{(p)}} \prod_{u\in\mathrm{P}_v} \gamma(\tfrac{1+r}{2}, \iota(\underline{\pi_u} \otimes \chi_v), \psi_{F,v})^{-1} \cdot L(\tfrac{1}{2}, \mathrm{BC}(\iota\pi^p) \otimes (\iota\chi^p \circ \mathrm{Nm}_{E/F}))$$

holds. As $\mathrm{BC}(\iota\hat{\pi}_v) \simeq \mathrm{BC}(\iota\pi_v^{\vee}) \simeq \mathrm{BC}(\iota\pi_v) \circ \mathtt{c}$ for every $v \in \mathtt{V}_F$, we have the functional equation

$$\iota\mathscr{L}_p(\pi)(\chi) = \epsilon(\mathrm{BC}({}^{\iota}\pi) \otimes \iota\chi \circ \mathrm{Nm}_{E/F}) \cdot \iota\mathscr{L}_p(\pi)(\chi^{-1}) \tag{3.18}$$

(Definition 1.1) for the root number

$$\epsilon(\mathrm{BC}({}^{\iota}\pi) \otimes \iota\chi \circ \mathrm{Nm}_{E/F}) = \prod_{v\in\mathtt{V}_F} \epsilon(\mathrm{BC}({}^{\iota}\pi_v) \otimes \iota\chi_v \circ \mathrm{Nm}_{E_v/F_v}) \in \{\pm1\}.$$

It is clear that for $v \in \mathtt{V}_F^{(\infty)}$, $\epsilon(\mathrm{BC}({}^{\iota}\pi_v) \otimes \iota\chi_v \circ \mathrm{Nm}_{E_v/F_v}) = (-1)^r$; and for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\epsilon(\mathrm{BC}({}^{\iota}\pi_v) \otimes \iota\chi_v \circ \mathrm{Nm}_{E_v/F_v}) = \epsilon(\pi_v)$ by definition.

To summarize, if we denote by $\vee$ the involution on $\mathbb{Z}_p[[\Gamma_{F,p}]] \otimes_{\mathbb{Z}_p[\Gamma^{\diamond}_{F,p}]} \mathbb{L}(\Gamma^{\diamond}_{F,p})$ induced by the inverse homomorphism of $\Gamma_{F,p}$, then (3.18) implies the functional equation

$$\mathscr{L}_p(\pi) = (-1)^{r[F:\mathbb{Q}]}\epsilon(\pi) \cdot \mathscr{L}_p(\pi)^{\vee}.$$

It follows that the vanishing order of $\mathscr{L}_p(\pi)$ at $\mathbf{1}$ has the same parity as $r[F:\mathbb{Q}] + \frac{1-\epsilon(\pi)}{2}$. The proposition is then proved since the vanishing order of $\mathscr{L}_p^{\diamond}(\pi)$ at $\mathbf{1}$ is same as that of $\mathscr{L}_p(\pi)$. □

Remark 3.40. We expect that the p-adic L-function $\mathscr{L}_p(\pi)$ constructed in the proof of Proposition 3.39 is again a p-adic measure, that is, an element of $\mathbb{Z}_p[[\Gamma_{F,p}]] \otimes_{\mathbb{Z}_p} \mathbb{L}$.

3.6. **Remarks on p-adic measures.** In this subsection, we review some facts about derivatives of p-adic measures and make some remarks that will be used in the next section. For $d \geqslant 1$, we denote by U_d the image of $1 + O_F \otimes (p^d\mathbb{Z}_p)$ in $\Gamma_{F,p}$, which is an open subgroup of finite index.

Let $\mathrm{d}\mu$ be an $\mathbb{L}$-valued p-adic measure on $\Gamma_{F,p}$ (for a finite extension $\mathbb{L}/\mathbb{Q}_p$). For every continuous character $\chi\colon \Gamma_{F,p} \to R^{\times}$ for a complete $\mathbb{L}$-ring R, we put

$$\mu(\chi) \coloneqq \int_{\Gamma_{F,p}} \chi\,\mathrm{d}\mu \coloneqq \lim_{d\to\infty} \sum_{x\in\Gamma_d} \chi(x)\,\mathrm{vol}(xU_d,\,\mathrm{d}\mu)$$

where $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is an arbitrary increasing chain of sets of representatives of $\Gamma_{F,p}/U_d$ for $d = 1, 2, \dots$. Then $\mu(\chi)$ does not depend on $(\Gamma_d)_d$ and defines a bounded rigid analytic function μ on $\mathscr{X}_{F,p} \otimes_{\mathbb{Q}_p} \mathbb{L}$, or equivalently an element in $\mathbb{Z}_p[[\Gamma_{F,p}]] \otimes_{\mathbb{Z}_p} \mathbb{L}$. We consider its derivative $\partial\mu(\mathbf{1})$ at $\mathbf{1}$, which is an element in $\Gamma_{F,p} \otimes_{\mathbb{Z}_p} \mathbb{L}$ – the cotangent space of $\mathscr{X}_{F,p} \otimes_{\mathbb{Q}_p} \mathbb{L}$ at $\mathbf{1}$. More precisely, $\partial\mu(\mathbf{1})$ is the linear functional in $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p), \mathbb{L})$ that sends $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ to

$$\partial_\lambda\mu(\mathbf{1}) \coloneqq \lim_{c\to\infty} \frac{1}{p^c} \left(\mu(\exp(p^c\lambda)) - \mu(\mathbf{1})\right) = \lim_{c\to\infty} \frac{1}{p^c} \lim_{d\to\infty} \sum_{x\in\Gamma_d} (\exp(p^c\lambda(x)) - 1)\,\mathrm{vol}(xU_d,\,\mathrm{d}\mu).$$

Since

$$\frac{1}{p^c}(\exp(p^c\lambda(x)) - 1) = \lambda(x) + \frac{p^c\lambda(x)^2}{2!} + \frac{p^{2c}\lambda(x)^3}{3!} + \cdots,$$

and $\mathrm{vol}(xU_d,\,\mathrm{d}\mu)$ is bounded independent of x and d, we have

$$\partial_\lambda\mu(\mathbf{1}) = \lim_{d\to\infty} \sum_{x\in\Gamma_d} \lambda(x)\,\mathrm{vol}(xU_d,\,\mathrm{d}\mu). \tag{3.19}$$

Definition 3.41. We say that an $\mathbb{L}$-valued p-adic measure $\mathrm{d}\mu$ on $\Gamma_{F,p}$ is *integral* if $\mathrm{vol}(\Omega,\,\mathrm{d}\mu) \in O_{\mathbb{L}}$ for every open compact subset $\Omega \subseteq \Gamma_{F,p}$, that is, μ belongs to $O_{\mathbb{L}}[[\Gamma_{F,p}]]$.

Lemma 3.42. *Let* $\mathrm{d}\mu$ *be an integral* $\mathbb{L}$*-valued* p*-adic measure on* $\Gamma_{F,p}$*. Then for every* $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ *and every* $d \geqslant 1$*, we have*

$$\partial_\lambda \mu(\mathbf{1}) - \sum_{x \in \Gamma_d} \lambda(x) \operatorname{vol}(xU_d,\ \mathrm{d}\mu) \in p^d O_{\mathbb{L}}.$$

In particular, $\partial\mu(\mathbf{1}) \in \Gamma^{\mathrm{fr}}_{F,p} \otimes_{\mathbb{Z}_p} O_{\mathbb{L}}$*.*

Proof. Since $U_d \subseteq p^d\Gamma_{F,p}$, we have $\lambda(x) - \lambda(x') \in p^d O_{\mathbb{L}}$ if $x = x'$ in $\Gamma_{F,p}/U_d$. Then the lemma follows from (3.19) since $\mathrm{d}\mu$ is integral. □

The discussion of this subsection can be easily generalized to p-adic measures valued in a finite product of finite extensions of $\mathbb{L}$.

4. Selmer theta lifts and their p-adic heights

In this section, we introduce Selmer theta lifts and study their p-adic heights. We fix an embedding $E \hookrightarrow \mathbb{C}$ and regard E as a subfield of $\mathbb{C}$ and regard $\overline{E}$ as the algebraic closure of E in $\mathbb{C}$. Fix an even positive integer $n = 2r$.

4.1. Hermitian spaces and Weil representations.

Let π be a relevant $\mathbb{L}$-representation of $G_r(\mathbb{A}^\infty_F)$ for some finite extension $\mathbb{L}/\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$.

Choose a finite set $\diamond$ of places of $\mathbb{Q}$ containing $\{\infty, p\}$ such that π_v is unramified (hence $v \notin \mathtt{V}^{\mathrm{ram}}_F$) for every $v \in \mathtt{V}_F \setminus \mathtt{V}^{(\diamond)}_F$.

Let $V, (\ ,\ )_V$ be a hermitian space (that is nondegenerate and E-linear in the second variable) over E of rank n that is split at every $v \in \mathtt{V}_F \setminus \mathtt{V}^{(\diamond)}_F$, has signature $(n-1, 1)$ along the induced inclusion $F \subseteq \mathbb{R}$ and signature $(n, 0)$ at other archimedean places of F. We introduce the following sets of notation.

(H1) For every F-ring R and every integer $m \geqslant 0$, we denote by

$$T(x) := \Big((x_i, x_j)_V\Big)_{i,j} \in \mathrm{Herm}_m(R)$$

the moment matrix of an element $x = (x_1, \dots, x_m) \in V^m \otimes_F R$.

(H2) For every $v \in \mathtt{V}_F$, we put $\epsilon_v := \eta_{E/F}((-1)^r \det V_v) \in \{\pm 1\}$. In particular, $\epsilon_v = 1$ for $v \notin \mathtt{V}^{(\diamond)}_F$.

(H3) Let $v \in \mathtt{V}^{\mathrm{fin}}_F$ be an element and $m \geqslant 0$ an integer.

- For $T \in \mathrm{Herm}_m(F_v)$, we put $(V^m_v)_T := \{x \in V^m_v \mid T(x) = T\}$, and

$$(V^m_v)_{\mathrm{reg}} := \bigcup_{T \in \mathrm{Herm}^\circ_m(F_v)} (V^m_v)_T,$$

where we recall Herm°_m from §2.1(F3).
- For every $\mathbb{Z}[p_v^{-1}]\langle p_v\rangle$-ring $\mathbb{M}$, we have a Fourier transform map $\widehat{\ }\colon \mathscr{S}(V^m_v, \mathbb{M}) \to \mathscr{S}(V^m_v, \mathbb{M})$ sending ϕ to $\widehat{\phi}$ defined by the formula

$$\widehat{\phi}(x) := \int_{V^m_v} \phi(y)\psi_{F,v}\left(\mathrm{Tr}_{E_v/F_v} \sum_{i=1}^m (x_i, y_i)_V\right) \mathrm{d}y,$$

which is in fact a finite sum, where $\mathrm{d}y$ is the self-dual Haar measure on V^m_v with respect to $\psi_{F,v}$. In what follows, we will always use this self-dual Haar measure on V^m_v.

(H4) Put $H := \mathrm{U}(V)$, which is a reductive group over F.

(H5) For $v \in \mathtt{V}^{\mathrm{fin}}_F \setminus \{v \in \mathtt{V}^{\mathrm{ram}}_F \mid \text{either } \epsilon_v = -1 \text{ or } v \mid 2\}$, a *good lattice* of V_v is an O_{E_v}-lattice Λ_v of V_v that is a subgroup of $\Lambda^\vee_v$ of index $q_v^{1-\epsilon_v}$, where

$$\Lambda^\vee_v := \{x \in V_v \mid \mathrm{Tr}_{E_v/F_v}(x, y)_V \in \mathfrak{p}_v^{-d_v} \text{ for every } y \in \Lambda_v\}.$$

We say that

- an open compact subgroup $L^\diamond$ of $H(\mathbb{A}^\diamond_F)$ is *good* if it is the product of the stabilizers of good lattices at $v \notin \mathtt{V}_F \setminus \mathtt{V}^{(\diamond)}_F$;
- a Schwartz function $\phi^\diamond \in \mathscr{S}(V^m_r \otimes_F \mathbb{A}^\diamond_F)$ is *good* if the it is the product of $\mathbf{1}_{\Lambda^m_v}$ in which Λ_v is a good lattice of V_v.

(H6) Denote by $\mathbb{T}^\diamond$ the (abstract) spherical Hecke algebra of rank-n unitary groups over E away from $\diamond$, and $\mathbb{S}^\diamond$ its subring consisting of Hecke operators supported on places split in E. In particular, for every good open compact subgroup $L^\diamond$ of $H(\mathbb{A}_F^\diamond)$, we have canonical isomorphisms

$$\mathbb{T}^\diamond = \mathbb{Z}[L^\diamond\backslash H(\mathbb{A}_F^\diamond)/L^\diamond], \quad \mathbb{S}^\diamond = \varinjlim_{\substack{\mathtt{T}\subseteq\mathtt{V}_F^{\mathrm{spl}}\backslash\mathtt{V}_F^{(\diamond)} \\ |\mathtt{T}|<\infty}} \mathbb{Z}[L_\mathtt{T}^\diamond\backslash H(F_\mathtt{T})/L_\mathtt{T}^\diamond]\otimes\mathbf{1}_{L^{\diamond\mathtt{T}}}$$

of commutative rings.

(H7) For every integer $m \geqslant 1$, every $v \in \mathtt{V}_F^{\mathrm{fin}}$ and every $\mathbb{Z}[p_v^{-1}]\langle p_v\rangle$-ring $\mathbb{M}$, we have the Weil representation $\omega_{m,v}$ of $G_m(F_v) \times H(F_v)$ on $\mathscr{S}(V_v^m, \mathbb{M})$ given by the following formulae:

- for $a \in \mathrm{GL}_m(E_v)$ and $\phi \in \mathscr{S}(V_v^m, \mathbb{M})$, we have
$$\omega_{m,v}(m(a))\phi(x) = |\det a|_{E_v}^r \cdot \phi(xa);$$
- for $b \in \mathrm{Herm}_m(F_v)$ and $\phi \in \mathscr{S}(V_v^m, \mathbb{M})$, we have
$$\omega_{m,v}(n(b))\phi(x) = \psi_{F,v}(\mathrm{tr}\, bT(x))\phi(x);$$
- for $\phi \in \mathscr{S}(V_v^m, \mathbb{M})$, we have
$$\omega_{m,v}\,(\mathtt{w}_m)\,\phi(x) = \gamma_{V_v,\psi_{F,v}}^m \cdot \widehat{\phi}(x),$$
where $\gamma_{V_v,\psi_{F,v}} \in \{\pm 1\}$ is the Weil constant of V_v with respect to $\psi_{F,v}$;
- for $h \in H(F_v)$ and $\phi \in \mathscr{S}(V_v^m, \mathbb{M})$, we have
$$\omega_{m,v}(h)\phi(x) = \phi(h^{-1}x).$$

(H8) When $m = n = 2r$, we have the *Siegel–Weil section* map

$$f_-^{\mathrm{SW}}\colon \mathscr{S}(V_v^{2r}) \to \mathrm{I}_{r,v}^\Box(\mathbf{1})$$

for $v \in \mathtt{V}_F^{\mathrm{fin}}$ sending Φ to f_Φ^{SW} defined by the formula

$$f_\Phi^{\mathrm{SW}}(g) = (\omega_{2r,v}(g)\Phi)\,(0), \quad g \in G_{2r}(F_v) = G_r^\Box(F_v).$$

(H9) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, there is a unique $\mathbb{Q}$-valued Haar measure $\mathrm{d}h_v$ on $H(F_v)$, called the *Siegel–Weil measure*, satisfying that for every $T^\Box \in \mathrm{Herm}_{2r}^\circ(F_v)$ and every $\Phi \in \mathscr{S}(V_v^{2r})$,[11]

$$I_{T^\Box}(\Phi) := \int_{H(F_v)} \Phi(h_v^{-1}x)\,\mathrm{d}h_v = b_{2r,v}(\mathbf{1}) \cdot W_{T^\Box}(f_\Phi^{\mathrm{SW}}),$$

where x is an arbitrary element in $(V_v^{2r})_{T^\Box}$. When v is unramified over $\mathbb{Q}$ and $H \otimes_F F_v$ is unramified, the measure $\mathrm{d}h_v$ gives volume 1 to every hyperspecial maximal subgroup of $H(F_v)$. For $\Phi \in \mathscr{S}(V_v^{2r}, R)$ with R a general $\mathbb{Q}$-ring, $I_{T^\Box}(\Phi)$ is well-defined and belongs to R.

(H10) Let $\iota\colon \mathbb{L} \to \mathbb{C}$ be an embedding. For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, put

$$\theta(\iota\pi_v) := \mathrm{Hom}_{G_r(F_v)}(\mathscr{S}(V_{\pi_v}^r), \iota\pi_v)$$

as a complex representation of $H(F_v)$. Then put

$$\theta(\iota\pi) := \otimes'_{v\in\mathtt{V}_F^{\mathrm{fin}}}\theta(\iota\pi_v)$$

as a complex representation of $H(\mathbb{A}_F^\infty)$.

Lemma 4.1. *For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, there exists a (unique up to isomorphism) hermitian space V_{π_v} over E_v of rank n such that for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, $\theta(\iota\pi_v) \neq 0$ if and only if $V_v \simeq V_{\pi_v}$. When $V_v \simeq V_{\pi_v}$, $\theta(\iota\pi_v)$ is a tempered irreducible admissible representation of $H(F_v)$, satisfying*

$$\mathrm{Hom}_{H(F_v)}\left(\mathscr{S}(V_{\pi_v}^r), \theta(\iota\pi_v)\right) \simeq \iota\pi_v$$

as $\mathbb{C}[G_r(F_v)]$-modules.

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[11] We recall our convention from §1.6 that $\mathscr{S}(V_v^{2r})$ means $\mathscr{S}(V_v^{2r}, \mathbb{C})$.

Proof. For every fixed embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, the existence and the uniqueness of V_{π_v} follow from the local theta dichotomy [GG11, Theorem 3.10] (see also [HKS96, Corollary 4.4] and [Har07, Theorem 2.1.7]). As V_{π_v} does not change if we twist the additive character $\psi_{F,v}$ by automorphisms of $\mathbb{C}$, it is the same for all ι.

Since $\iota\pi_v$ is tempered, the irreducibility and the temperedness of $\theta(\iota\pi_v)$ follow from [GI16, Theorem 4.1(v)] and (the same argument for) [GS12, Theorem 1.3(ii)], respectively. The last isomorphism follows from the dual statements. □

Definition 4.2. We say that V (as above) is *π-coherent* if $V_v \simeq V_{\pi_v}$ for every $v \in \mathtt{V}_F^{\mathrm{fin}}$.

Remark 4.3. We have the following remarks.

(1) The following conditions are equivalent for V as above: π-coherent, $\pi^\vee$-coherent, $\pi^\dagger$-coherent, $\hat{\pi}$-coherent.
(2) There exists a hermitian space V as above that is π-coherent if and only if

$$\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty\})}} \eta_{E/F}\left((-1)^r \det V_{\pi_v}\right) = -(-1)^{r[F:\mathbb{Q}]}. \tag{4.1}$$

Moreover, when (4.1) holds, $\mathscr{L}_p^\diamond(\pi)$ vanishes at $\mathbf{1}$.

In the rest of this subsection, we discuss the rationality of local theta liftings. For readers who are willing to fix an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and do not care about the rationality of the coefficients of the Selmer theta lifts below, this discussion may be ignored.

Take a place $v \in \mathtt{V}_F^{\mathrm{fin}}$. We say that π_v is *symmetric* if for every element $a \in F_v^\times$, $\pi_v^{\dagger_a} \simeq \pi_v$, where $\dagger_a$ is the automorphism of $G_r(F_v)$ given by the conjugation of the element $\begin{pmatrix} 1_r & \\ & a1_r \end{pmatrix} \in \mathrm{GL}_{2r}(E_v)$.[12] Denote by $\mathtt{U}_\pi$ the subset of $\mathtt{V}_F^{\mathrm{fin}}$ consisting of v such that π_v is *not* symmetric.

It is easy to see that

$$\left\{(a_w)_w \in \prod_{w<\infty} \mathbb{Z}_w^\times = \widehat{\mathbb{Z}}^\times \,\middle|\, a_w \in \bigcap_{v\in\mathtt{U}_\pi\cap\mathtt{V}_F^{(w)}} \mathrm{Nm}_{E_v/F_v} E_v^\times\right\} \tag{4.2}$$

is an open subgroup of $\widehat{\mathbb{Z}}^\times$. Thus, we may define $\mathbb{Q}_\pi$ to be the finite abelian extension of $\mathbb{Q}$ contained in $\mathbb{C}$ determined by this subgroup via the global class field theory.

Remark 4.4. We have the following remarks concerning $\mathbb{Q}_\pi$.

(1) It is clear that $\mathbb{Q}_{\hat{\pi}} = \mathbb{Q}_\pi$ since $\mathtt{U}_{\hat{\pi}} = \mathtt{U}_{\pi^\dagger} = \mathtt{U}_\pi$.
(2) It is clear that in (4.2), we may replace $\mathtt{U}_\pi$ by $\mathtt{U}_\pi \cap \mathtt{V}_F^{\mathrm{ram}}$.
(3) Suppose that we are in the situation of Assumption 1.6. For every $v \in \mathtt{V}_F^{\mathrm{ram}}$ and every $a \in O_{F_v}^\times$, since $\dagger_a$ preserves $K_{r,v}$, we have $\pi_v^{\dagger_a} \simeq \pi_v$, that is, π_v is symmetric. In other words, $\mathtt{U}_\pi \cap \mathtt{V}_F^{\mathrm{ram}} = \emptyset$ hence $\mathbb{Q}_\pi = \mathbb{Q}$. For this reason, the readers may just assume $\mathbb{Q}_\pi = \mathbb{Q}$ for further reading.
(4) Since every p-adic place of F splits in E, p is unramified in $\mathbb{Q}_\pi$.

Lemma 4.5. *For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, and every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}_\pi)$, $\theta(\sigma\iota\pi_v)$ is isomorphic to $\sigma\theta(\iota\pi_v)$.*

Proof. We have $\sigma\theta(\iota\pi_v) = \mathrm{Hom}_{G_r(F_v)}(\sigma\mathscr{S}(V_{\pi_v}^r), \sigma\iota\pi_v)$. The representation $\sigma\mathscr{S}(V_{\pi_v}^r)$ has the same formulae of definition as $\mathscr{S}(V_{\pi_v}^r)$ except that $n(b)$ sends ϕ to the function $x \mapsto \psi_{F,v}(a \operatorname{tr} bT(x))\phi(x)$ for some element $a \in \mathbb{Z}_{p_v}^\times$ (resp. $a \in \mathbb{Z}_{p_v}^\times \cap \mathrm{Nm}_{E_v/F_v} E_v^\times$) if $v \notin \mathtt{U}_\pi$ (resp. $v \in \mathtt{U}_\pi$). It follows that

$$\mathrm{Hom}_{G_r(F_v)}(\sigma\mathscr{S}(V_{\pi_v}^r), \sigma\iota\pi_v) \simeq \mathrm{Hom}_{G_r(F_v)}(\mathscr{S}(V_{\pi_v}^r), \sigma\iota\pi_v^{\dagger_a}) = \theta(\sigma\iota\pi_v^{\dagger_a}).$$

By definition, we have $\pi_v^{\dagger_a} \simeq \pi_v$. The lemma follows. □

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[12] Note that $\pi_v^{\dagger_a} \simeq \pi_v$ when $a \in \mathrm{Nm}_{E_v/F_v} E_v^\times$.

4.2. **p-adic height pairing on unitary Shimura varieties.** From this subsection, we will assume $F \neq \mathbb{Q}$. Put $\mathbb{L}_\pi := \mathbb{L} \otimes_{\mathbb{Q}} \mathbb{Q}_\pi$.

Back to the setup in §4.1, we have the projective system of Shimura varieties X_L associated with $\mathrm{Res}_{F/\mathbb{Q}} H$ indexed by neat open compact subgroups $L \subseteq H(\mathbb{A}_F^\infty)$, which are smooth *projective* schemes over E of dimension $n-1$. Put $\overline{X}_L := X_L \otimes_E \overline{E}$.

Lemma 4.6. *For every L, there exists a unique decomposition*

$$\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r)) = \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi] \oplus \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\widehat{\theta_\pi}]$$

of $\mathbb{L}_\pi[L\backslash H(\mathbb{A}_F^\infty)/L]$-modules such that for every homomorphism $\iota\colon \mathbb{L}_\pi \to \mathbb{C}$ extending the inclusion $\mathbb{Q}_\pi \subseteq \mathbb{C}$, $\iota\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]$ is isomorphic to a finite sum of copies of $\theta(\iota\pi)^L$ (§4.1(H10)) and $\iota\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\widehat{\theta_\pi}]$ does not contain $\theta(\iota\pi)^L$ as a subquotient.[13]

In what follow, we put $\mathrm{V}_{\pi,L} := \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]$. It is clear that $\mathrm{V}_{\pi,L}$ is nonzero only if V is π-coherent (Definition 4.2).

Proof. For every given ι, the existence of such a decomposition follows from Matsushima's formula. It follows from Lemma 4.5 that these decompositions are the same for all ι. □

The Hochschild–Serre spectral sequence in [Jan88, Corollary 3.4] induces a decreasing filtration

$$\cdots \subseteq \mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) \subseteq \mathrm{F}^1\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) \subseteq \mathrm{F}^0\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) = \mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$$

of $\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$ in the category of $\mathbb{L}_\pi[L\backslash H(\mathbb{A}_F^\infty)/L]$-modules such that there is a canonical isomorphism

$$\frac{\mathrm{F}^i\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))}{\mathrm{F}^{i+1}\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))} \simeq \mathrm{H}^i\left(E, \mathrm{H}^{2r-i}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))\right).$$

Lemma 4.7. *There exists a unique map of $\mathbb{L}_\pi[L\backslash H(\mathbb{A}_F^\infty)/L]$-modules*

$$\wp_\pi\colon \mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) \to \mathrm{H}^1\left(E, \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]\right)$$

such that it vanishes on $\mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$ and induces the identity map on

$$\mathrm{H}^1\left(E, \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]\right) \subseteq \frac{\mathrm{F}^1\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))}{\mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))}.$$

Proof. By [LL21, Proposition 6.9(1)], we have

$$\mathrm{Hom}_{\mathbb{L}_\pi[L\backslash H(\mathbb{A}_F^\infty)/L]}\left(\mathrm{H}^i_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r)), \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]\right) = 0 \tag{4.3}$$

as long as $i \neq 2r-1$. In particular, we have

$$\mathrm{Hom}_{\mathbb{L}_\pi[L\backslash H(\mathbb{A}_F^\infty)/L]}\left(\mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)), \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]\right) = 0.$$

By Lemma 4.6, we have a unique map

$$\wp^1_\pi\colon \mathrm{F}^1\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) \to \mathrm{H}^1\left(E, \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]\right)$$

satisfying the property in the lemma. It remains to show that $\wp^1_\pi$ extends uniquely to $\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$. The uniqueness follows from (4.3) with $i = 2r$. For the existence, note that [LL21, Proposition 6.9(1)] actually implies that there exists an element $\mathrm{s} \in \mathbb{S}^\diamond_{\mathbb{L}}$ (by possibly enlarging $\diamond$) such that s^* annihilates $\mathrm{H}^{2r}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))$ and acts by the identity map on $\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]$. In particular, $\wp_\pi := \wp^1_\pi \circ \mathrm{s}^*$ is such an extension. □

Denote by $\mathbb{S}^\diamond_{\pi,L}$ the subset of $\mathbb{S}^\diamond_{\mathbb{L}_\pi}$ consisting of elements s such that s^* annihilates $\mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$ and the induced endomorphism of $\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))/\mathrm{F}^2\mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r))$ has image in $\mathrm{H}^1(E, \mathrm{V}_{\pi,L})$. It is clear that $\mathbb{S}^\diamond_{\pi,L}$ is an ideal. On the other hand, we have the Hecke character

$$\chi^\diamond_\pi\colon \mathbb{S}^\diamond_{\mathbb{L}} \to \mathbb{L}$$

given by its action on π.

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[13] We warn the readers that the statement could be wrong if we replace $\mathbb{L}_\pi$ by $\mathbb{L}$.

Lemma 4.8. *Suppose that L is of the form $L_\diamond L^\diamond$ in which $L^\diamond$ is good (§4.1(H5)).*

(1) For every $\mathrm{s} \in \mathbb{S}^\diamond_{\pi,L}$, we have $\mathrm{s}^* = \chi^\diamond_{\hat{\pi}}(\mathrm{s}) \cdot \wp_\pi \colon \mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{L}_\pi(r)) \to \mathrm{H}^1(E, \mathrm{V}_{\pi,L})$.

(2) If $L_\diamond$ is of the form $\prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty\})}} L_v$ in which for every $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty\})} \setminus \mathtt{V}_F^{\mathrm{spl}}$, L_v is special maximal and $\theta(\iota\pi_v)^{L_v} \neq 0$ for every embedding $\iota \colon \mathbb{L} \to \mathbb{C}$, then the restriction of $\chi^\diamond_{\hat{\pi}}$ to $\mathbb{S}^\diamond_{\pi,L}$ is surjective.

Proof. By [Liu11a, Corollary A.6(2)], for every embedding $\iota \colon \mathbb{L} \to \mathbb{C}$ and every $v \in \mathtt{V}_F^{\mathrm{spl}} \setminus \mathtt{V}_F^{(\diamond)}$, we have $\theta(\iota\pi_v) \simeq \hat{\pi}_v \otimes_{\mathbb{L},\iota} \mathbb{C}$. This already implies (1).

For (2), For every embedding $\iota \colon \mathbb{L} \to \mathbb{C}$ and every $i \in \mathbb{Z}$, we have

$$\mathrm{H}^i_{\text{ét}}(\overline{X}_L, \mathbb{L}(r)) \otimes_{\mathbb{L},\iota} \mathbb{C} \simeq \bigoplus_{\pi'} (\pi')^L \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[L\backslash H(\mathbb{A}_F^\infty)/L]}\left((\pi')^L, \mathrm{H}^i_{\text{ét}}(\overline{X}_L, \mathbb{L}(r)) \otimes_{\mathbb{L},\iota} \mathbb{C}\right) \tag{4.4}$$

in which π' runs over all irreducible admissible (complex) representations of $H(\mathbb{A}_F^\infty)$, by Matsushima's formula. By [LL21, Proposition 6.9(1)], we may find $\mathrm{s} \in \mathbb{S}^\diamond_{\mathbb{L}}$ that annihilates $\mathrm{H}^i_{\text{ét}}(\overline{X}_L, \mathbb{L}(r))$ for every $i \neq 2r-1$ and such that $\chi^\diamond_{\hat{\pi}}(\mathrm{s}) = 1$. It remains to show that if π' contributes nontrivially to $\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}(r))$ in (4.4) satisfying that $\pi'_v \simeq \theta(\iota\pi_v)$ for every $v \in \mathtt{V}_F^{\mathrm{spl}} \setminus \mathtt{V}_F^{(\diamond)}$, then the same must hold for every $v \in \mathtt{V}_F^{\mathrm{fin}}$. Indeed, by the strong multiplicity one property [Ram, Theorem A] and the local-global compatibility of base change [KMSW], we already have the isomorphism for $v \in \mathtt{V}_F^{\mathrm{spl}}$ and that $\mathrm{BC}(\pi'_v) \simeq \mathrm{BC}(\theta(\iota\pi_v))$ for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^{\mathrm{spl}}$. Now take an element $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^{\mathrm{spl}}$. Since both π'_v and $\theta(\iota\pi_v)$ have nontrivial L_v-invariants and L_v is special maximal, they are constituents of the same principal series representation ρ of $H(F_v)$. Since ρ^{L_v} is one-dimensional, they must be the same constituent. Thus, (2) follows. □

Remark 4.9. In both [LL21] and [LL22], the authors mistakenly identified $\chi^{\mathrm{R}}_{\pi^\vee}$ with $(\chi^{\mathrm{R}}_\pi)^{\mathtt{c}}$, where $\chi^{\mathrm{R}}_\pi \colon \mathbb{T}^{\mathrm{R}}_{\mathbb{Q}^{\mathrm{ac}}} \to \mathbb{Q}^{\mathrm{ac}}$ is the Hecke character in [LL21, Definition 6.8] (and similarly for $\chi^{\mathrm{R}}_{\pi^\vee}$); in fact, they only coincide when restricted to $\mathbb{T}^{\mathrm{R}}_{\mathbb{Q}^{\mathrm{ac}}\cap\mathbb{R}}$. As a consequence, one should replace $\chi^{\mathrm{R}}_\pi(\mathrm{s})^{\mathtt{c}}$ by $\chi^{\mathrm{R}}_{\pi^\vee}(\mathrm{s})$ in [LL21, Proposition 6.10(1)]; and whenever one asks for two elements in $\mathbb{S}^{\mathrm{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}^{\mathrm{R}}_\pi$, they should actually be in $\mathbb{S}^{\mathrm{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}^{\mathrm{R}}_{\pi^\vee}$. Such modifications do not affect the proof of the results.

Lemma 4.10. *For $v \in \mathtt{V}_F^{(p)}$, if π_v is unramified, then $\mathrm{V}_{\pi,L}$ is crystalline at every place u of E above v.*

Proof. If π_v is unramified, then its local theta lift is also an unramified representation of $H(F_v)$. In particular, we may assume that L is of the form L_vL^v in which L_v is hyperspecial maximal. By [RSZ20, Theorem 4.5] (or a more closely related discussion after [LL21, Proposition 7.1]), X_L admits a finite étale cover that has smooth reduction at every place u of E above v. Thus, $\mathrm{V}_{\pi,L}$ is crystalline at u. □

Lemma 4.11. *There is a unique up to isomorphism semisimple continuous representation ρ_π of $\mathrm{Gal}(\overline{E}/E)$ of dimension n with coefficients in $\overline{\mathbb{Q}}_p$ such that for every place u of E not above $\diamond$ that is split over F, ρ_π is unramified at u and a geometric Frobenius at u acts with a characteristic polynomial that coincides with the Satake polynomial of π_u, regarded as an unramified representation of $\mathrm{GL}_n(E_u)$. Moreover, we have $\rho_{\hat{\pi}} \simeq \rho_\pi^{\mathtt{c}} \simeq \rho_\pi^\vee(1-n)$.*

Proof. The uniqueness of ρ_π follows from its property and the Chebotarev density theorem; and the last statement follows from the uniqueness. It remains to show the existence of ρ_π.

Choose an isomorphism $\iota \colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. By [Mok15], the automorphic base change of $\otimes_{v \in \mathtt{V}_F^{(\infty)}} \pi_v^{[r]} \otimes \iota\pi$ is an isobaric sum of distinct unitary cuspidal automorphic representations Π_j of $\mathrm{GL}_{n_j}(\mathbb{A}_E)$ for some partition $n = n_1 + \cdots + n_s$. By [CH13, Theorem 3.2.3], for each $1 \leqslant j \leqslant s$, we have a semisimple representation ρ_{Π_j} of $\mathrm{Gal}(\overline{E}/E)$ such that for every place u of E not above $\diamond$ that is split over F, the restriction of ρ_{Π_j} to the place u is unramified and corresponds to the irreducible admissible representation $\left(\Pi_{j,u} \otimes |\ |_{E_u}^{\frac{1-n}{2}}\right) \otimes_{\mathbb{C},\iota^{-1}} \overline{\mathbb{Q}}_p$ of $\mathrm{GL}_{n_j}(E_u)$ under the unramified local Langlands correspondence. Then $\rho_\pi := \bigoplus_{j=1}^s \rho_{\Pi_j}$ does the job. □

Hypothesis 4.12. *For every homomorphism $\iota \colon \mathbb{L}_\pi \to \overline{\mathbb{Q}}_p$ over $\mathbb{L}$ and every irreducible $\overline{\mathbb{Q}}_p[\mathrm{Gal}(\overline{E}/E)]$-module ρ that is a subquotient of $\mathrm{V}_{\pi,L} \otimes_{\mathbb{L}_\pi,\iota} \overline{\mathbb{Q}}_p$, ρ is a direct summand of $\rho_\pi(r)$.*

Remark 4.13. We have the following remarks concerning Hypothesis 4.12.

(1) Hypothesis 4.12 is equivalent to the parallel statement for $\hat{\pi}$.
(2) We understand that Hypothesis 4.12 will follow from a sequel of the work [KSZ].
(3) A precise prediction of the semisimplification of $\mathrm{V}_{\pi,L}\otimes_{\mathbb{L}_\pi,\iota}\overline{\mathbb{Q}}_p$, if not zero, can be found in [LL21, Hypothesis 6.6]. Such prediction is independent of ι.
(4) It is conjectured that $\mathrm{V}_{\pi,L}\otimes_{\mathbb{L}_\pi,\iota}\overline{\mathbb{Q}}_p$ is irreducible as an $\overline{\mathbb{Q}}_p[L\backslash H(\mathbb{A}_F^\infty)/L][\mathrm{Gal}(\overline{E}/E)]$-module. However, this does not seem reachable at this moment.

From this moment, we will *assume Hypothesis 4.12 without further mentioning*.

Lemma 4.14. *For every finite place u of E not above p, we have*

$$\mathrm{H}^i(E_u,\mathrm{V}_{\pi,L})=\mathrm{H}^i(E_u,\mathrm{V}_{\hat{\pi},L})=0$$

for every $i\in\mathbb{Z}$.

Proof. By symmetry, we only need to consider $\mathrm{V}_{\pi,L}$. By Hypothesis 4.12, it suffices to show that $\mathrm{H}^1(E_u,\rho_\pi(r))=0$ for such u. By [Car12, Theorem 1.1] and [TY07, Lemma 1.4(3)], we know that the associated Weil–Deligne representation of $\rho_\pi(r)|_{E_u}$ is pure of weight -1, which implies that $\mathrm{H}^i(E_u,\rho_\pi(r))=0$ by (the proof of) [Nek00, Proposition 2.5]. □

Lemma 4.15. *Take $v\in\mathtt{V}_F^{(p)}$. If π_v is Panchishkin unramified (Definition 3.21), then both $\mathrm{V}_{\pi,L}|_{E_u}$ and $\mathrm{V}_{\hat{\pi},L}|_{E_u}$ satisfy the Panchishkin condition (Definition A.12) and are pure of weight -1 for u above v.*

Proof. By symmetry and Lemma 3.22, we only need to consider $\mathrm{V}_{\pi,L}$. We will use the results and notational conventions introduced in §A.6. Since $\mathrm{V}_{\pi,L}$ is crystalline (Lemma 4.10), by Lemma A.14 and Hypothesis 4.12, it suffices to show that $\rho_\pi(r)|_{E_u}$ satisfies the Panchishkin condition and is pure of weight -1 for u above v. By [Car14, Theorem 1.1], we know that for every embedding $\tau\colon E_u\to\overline{\mathbb{Q}}_p$,
(1) $\rho_\pi(r)|_{E_u}$ is crystalline and has Hodge–Tate weights $\{-r,-r+1,\dots,r-1\}$ at τ;
(2) the associated Weil–Deligne representation $\mathrm{WD}(\rho_\pi(r)|_{E_u})_\tau$ (see §A.6) is unramified and its multiset of generalized geometric Frobenius eigenvalues is $\{\alpha_{v,1}\sqrt{q_v}^{-1},\dots,\alpha_{v,n}\sqrt{q_v}^{-1}\}$.

By (2), we know that $\rho_\pi(r)|_{E_u}$ is pure of weight -1. Moreover, by Lemma 4.10 and Remark A.10, the multiset of generalized φ-eigenvalues on $\mathbb{D}:=\mathbb{D}_{\mathrm{cris}}(\rho_\pi(r)|_{E_u})$ is $\{\alpha_{v,1}\sqrt{q_v}^{-1},\dots,\alpha_{v,n}\sqrt{q_v}^{-1}\}$ as well.

For the Panchishkin condition, by Lemma 3.22, we may assume that the unique subset J of $\{1,\dots,n\}$ with $|J|=r$ such that $\sqrt{q_v}^{r^2}\prod_{j\in J}\alpha_{v,j}\in O_{\mathbb{L}}^\times$ is $\{1,\dots,r\}$ without loss of generality. Then $\alpha_{v,j}\sqrt{q_v}^{-1}$ belongs to $\overline{\mathbb{Z}}_p$ if and only if $i\geqslant r+1$. Let $\mathbb{D}^+\subset\mathbb{D}$ be the $\mathbb{L}\otimes_{\mathbb{Q}_p}E_{u,0}$-submodule spanned by the generalized eigenspaces with respect to the crystalline Frobenius for the eigenvalues $\{\alpha_{v,j}\sqrt{q_v}^{-1}\mid 1\leqslant j\leqslant r\}$, which is the negative-slope submodule defined in general in Lemma A.13. By the weak admissibility of $\mathbb{D}$ and rank counting, the map (A.4) for $\mathbb{D}^+$ is an isomorphism, and by inspection of the Newton and Hodge polygons, $\mathbb{D}^+$ is weakly admissible. It follows that the equivalent Panchishkin condition of Lemma A.13 is satisfied. □

If π_v is Panchishkin unramified for every $v\in\mathtt{V}_F^{(p)}$, then we may apply §A.7 to the case where $K=E$, $X=X_L$, $d=d'=r$, $\mathbb{L}=\mathbb{L}_\pi$, $\mathrm{V}=\mathrm{V}_{\pi,L}$ and $\mathrm{V}'=\mathrm{V}_{\hat{\pi},L}$. Indeed, (V1) is due to Lemma 4.14; (V2) and (V3) are due to Lemma 4.10 and Lemma 4.15. Consequently, we have a canonical p-adic height pairing

$$\langle\,,\,\rangle_{(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat{\pi},L}),E}\colon\mathrm{H}^1_f(E,\mathrm{V}_{\pi,L})\times\mathrm{H}^1_f(E,\mathrm{V}_{\hat{\pi},L})\to\Gamma_{E,p}\otimes_{\mathbb{Z}_p}\mathbb{L}_\pi. \tag{4.5}$$

4.3. Selmer theta lifts.

We take a finite set $\blacklozenge$ of places of $\mathbb{Q}$ containing $\{\infty\}$ and a subfield $\mathbb{M}$ of $\mathbb{C}$ containing $\mathbb{Q}\langle\prod_{w\in\blacklozenge\setminus\{\infty\}}w\rangle$ and $\mathbb{Q}_\pi$.

Consider a neat open compact subgroup $L\subseteq H(\mathbb{A}_F^\infty)$. Recall that for every element $x\in V^m\otimes_F\mathbb{A}_F^\infty$, we have *Kudla's special cycle* $Z(x)_L\in\mathrm{Z}^m(X_L)$ if $T(x)\in\mathrm{Herm}_m^\circ(F)^+$ and $Z(x)_L\in\mathrm{CH}^m(X_L)_{\mathbb{Q}}$ in general. See [LL21, Section 4] for more details in our setting. For every $\phi\in\mathscr{S}(V^m\otimes_F\mathbb{A}_F^\infty,\mathbb{M})^{K_m^\blacklozenge\times L}$ and every $T\in\mathrm{Herm}_m(F)$, we put

$$Z_T(\phi)_L\coloneqq\sum_{\substack{x\in L\backslash V^m\otimes_F\mathbb{A}_F^\infty\\ T(x)=T}}\phi(x)Z(x)_L$$

as an element in $\mathrm{Z}^m(X_L)\otimes\mathbb{M}$ if $T\in\mathrm{Herm}_m^\circ(F)^+$ and in $\mathrm{CH}^m(X_L)\otimes\mathbb{M}$ in general. Denote by

$$Z_T^{\text{ét}}(\phi)_L\in\mathrm{H}^{2m}_{\text{ét}}(X_L,\mathbb{Q}_p(m))\otimes_{\mathbb{Q}}\mathbb{M}$$

the image of $Z_T(\phi)_L$ under the (absolute) cycle class map $\mathrm{Z}^m(X_L) \to \mathrm{CH}^m(X_L) \to \mathrm{H}^{2m}_{\text{ét}}(X_L, \mathbb{Q}_p(m))$.

Definition 4.16. Suppose that $m = r$. We define the *π-Selmer generating function* to be

$$Z^\pi_\phi(g)_L := \sum_{T \in \mathrm{Herm}_r(F)^+} \wp_\pi\left(Z^{\text{ét}}_T(\omega_r(g)\phi)_L\right) \cdot q^T \in \mathrm{H}^1(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi} \mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C}), \quad g \in G_r(\mathbb{A}^\infty_F).$$

Here, ω_r is the restricted tensor product of $\omega_{r,v}$ (§4.1(H7)) over all $v \in \mathtt{V}^{\text{fin}}_F$; and $\wp_\pi$ is the map in Lemma 4.7.[14]

Hypothesis 4.17 (Modularity of π-Selmer generating functions)**.** *For every $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{C})^{K^\blacklozenge_r \times L}$, there exists an element*

$$Z^\pi_{\phi,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \mathcal{A}^{[r]}_{r,\text{hol}}$$

such that $(1 \otimes \boldsymbol{q}^{\text{an}}_r)(g \cdot Z^\pi_{\phi,L}) = Z^\pi_\phi(g)_L$ holds in $\mathrm{H}^1(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi} \mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C})$ for every $g \in G_r(\mathbb{A}^\infty_F)$, where $\boldsymbol{q}^{\text{an}}_r$ is the analytic q-expansion map (Definition 2.5).

Remark 4.18. Hypothesis 4.17 is implied by [LL21, Hypothesis 4.5].

We warn the readers that Hypothesis 4.17 is stronger than the following statement: For every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, there exists an element $Z^{\pi,\iota}_{\phi,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi,\iota\times 1} \mathcal{A}^{[r]}_{r,\text{hol}}$ such that for every $g \in G_r(\mathbb{A}^\infty_F)$, $(1 \otimes \boldsymbol{q}^{\text{an}}_r)(g \cdot Z^{\pi,\iota}_{\phi,L})$ coincides with the natural image of $Z^\pi_\phi(g)_L$ in $\mathrm{H}^1(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi,\iota\times 1} \mathrm{SF}_r(\mathbb{C})$ induced by ι.[15] The stronger statement in Hypothesis 4.17 reflects, in some sense, the conjecture that the image of the absolute cycle class map $\mathrm{CH}^r(X_L) \to \mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{Q}_p(r))$ is a finitely generated abelian group.

Recall from Definition 2.3 the $\mathbb{Q}_p$-vector space $\mathcal{H}^{[r]}_r$ and the subspaces $\mathcal{V}_\pi, \mathcal{V}_{\hat{\pi}}$ of $\mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}_p} \mathbb{L}$ introduced after Lemma 3.14.

Proposition 4.19. *Assume that Hypothesis 4.17 holds for π. Then for every $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{M})^{K^\blacklozenge_r \times L}$, there exists a unique element*

$$\mathcal{Z}^\pi_{\phi,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi} \left(\mathcal{V}_\pi \otimes_{\mathbb{Q}} \mathbb{M}\right)$$

such that for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, $(\mathcal{Z}^\pi_{\phi,L})^\iota$, regarded as an element in $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \mathcal{A}^{[r]}_{r,\text{hol}}$ via the inclusion $\mathbb{M} \subseteq \mathbb{C}$, coincides with $Z^\pi_{\phi,L}$.

Proof. We first explain that it suffices to find the element $\mathcal{Z}^\pi_{\phi,L}$ in $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_\pi} \left(\mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}} \mathbb{M}\right)$. Indeed, if we can find such elements, then the assignment $\phi \mapsto \mathcal{Z}^\pi_{\phi,L}$ defines a functional in

$$\mathrm{Hom}_{\mathbb{Q}_\pi[K^\blacklozenge_r \backslash G_r(\mathbb{A}^\infty_F)/K^\blacklozenge_r][L \backslash H(\mathbb{A}^\infty_F)/L]}\left(\mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{M})^{K^\blacklozenge_r \times L}, \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_\pi} \left(\mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}} \mathbb{M}\right)\right).$$

By the definition of $\mathrm{V}_{\pi,L} = \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}_\pi(r))[\theta_\pi]$ from Lemma 4.6 and Lemma 4.1, the functional $\mathcal{Z}^\pi_{-,L}$ must take values in the subspace $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}_\pi} \left(\mathcal{V}_\pi \otimes_{\mathbb{Q}} \mathbb{M}\right)$ (which could be zero).

Now we show the existence of $\mathcal{Z}^\pi_{\phi,L}$ as an element in $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_\pi} \left(\mathcal{H}^{[r]}_r \otimes_{\mathbb{Q}} \mathbb{M}\right)$. Put $G'_r := \mathrm{Res}_{F/\mathbb{Q}}\, G_r$, which has been regarded as a subgroup of $\widetilde{G}_r$ in Remark 2.7. For every $w \notin \blacklozenge$, choose a nonnegative power Δ_w of w such that the intersection of

$$\widetilde{K}_{r,w} := \widetilde{\mathcal{G}}_r(\mathbb{Z}_w) \times_{\widetilde{\mathcal{G}}_r(\mathbb{Z}_w/\Delta_w)} \widetilde{\mathcal{P}}_r(\mathbb{Z}_w/\Delta_w)$$

with $G'_r(\mathbb{Q}_w)$ is contained in $K_{r,w}$ (and we may take $\Delta_w = 1$ when w is unramified in E). Put $\widetilde{K}^\blacklozenge_r := \prod_{w \notin \blacklozenge} \widetilde{K}_{r,w}$ and $K'^\blacklozenge_r := G'_r(\mathbb{A}^\blacklozenge) \cap \widetilde{K}^\blacklozenge_r \subseteq K^\blacklozenge_r$.

We claim that for every open compact subgroup K' of $\prod_{w \in \blacklozenge} G'_r(\mathbb{Q}_w)$, there exists an open compact subgroup $\widetilde{K}$ of $\prod_{w \in \blacklozenge} \widetilde{G}_r(\mathbb{Q}_w)$ containing K' such that the natural map

$$G'_r(\mathbb{Q}) \backslash G'_r(\mathbb{R})^{\text{ad}} \times G'_r(\mathbb{A}^\infty)/K'K'^\blacklozenge_r \to \widetilde{G}_r(\mathbb{Q}) \backslash \widetilde{G}_r(\mathbb{R})^{\text{ad}} \times \widetilde{G}_r(\mathbb{A}^\infty)/\widetilde{K}\widetilde{K}^\blacklozenge_r$$

is injective, and hence an open and closed immersion. In fact, since $\widetilde{G}_r(\mathbb{Q})$ is discrete in $\widetilde{G}_r(\mathbb{A}^\infty)$, we have

$$\varprojlim_{K' \subseteq \widetilde{K}} \widetilde{G}_r(\mathbb{Q}) \backslash \widetilde{G}_r(\mathbb{R})^{\text{ad}} \times \widetilde{G}_r(\mathbb{A}^\infty)/\widetilde{K}\widetilde{K}^\blacklozenge_r = \widetilde{G}_r(\mathbb{Q}) \backslash \widetilde{G}_r(\mathbb{R})^{\text{ad}} \times \widetilde{G}_r(\mathbb{A}^\infty)/K'\widetilde{K}^\blacklozenge_r$$

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[14] Since $\mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{M})$ *strictly* contains $\mathbb{L} \otimes_{\mathbb{Q}} \mathrm{SF}_r(\mathbb{C})$, *a priori* we do not know whether $Z^\pi_\phi(g)_L$ belongs to $\mathrm{H}^1(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \mathrm{SF}_r(\mathbb{C})$.

[15] In particular, Theorem 4.21 below is stronger than Theorem 1.7.

Then the claim follows from the obvious injectivity of the map

$$G'_r(\mathbb{Q})\backslash G'_r(\mathbb{R})^{\mathrm{ad}} \times G'_r(\mathbb{A}^\infty)/K'K'^{\blacklozenge}_r \to \widetilde{G}_r(\mathbb{Q})\backslash\widetilde{G}_r(\mathbb{R})^{\mathrm{ad}} \times \widetilde{G}_r(\mathbb{A}^\infty)/K'\widetilde{K}^{\blacklozenge}_r.$$

Choose a sufficiently small K' as above such that $Z^\pi_{\phi,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \mathcal{A}^{[r]}_{r,\mathrm{hol}}(K'K^{\blacklozenge}_r)$. By the above claim, we may extend $Z^\pi_{\phi,L}$ by zero to obtain an element $\widetilde{Z}^\pi_{\phi,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \widetilde{\mathcal{A}}^{[r]}_{r,\mathrm{hol}}(\widetilde{K}\widetilde{K}^{\blacklozenge}_r)$ for some $\widetilde{K}$ as above.

Note that for every $\mathbb{L}_\pi$-module M, the commutative diagram

$$\begin{tikzcd}
\mathrm{M} \otimes_{\mathbb{Q}_\pi} \mathrm{SF}_r(\mathbb{M}) \arrow[r] \arrow[d] & \mathrm{M} \otimes_{\mathbb{L}_\pi} \mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{M}) \arrow[d] \\
\mathrm{M} \otimes_{\mathbb{Q}_\pi} \mathrm{SF}_r(\mathbb{C}) \arrow[r] & \mathrm{M} \otimes_{\mathbb{L}_\pi} \mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C})
\end{tikzcd}$$

in the category of $\mathbb{L}_\pi$-modules, in which all arrows are natural inclusions, is Cartesian. Thus, by Lemma 2.11, we have

$$\widetilde{\boldsymbol{h}}_r\left(\widetilde{Z}^\pi_{\phi,L}\right) \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p\otimes_{\mathbb{Q}}\mathbb{Q}_\pi} \left(\widetilde{\mathcal{H}}^{[r]}_r \otimes_{\mathbb{Q}} \mathbb{M}\right).$$

It follows from the construction that, in view of (2.6), the element

$$\xi_{r*}\zeta^*_r\widetilde{\boldsymbol{h}}_r\left(\widetilde{Z}^\pi_{\phi,L}\right) \in \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p\otimes_{\mathbb{Q}}\mathbb{Q}_\pi} \left(\mathrm{H}^0(\boldsymbol{\Sigma}_r(K'K'^{\blacklozenge}_r), \xi_{r*}(\boldsymbol{\omega}^\delta_r)^{\otimes r}) \otimes_{\mathbb{Q}} \mathbb{M}\right)$$

belongs to the subspace

$$\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_p\otimes_{\mathbb{Q}}\mathbb{Q}_\pi} \left(\mathrm{H}^0(\boldsymbol{\Sigma}_r(K'K^{\blacklozenge}_r), \boldsymbol{\omega}^{\otimes r}_r) \otimes_{\mathbb{Q}} \mathbb{M}\right)$$

(along the canonical subbundle $\boldsymbol{\omega}^{\otimes r}_r \subseteq \xi_{r*}(\boldsymbol{\omega}^\delta_r)^{\otimes r}$). Then we define $\mathcal{Z}^\pi_{\phi,L}$ to be $\xi_{r*}\zeta^*_r\widetilde{\boldsymbol{h}}_r\left(\widetilde{Z}^\pi_{\phi,L}\right)$, which satisfies the requirement. The proposition is proved. □

Definition 4.20 (Selmer theta lift)**.** Suppose that Hypothesis 4.17 holds for π. For every $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{M})^{K^{\blacklozenge}_r \times L}$ and every $\varphi \in \mathcal{V}_{\hat{\pi}}$, we put

$$\Theta^{\mathrm{Sel}}_\phi(\varphi)_L := \langle \varphi^\dagger, \mathcal{Z}^\pi_{\phi,L}\rangle_\pi$$

(see Notation 3.15 for the pairing) as an element of $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{Q}_\pi} \mathbb{M}$, called a *Selmer theta lift* of π. It is clear from the construction that $\Theta^{\mathrm{Sel}}_\phi(\varphi)_L$ is compatible under pullbacks with respect to L.

At last, we state our theorem concerning Hypothesis 4.17, whose proof will be given in §4.8.

Theorem 4.21. *Suppose that we are in the situation of Assumption 1.6 and $n < p$. If the vanishing order of $\mathscr{L}^\diamond_p(\pi)$ at $\mathbf{1}$ is one, then Hypothesis 4.17 holds for π.*

4.4. **A p-adic arithmetic inner product formula.** Recall from [LL21, Definition 3.8] that we have a canonical volume $\mathrm{vol}^\natural(L) \in \mathbb{Q}_{>0}$, which in fact equals the product of the constant W_{2r} in Lemma 3.2 and the volume of L under the Siegel–Weil measure in §4.1(H9). If Hypothesis 4.17 holds for both π and $\hat{\pi}$, then for every $\varphi_1 \in \mathcal{V}_{\hat{\pi}}$, every $\varphi_2 \in \mathcal{V}_\pi$ and every pair $\phi_1, \phi_2 \in \mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{M})^{K^{\blacklozenge}_r \times L}$, we have the height

$$\mathrm{vol}^\natural(L) \cdot \langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1)_L, \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)_L\rangle_{(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat{\pi},L}),E} \in \Gamma_{E,p} \otimes_{\mathbb{Z}_p} \mathbb{L}_\pi \otimes_{\mathbb{Q}_\pi} \mathbb{M} = \Gamma_{E,p} \otimes_{\mathbb{Z}_p} \mathbb{L} \otimes_{\mathbb{Q}} \mathbb{M}$$

from (4.5), which is independent of L. Denote the above canonical value as $\langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^\natural_{\pi,E}$, and then put

$$\langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^\natural_{\pi,F} := \mathrm{Nm}_{E/F}\langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^\natural_{\pi,E} \in \Gamma_{F,p} \otimes_{\mathbb{Z}_p} \mathbb{L} \otimes_{\mathbb{Q}} \mathbb{M}.$$

Now we can state our *p-adic arithmetic inner product formula*, whose proof will be given in §4.9.

Theorem 4.22. *Suppose that we are in the situation of Assumption 1.6 and $n < p$.*

(1) *If the vanishing order of $\mathscr{L}^\diamond_p(\pi)$ at $\mathbf{1}$ is one (so that Hypothesis 4.17 holds for both π and $\hat{\pi}$ by Theorem 4.21 and Remark 1.5(3)), then for every choice of elements*
  - *$\varphi_1 = \otimes_v\varphi_{1,v} \in \mathcal{V}_{\hat{\pi}}$ and $\varphi_2 = \otimes_v\varphi_{2,v} \in \mathcal{V}_\pi$ both fixed by $K^\diamond_r$ such that $\langle\varphi_{1,v}, \varphi_{2,v}\rangle_{\pi_v} = 1$ for every $v \in \mathtt{V}_F \setminus \mathtt{V}^{(\diamond)}_F$,*
  - *$\phi_1 = \otimes_v\phi_{1,v}, \phi_2 = \otimes_v\phi_{2,v} \in \mathscr{S}(V^r \otimes_F \mathbb{A}^\infty_F, \mathbb{M})^{K^{\blacklozenge}_r}$ with $\phi^\diamond_1 = \phi^\diamond_2$ good (§4.1(H5)),*

the identity

$$\langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^{\natural}_{\pi,F} = \partial\mathscr{L}^{\Diamond}_p(\pi)(\mathbf{1}) \cdot \prod_{v\in\mathtt{V}^{(p)}_F}\prod_{u\in \mathrm{P}_v}\gamma(\tfrac{1+r}{2}, \underline{\pi_u}, \psi_{F,v}) \cdot \prod_{v\in\mathtt{V}^{(\Diamond\setminus\{\infty\})}_F} Z(\varphi^{\dagger}_{1,v}\otimes\varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}) \tag{4.6}$$

holds in $\Gamma_{F,p}\otimes_{\mathbb{Z}_p}\mathbb{L}\otimes_{\mathbb{Q}}\mathbb{M}$, *where*

- $\gamma(\frac{1+r}{2}, \underline{\pi_u}, \psi_{F,v})$ *is the unique element in* $\mathbb{L}^{\times}$ *satisfying* $\iota\gamma(\frac{1+r}{2}, \underline{\pi_u}, \psi_{F,v}) = \gamma(\frac{1+r}{2}, \iota\underline{\pi_u}, \psi_{F,v})$ *for every embedding* $\iota\colon \mathbb{L}\to\mathbb{C}$;
- *the term* $Z(\varphi^{\dagger}_{1,v}\otimes\varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}) \in \mathbb{L}\otimes_{\mathbb{Q}}\mathbb{M}$ *is from Lemma 3.30.*

(2) *If the vanishing order of* $\mathscr{L}^{\Diamond}_p(\pi)$ *at* $\mathbf{1}$ *is not one, then*

$$\mathrm{Nm}_{E/F}\left\langle \wp_{\pi}\left(Z^{\text{ét}}_{T_1}(\phi_1)_L\right), \wp_{\hat\pi}\left(Z^{\text{ét}}_{T_2}(\phi_2)_L\right)\right\rangle_{(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat\pi,L}),E} = 0$$

for every $\phi_1,\phi_2\in\mathscr{S}(V^r\otimes_F\mathbb{A}^{\infty}_F,\mathbb{C})^L$ *and* $T_1,T_2\in\mathrm{Herm}_r(F)^+$.

Remark 4.23. We have the following remarks concerning Theorem 4.22.

(1) By the interpolation property of $\mathscr{L}^{\Diamond}_p(\pi)$ and Lemma 3.31, the right-hand side of (4.6) does not change when enlarging $\Diamond$. In particular, we may enlarge $\Diamond$ to prove the theorem.

(2) Note that when we vary $\varphi_{1,v},\varphi_{2,v},\phi_{1,v},\phi_{2,v}$ for $v\in\mathtt{V}^{(\Diamond\setminus\{\infty\})}_F$, both sides of (4.6) define elements in the space

$$\bigotimes_{v\in\mathtt{V}^{(\Diamond\setminus\{\infty\})}_F}\mathrm{Hom}_{G_r(F_v)\times G_r(F_v)}\left(\mathrm{I}^{\Box}_{r,v}(\mathbf{1}), \pi_v\boxtimes\hat\pi_v\right),$$

which is one-dimensional if V is π-coherent [LL22, Proposition 4.8(1)][16] and vanishes if not. In particular, when V is not π-coherent, all quantities in the theorem are trivially zero.

(3) In the situation of Assumption 1.6, we have $\epsilon(\pi_v) = -1$ (resp. $\epsilon(\pi_v) = 1$) if $v\in\mathtt{S}_{\pi}$ (resp. $v\in\mathtt{V}^{\mathrm{fin}}_F\setminus\mathtt{S}_{\pi}$), where $\epsilon(\pi_v)$ is introduced before Proposition 3.39. By [Liu22, Theorem 1.2] and [LL22, Proposition 3.9], V is π-coherent if and only if $\eta_{E/F}((-1)^r\det V_v) = \epsilon(\pi_v)$ for every $v\in\mathtt{V}^{\mathrm{fin}}_F$. In particular, the theorem is trivial unless $r[F:\mathbb{Q}]+|\mathtt{S}_{\pi}|$ is odd by Remark 4.3(2).

(4) It is clear that Theorem 4.22(2) implies Theorem 1.8(2).

(5) The role of the set $\blacklozenge$ is only to control the coefficient field $\mathbb{M}$ (the smaller $\blacklozenge$ is, the smaller $\mathbb{M}$ we can take). For the proof of the theorem, we may just take $\mathbb{M}=\mathbb{C}$ and ignore the choice of $\blacklozenge$.

Proof of Corollary 1.9 assuming Theorem 4.22. When the vanishing order of $\mathscr{L}^{\Diamond}_p(\pi)$ at $\mathbf{1}$ is one, we may choose V that is π-coherent by Proposition 3.39 and Remark 4.23(3). In particular, we may find data $\varphi_1,\varphi_2,\phi_1,\phi_2$ such that the right-hand side of (4.6) is nonzero. Thus, $\Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1)\neq 0$, which implies that $\mathrm{H}^1_f(E,\mathrm{V}_{\pi,L})\neq 0$. Then the corollary follows from Hypothesis 4.12. □

The rest of this section is devoted to the proof of Theorem 4.21 and Theorem 4.22. Once again, for the proof of these theorems, we may assume $\mathbb{M}=\mathbb{C}$ hence the choice of $\blacklozenge$ is irrelevant.

From now on, we will assume that we are in the situation of Assumption 1.6. In particular, $\mathbb{Q}_{\pi}=\mathbb{Q}$ (Remark 4.4(3)). We may also that V is π-coherent (Definition 4.2) hence the vanishing order of $\mathscr{L}_p(\pi)$ at $\mathbf{1}$ is at least one (Remark 4.3), since otherwise both theorems are trivial.

To shorten notation, we put

$$\mathtt{R} := \mathtt{V}^{(\Diamond\setminus\{p\})}_F\cap\mathtt{V}^{\mathrm{spl}}_F\cap\mathtt{V}^{\heartsuit}_F, \quad \mathtt{T} := \mathtt{V}^{\mathrm{fin}}_F\setminus(\mathtt{V}^{(\Diamond)}_F\cap\mathtt{V}^{\mathrm{spl}}_F\cap\mathtt{V}^{\heartsuit}_F),$$

so that $\mathtt{V}^{(p)}_F\cup\mathtt{R}\cup\mathtt{T}$ is a partition of $\mathtt{V}^{\mathrm{fin}}_F$. By enlarging $\Diamond$, we also assume that

$$\text{The set of primes of } E \text{ above } \mathtt{R} \text{ is nonempty and generate the relative class group of } E/F. \tag{4.7}$$

---

[16] Here, we regard $G_r\times G_r$ as a subgroup of G_{2r} through (3.2) rather than ι from §3.1, which explains the change from $\pi^{\vee}_v$ to $\hat\pi_v$ (see Remark 3.1).

4.5. **Strategy for the modularity.** We first reduce Hypothesis 4.17 to a problem about height pairing. The lemma below is the starting point.

Lemma 4.24. *If π_v is unramified for every $v \in \mathtt{V}_F^{(p)}$, then the image of the composite map*

$$\mathrm{CH}^r(X_L) \to \mathrm{H}^{2r}_{\text{ét}}(X_L, \mathbb{Q}_p(r)) \xrightarrow{\wp_\pi} \mathrm{H}^1(E, \mathrm{V}_{\pi,L})$$

is contained in $\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L})$ for every neat open compact subgroup L of $H(\mathbb{A}_F^\infty)$.

Proof. By Lemma 4.14 (which relies on Hypothesis 4.12), it suffices to show that the image of the above composite map is a crystalline class at every p-adic place of E. This then follows from Lemma 4.10 and [Nek00, Theorem 3.1]. □

Definition 4.25. We say that an element $\varphi \in \mathbb{L} \otimes_{\mathbb{Q}} \mathcal{A}^{[r]}_{r,\mathrm{hol}}$ is *strongly nonzero* if for every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$, the induced element $\iota\varphi \in \mathcal{A}^{[r]}_{r,\mathrm{hol}}$ is nonzero.

Lemma 4.26. *Suppose that we can find*

- *a neat open compact subgroup L of $H(\mathbb{A}_F^\infty)$*
- *an element $\phi_1 \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$,*
- *an element $\zeta \in \mathrm{H}^1_f(E, \mathrm{V}_{\hat{\pi},L})$,*
- *an element $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{E,p}, \mathbb{Z}_p)$,*
- *a strongly nonzero element $\varphi_1 \in \mathbb{L} \otimes_{\mathbb{Q}} \mathcal{A}^{[r]}_{r,\mathrm{hol}}$,*

such that

$$(1 \times \boldsymbol{q}_r^{\mathrm{an}})(g \cdot \varphi_1) = \sum_{T \in \mathrm{Herm}_r(F)^+} \lambda \left\langle \wp_\pi \left(\mathrm{Z}^{\text{ét}}_T(\omega_r(g)\phi_1)_L \right), \zeta \right\rangle_{(\mathrm{V}_{\pi,L}, \mathrm{V}_{\hat{\pi},L}),E} \cdot q^T \in \mathrm{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C})$$

for every $g \in G_r(\mathbb{A}_F^\infty)$ (in which the height pairing makes sense by Lemma 4.24). Then Hypothesis 4.17 holds for every $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)$.

Proof. First, we note that π can actually be defined over a number field $\mathbb{E}$ contained in $\mathbb{L}$ and we will assume that π has coefficients in $\mathbb{E}$ in this proof. For every embedding $\varepsilon\colon \mathbb{E} \to \mathbb{C}$, put

$$\mathrm{V}_\pi^\varepsilon := \mathrm{Hom}_{H(\mathbb{A}_F^\infty)} \left(\theta(\varepsilon\pi), \varinjlim_{L} \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}(r)) \otimes_{\mathbb{E},\varepsilon} \mathbb{C} \right)$$

as an $(\mathbb{L} \otimes_{\mathbb{E},\varepsilon} \mathbb{C})[\mathrm{Gal}(\overline{E}/E)]$-module, where we recall $\theta(\varepsilon\pi)$ from §4.1(H10). Then for each individual L, $\mathrm{V}_{\pi,L} \otimes_{\mathbb{E},\varepsilon} \mathbb{C} = \theta(\varepsilon\pi)^L \otimes_{\mathbb{C}} \mathrm{V}_\pi^\varepsilon$, so that

$$\mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{E},\varepsilon} \mathbb{C} = \theta(\varepsilon\pi)^L \otimes_{\mathbb{C}} \mathrm{H}^1_f(E, \mathrm{V}_\pi^\varepsilon).$$

For every neat open compact subgroup L of $H(\mathbb{A}_F^\infty)$, the assignment

$$\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L \mapsto Z^\pi_\phi(-)_L$$

(Definition 4.16) defines a functional in

$$Z \in \mathrm{Hom}_{\mathbb{C}[G_r(\mathbb{A}_F^\infty)][L\backslash H(\mathbb{A}_F^\infty)/L]} \left(\mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L, \mathrm{H}^1_f(E, \mathrm{V}_{\pi,L}) \otimes_{\mathbb{L}} \mathcal{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C}) \right)$$

(Definition 2.6). For the lemma, it suffices to show that for every embedding $\varepsilon\colon \mathbb{E} \to \mathbb{C}$, the functional Z^ε in

$$\mathrm{Hom}_{\mathbb{C}[G_r(\mathbb{A}_F^\infty)][L\backslash H(\mathbb{A}_F^\infty)/L]} \left(\mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L, \theta(\varepsilon\pi)^L \otimes_{\mathbb{C}} \mathrm{H}^1_f(E, \mathrm{V}_\pi^\varepsilon) \otimes_{\mathbb{L}\otimes_{\mathbb{Q}}\mathbb{C}} \mathcal{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C}) \right)$$

factors through the subspace $\theta(\varepsilon\pi)^L \otimes_{\mathbb{C}} \mathrm{H}^1_f(E, \mathrm{V}_\pi^\varepsilon) \otimes_{\mathbb{C}} \left(\boldsymbol{q}_r^\infty \mathcal{A}^{[r]}_{r,\mathrm{hol}} \right)$ (Definition 2.6) of the target. By Lemma 4.1, there exists an irreducible $\mathbb{C}[G_r(\mathbb{A}_F^\infty)]$-submodule $\mathcal{M}$ of $\mathrm{H}^1_f(E, \mathrm{V}_\pi^\varepsilon) \otimes_{\mathbb{L}\otimes_{\mathbb{Q}}\mathbb{C}} \mathcal{SF}_r(\mathbb{L} \otimes_{\mathbb{Q}} \mathbb{C})$ such that Z^ε takes values in $\theta(\varepsilon\pi)^L \otimes_{\mathbb{C}} \mathcal{M}$. Thus, it suffices to show that $\mathcal{M}$ and $\mathrm{H}^1_f(E, \mathrm{V}_\pi^\varepsilon) \otimes_{\mathbb{C}} \left(\boldsymbol{q}_r^\infty \mathcal{A}^{[r]}_{r,\mathrm{hol}} \right)$ have nonzero intersection, which is implied by the situation of the lemma. □

In practice, we are not able to study the height pairing $\left\langle \wp_\pi \left(\mathrm{Z}^{\text{ét}}_T(\omega_r(g)\phi_1)_L \right), \zeta \right\rangle_{(\mathrm{V}_{\pi,L}, \mathrm{V}_{\hat{\pi},L}),E}$ for all $g \in G_r(\mathbb{A}_F^\infty)$ for given ϕ_1 and ζ. However, the following lemma shows that it suffices to consider a much smaller set of g. Recall the subgroups $M_r \subseteq P_r \subseteq G_r$ from §2.1(G3,G4).

Lemma 4.27. *Let $\ell\colon \mathrm{CH}^r(X_L)_{\mathbb{Q}} \to \mathbb{L}$ be a $\mathbb{Q}$-linear map. For every $\phi \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$ that is fixed under $\prod_{v\in \mathtt{V}_F^{\mathrm{fin}}\setminus \mathtt{R}}(K_{r,v}\cap M_r(F_v))$ and every $\varphi \in \mathbb{L}\otimes_{\mathbb{Q}} \mathcal{A}^{[r]}_{r,\mathrm{hol}}$, if*

$$(1\times \boldsymbol{q}_r^{\mathrm{an}})(g\cdot\varphi) = \sum_{T\in \mathrm{Herm}_r(F)^+} \ell\left(Z_T(\omega_r(g)\phi)_L\right)\cdot q^T \in \mathrm{SF}_r(\mathbb{L}\otimes_{\mathbb{Q}}\mathbb{C})$$

holds for every $g \in M_r(F_{\mathtt{R}})$, then it holds for all $g \in G_r(\mathbb{A}_F^\infty)$.

Proof. To prove the lemma, it suffices to show the identity for every embedding $\iota\colon \mathbb{L}\to\mathbb{C}$. Thus, we may assume that the coefficients are in $\mathbb{C}$ instead of $\mathbb{L}\otimes_{\mathbb{Q}}\mathbb{C}$ and that $\ell\colon \mathrm{CH}^r(X_L)_{\mathbb{C}} \to \mathbb{C}$ is a complex linear functional.

We first turn the generating function into the automorphic setting. For every $v \in \mathtt{V}_F^{(\infty)}$, let $V_v^{\mathrm{std}} := (E_v)^{2r}$ be the standard positive definite hermitian space defined by the identity matrix 1_{2r}, ϕ_v^{std} the standard Gaussian function on $(V_v^{\mathrm{std}})^r$, and $\omega_{r,v}$ the Weil representation of $G_r(F_v)$ generated by ϕ_v^{std} in which every function factors through the moment map $T\colon (V_v^{\mathrm{std}})^r \to \mathrm{Herm}_r(F_v)$. Put $\boldsymbol{\omega}_r := \otimes_{v\in\mathtt{V}_F}\omega_{v,r}$ and $\boldsymbol{\phi} := (\otimes_{v\in\mathtt{V}_F^{(\infty)}}\phi_v^{\mathrm{std}})\otimes\phi$. For every $T\in\mathrm{Herm}_r(F)^+$ and $g\in G_r(\mathbb{A}_F)$, put

$$Z_T(\boldsymbol{\omega}_r(g)\boldsymbol{\phi})_L := \sum_{\substack{x\in L\setminus V^r\otimes_F\mathbb{A}_F^\infty\\ T(x)=T}} (\boldsymbol{\omega}_r(g)\boldsymbol{\phi})(T,x)Z(x)_L \in \mathrm{CH}^r(X_L)_{\mathbb{C}}.$$

Denote $\mathscr{G}$ to be the subset of $G_r(\mathbb{A}_F^\infty)$ consisting of g such that for every $g_\infty\in G_r(F_\infty)$, the sum

$$\sum_{T\in\mathrm{Herm}_r(F)^+}\ell\left(Z_T(\boldsymbol{\omega}_r(g_\infty g)\boldsymbol{\phi})_L\right)$$

is absolutely convergent and equals $\varphi(g_\infty g)$. Thus, the lemma is equivalent to the following statement: If $M_r(F_{\mathtt{R}})\subseteq\mathscr{G}$, then $\mathscr{G} = G_r(\mathbb{A}_F^\infty)$.

Choose an open compact subgroup K of $G_r(\mathbb{A}_F^\infty)$ that fixes ϕ and contains $\prod_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{R}}(K_{r,v}\cap M_r(F_v))$. It is clear that $\mathscr{G}$ is preserved under the right translation by K. On the other hand, condition (4.7) implies that $M_r(F_{\mathtt{R}})$ maps surjectively to the double quotient $G_r(F)\backslash G_r(\mathbb{A}_F^\infty)/K$. Thus, it suffices to show the following claim:

($*$) If $g\in\mathscr{G}$, then $\gamma g\in\mathscr{G}$ for every $\gamma\in G_r(F)$.

The above claim is slightly stronger than the formal modularity property of Kudla's generating functions as proved in [Liu11a, Theorem 3.5] as we do not assume the absolute convergence *a priori*. Nevertheless, it can be proved by essentially the same argument. First note that $G_r(F)$ is generated by $P_r(F)$ and the element

$$w := \begin{pmatrix} 1_{r-1} & & & \\ & & & 1 \\ & & 1_{r-1} & \\ & -1 & & \end{pmatrix}.$$

Claim ($*$) is obvious for $\gamma\in P_r(F)$. Thus, it remains to consider $\gamma = w$. Denote by $\partial\colon \mathrm{Herm}_r\to\mathrm{Herm}_{r-1}$ the map that sends T to its upper-left block of size $r-1$. The proof of [Liu11a, Theorem 3.5(1)] indeed shows the following: If $g\in\mathscr{G}$, then for every $g_\infty\in G_r(F_\infty)$, the sum

$$\sum_{T'\in\mathrm{Herm}_{r-1}(F)^+}\left(\sum_{\substack{T\in\mathrm{Herm}_r(F)^+\\ \partial T=T'}}\ell\left(Z_T(\boldsymbol{\omega}_r(wg_\infty g)\boldsymbol{\phi})_L\right)\right)$$

is absolutely convergent *in order*, and equals $\varphi(wg_\infty g)$. It remains to show that the above sum is indeed absolutely convergent as a double sum. Since $\mathrm{Herm}_r(F)$ is dense in $\mathrm{Herm}_r(\mathbb{A}_F^\infty)$, it follows that

$$\varphi(n(b)wg_\infty g) = \sum_{T'\in\mathrm{Herm}_{r-1}(F)^+}\left(\sum_{\substack{T\in\mathrm{Herm}_r(F)^+\\ \partial T=T'}}\ell\left(Z_T(\boldsymbol{\omega}_r(n(b)wg_\infty g)\boldsymbol{\phi})_L\right)\right)$$

for every $b \in \mathrm{Herm}_r(\mathbb{A}_F)$, in which the right-hand side is again understood as a convergent sum in order. Then it is easy to see that for every $T \in \mathrm{Herm}_r^+(F)$,

$$\int_{\mathrm{Herm}_r(F)\backslash\mathrm{Herm}_r(\mathbb{A}_F)} \varphi(n(b)wg_\infty g)\psi_F^{-1}(\mathrm{tr}\, Tb)\,\mathrm{d}b = \ell\left(Z_T(\boldsymbol{\omega}_r(wg_\infty g)\boldsymbol{\phi})_L\right).$$

Thus, $\sum_{T\in\mathrm{Herm}_r(F)^+} \ell\left(Z_T(\boldsymbol{\omega}_r(wg_\infty g)\boldsymbol{\phi})_L\right)$ is absolutely convergent and equals $\varphi(wg_\infty g)$. In other words, $wg \in \mathscr{G}$. Claim $(*)$ hence the lemma are proved. □

The candidate ζ in Lemma 4.26 will also be (the limit of) elements of the form $\wp_{\tilde{\pi}}(Z^{\text{ét}}_{T_2}(\phi_2)_L)$ for some $T_2 \in \mathrm{Herm}_r^\circ(F)^+$ and $\phi_2 \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$. Next, we construct some pairs of Schwartz functions in $\mathscr{S}(V_v^r)$ for every $v \in \mathtt{V}_F^{\mathrm{fin}}$ that will be candidates in Lemma 4.26.

Notation 4.28. For every $v \in \mathtt{V}_F^{(p)}$, we denote by $\varepsilon_v \in \mathbb{N}^{\mathtt{P}_v}$ the element that takes value 1 on $\mathtt{P}_v \cap \mathtt{P}_{\mathrm{CM}}$ (§2.1(F2)) and value 0 on $\mathtt{P}_v \setminus \mathtt{P}_{\mathrm{CM}}$. Put $\varepsilon := (\varepsilon_v)_v \in \mathbb{N}^{\mathtt{P}}$.

For $v \in \mathtt{R}$, define

$$\mathscr{R}_v := \left\{(\phi_{v,1}, \phi_{v,2}) \in \mathscr{S}(V_v^r, \mathbb{Z}_{(p)})^2 \,\middle|\, \text{the support of } \phi_{v,1}\otimes\phi_{v,2} \text{ is contained in } (V_v^{2r})_{\mathrm{reg}} \text{ (§4.1(H3))}\right\} \tag{4.8}$$

which is stable under the action of $M_r(F_v) \times M_r(F_v)$.

- (S1) For $v \in \mathtt{R}$, choose an arbitrary pair $(\phi_{v,1}, \phi_{v,2}) \in \mathscr{R}_v$.
- (S2) For $v \in \mathtt{T}$, choose a good lattice Λ_v of V_v and put $\phi_{v,1} = \phi_{v,2} := \mathbf{1}_{\Lambda_v^r} \in \mathscr{S}(V_v^r, \mathbb{Z})$.
- (S3) For $v \in \mathtt{V}_F^{(p)}$, choose a good lattice Λ_v of V_v and a polarization $\Lambda_v = \Lambda_{v,1} \oplus \mathfrak{p}_v^{-d_v}\Lambda_{v,2}$ of free O_{E_v}-modules, namely, $\Lambda_{v,1}$ and $\Lambda_{v,2}$ are free isotropic O_{E_v}-submodules of Λ_v of rank r. For $e \in \mathbb{N}^{\mathtt{P}_v}$, put

$$\Lambda_{v,1}^{[e]} := \left\{x \in \left(\varpi_v^{-e-\varepsilon_v}\cdot\Lambda_{v,1} \oplus \varpi_v^{-e+\varepsilon_v^{\mathtt{c}}}\cdot\Lambda_{v,2}\right)^r \,\middle|\, T(x) \in \mathrm{Herm}_r(O_{F_v}) \text{ and } x \bmod \Lambda_{v,2}\otimes\mathbb{Q} \text{ generates } \varpi_v^{-e-\varepsilon_v}\cdot\Lambda_{v,1}\right\},$$
$$\Lambda_{v,2}^{[e]} := \left\{x \in (\varpi_v^{-e}\cdot\Lambda_{v,1} \oplus \varpi_v^{-e}\cdot\Lambda_{v,2})^r \,\middle|\, T(x) \in \mathrm{Herm}_r(O_{F_v}) \text{ and } x \bmod \Lambda_{v,1}\otimes\mathbb{Q} \text{ generates } \varpi_v^{-e}\cdot\Lambda_{v,2}\right\}.$$

  For $i = 1, 2$, let $\phi_{v,i}^{[e]} \in \mathscr{S}(V_v^r, \mathbb{Z})$ be the characteristic function of $\Lambda_{v,i}^{[e]}$.

For $i = 1, 2$ and $e \in \mathbb{N}^{\mathtt{P}}$, we put

$$\phi_i^{[e]} := \left(\bigotimes_{v\in\mathtt{V}_F^{(p)}} \phi_{v,i}^{[e_v]}\right) \otimes \left(\bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} \phi_{v,i}\right) \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty, \mathbb{Z}_{(p)}).$$

At last, we choose an open compact subgroup $L_v \subseteq H(F_v)$ for every $v \in \mathtt{V}_F^{\mathrm{fin}}$.

- For $v \in \mathtt{R}$, we choose some L_v that fixes $\phi_{v,i}$ for $i = 1, 2$.
- For $v \in \mathtt{T}$, we define L_v to be the stabilizer of Λ_v.
- For $v \in \mathtt{V}_F^{(p)}$, we define L_v to be the stabilizer of the lattice chain $\Lambda_{v,1} \oplus \mathfrak{p}_v\Lambda_{v,2} \subseteq \Lambda_{v,1} \oplus \Lambda_{v,2}$.

Put $L := \prod_v L_v \subseteq H(\mathbb{A}_F^\infty)$ so that $L^\diamond$ is good. We may assume that L is neat by shrinking L_v for $v \in \mathtt{R}(\neq \emptyset)$.

Lemma 4.29. *Take an element $v \in \mathtt{V}_F^{(p)}$.*

1. *For $i = 1, 2$ and $e, e' \in \mathbb{N}^{\mathtt{P}_v}$, we have $\mathrm{U}_v^{e'}\phi_{v,i}^{[e]} = \phi_{v,i}^{[e+e']}$.*
2. *For $i = 1, 2$ and $e \in \mathbb{N}^{\mathtt{P}_v}$, $\phi_{v,i}^{[e]}$ is fixed by L_v.*
3. *For every $(e_1, e_2) \in \mathbb{N}^{\mathtt{P}_v} \times \mathbb{N}^{\mathtt{P}_v}$, the support of $\phi_{v,1}^{[e_1]} \otimes \phi_{v,2}^{[e_2]}$ is contained in $(V_v^{2r})_{\mathrm{reg}}$ (§4.1(H3)); and we have*

$$f^{\mathrm{SW}}_{\phi_{v,1}^{[e_1]}\otimes\phi_{v,2}^{[e_2]}} = b_{2r,v}(\mathbf{1})^{-1}\,\mathrm{vol}(L_v, \mathrm{d}h_v)\cdot(\mathtt{f}_{\mathbf{1}_v}^{[e_1^{\mathtt{c}}+\varepsilon_v^{\mathtt{c}}+e_2]})^{\mathbf{1}_v}, \tag{4.9}$$

   where $\mathrm{vol}(L_v, \mathrm{d}h_v)$ *denotes the volume of L_v under the Siegel–Weil measure* $\mathrm{d}h_v$ *in §4.1(H9).*

Proof. For (1), by induction, it suffices to consider the case where $e' = 1_u$ for some $u \in \mathtt{P}_v$. We will prove the case where $i = 1$ and leave the other similar case to the reader. By definition, we have

$$\begin{aligned}
(\mathrm{U}_v^{1_u}\phi_{v,1}^{[e]})(x) &= \sum_{b\in\mathrm{Herm}_r(O_{F_v}/\varpi_v)} (\omega_{r,v}(n(\varpi_v^{-d_v}b^\sharp)m(\varpi_v^{1_u}))\phi_{v,1}^{[e]})(x) \\
&= (\omega_{r,v}(m(\varpi_v^{1_u}))\phi_{v,1}^{[e]})(x) \sum_{b\in\mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\mathrm{tr}\,\varpi_v^{-d_v}b^\sharp T(x)) \\
&= q_v^{-r^2}\phi_{v,1}^{[e]}(\varpi_v^{1_u}x) \sum_{b\in\mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\mathrm{tr}\,\varpi_v^{-d_v}b^\sharp T(x)).
\end{aligned} \tag{4.10}$$

Since

$$\sum_{b\in\mathrm{Herm}_r(O_{F_v}/\varpi_v)} \psi_{F,v}(\varpi_v^{-d_v}\,\mathrm{tr}\,b^\sharp T(x)) = \begin{cases} q_v^{r^2} & \text{if } T(x)\in\mathrm{Herm}_{2r}(O_{F_v}), \\ 0 & \text{if } T(x)\in\varpi_v^{-1}\mathrm{Herm}_{2r}(O_{F_v})\setminus\mathrm{Herm}_{2r}(O_{F_v}), \end{cases}$$

we have (4.10) $= \phi_{v,1}^{[e+1_u]}(x)$.

For (2), by (1), it suffices to consider the case where $e = 0$, for which the invariance under L_v is obvious.

For (3), it is easy to see that the image of $\Lambda_{v,1}^{[e_1]}\times\Lambda_{v,2}^{[e_2]}$ under the moment map $T\colon V_v^{2r}\to\mathrm{Herm}_{2r}(F_v)$ is contained in the set $\mathfrak{T}_v^{[e_1^{\mathtt{c}}+\varepsilon_v^{\mathtt{c}}+e_2]}$ in Construction 3.8, which is contained in $\mathrm{Herm}_{2r}^\circ(F_v)$. For (4.9), by (1) and Lemma 3.11, it suffices to consider the case where $e_1 = e_2 = 0$. In the definition of $\Lambda_{v,i}^{[0]}$, the condition that $T(x)\in\mathrm{Herm}_{2r}(O_{F_v})$ is automatic. Then it is a straightforward exercise in linear algebra that the image of $\Lambda_{v,1}^{[0]}\times\Lambda_{v,2}^{[0]}$ under the moment map T is exactly $\mathfrak{T}_v^{[\varepsilon_v^{\mathtt{c}}]}$; and that for every $x\in\Lambda_{v,1}^{[0]}\times\Lambda_{v,2}^{[0]}$, an element $h_v\in H(F_v)$ keeps x in $\Lambda_{v,1}^{[0]}\times\Lambda_{v,2}^{[0]}$ if and only if $h_v\in L_v$. It follows from §4.1(H9) that

$$W_{T^\square}(f^{\mathrm{SW}}_{\phi_{v,1}^{[0]}\otimes\phi_{v,2}^{[0]}}) = b_{2r,v}(\mathbf{1})^{-1}\,\mathrm{vol}(L_v,\,\mathrm{d}h_v)\cdot\mathbf{1}_{\mathfrak{T}_v^{[\varepsilon_v^{\mathtt{c}}]}}(T^\square)$$

for every $T^\square\in\mathrm{Herm}_{2r}^\circ(F_v)$, which implies (4.9) (when $e_1 = e_2 = 0$).

The lemma is proved. □

Recall the ideals $\mathbb{S}_{\pi,L}^\diamond$ and $\mathbb{S}_{\hat\pi,L}^\diamond$ of $\mathbb{S}_{\mathbb{L}}^\diamond$ introduced in front of Lemma 4.8. For $(T_1,T_2)\in\mathrm{Herm}_r^\circ(F)^+\times\mathrm{Herm}_r^\circ(F)^+$, $(\mathrm{s}_1,\mathrm{s}_2)\in\mathbb{S}_{\pi,L}^\diamond\times\mathbb{S}_{\hat\pi,L}^\diamond$ and $(e_1,e_2)\in\mathbb{N}^{\mathtt{P}}\times\mathbb{N}^{\mathtt{P}}$, we have $Z_{T_1}^{\text{ét}}(\mathrm{s}_1\phi_1^{[e_1]})_L = \mathrm{s}_1^*Z_{T_1}^{\text{ét}}(\phi_1^{[e_1]})_L$ and $Z_{T_2}^{\text{ét}}(\mathrm{s}_2\phi_2^{[e_2]})_L = \mathrm{s}_2^*Z_{T_2}^{\text{ét}}(\phi_2^{[e_2]})_L$ by [LL21, Lemma 4.4]. In particular,

$$Z_{T_1}^{\text{ét}}(\mathrm{s}_1\phi_1^{[e_1]})_L = \wp_\pi\left(\mathrm{s}_1^*Z_{T_1}^{\text{ét}}(\phi_1^{[e_1]})\right)\in\mathrm{H}^1_f(E,\mathrm{V}_{\pi,L}),\quad Z_{T_2}^{\text{ét}}(\mathrm{s}_2\phi_2^{[e_2]})_L = \wp_{\hat\pi}\left(\mathrm{s}_2^*Z_{T_2}^{\text{ét}}(\phi_2^{[e_2]})_L\right)\in\mathrm{H}^1_f(E,\mathrm{V}_{\hat\pi,L})$$

by Lemma 4.24. By [LL21, Lemma 6.4] (in which we may take $\mathtt{R}'$ as $\mathtt{R}\cup\mathtt{V}_F^{(p)}$ by Lemma 4.29(3)), the algebraic cycles $Z_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L$ and $Z_{T_2}(\mathrm{s}_2\phi_2^{[e_2]})_L$ do not intersect. Therefore, by the discussion in §A.7, we have a decomposition formula

$$\langle Z_{T_1}^{\text{ét}}(\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}^{\text{ét}}(\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat\pi,L}),E} = \sum_{u\nmid\infty}\langle Z_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat\pi,L}),E_u}\in\Gamma_{E,p}\otimes_{\mathbb{Z}_p}\mathbb{L} \tag{4.11}$$

for our p-adic height pairing. In what follows, to shorten notation, we will suppress the part $(\mathrm{V}_{\pi,L},\mathrm{V}_{\hat\pi,L})$ in the subscript of height pairings.

Notation 4.30. For a finite place u (resp. v) of E (resp. F) not above p, we denote by $[u]$ (resp. $[v]$) the image of an arbitrary uniformizer at u (resp. v) in $\Gamma_{E,p}$ (resp. $\Gamma_{F,p}$).

4.6. Local height away from p.

In this subsection and the next one, we study the local summands in (4.11).

Lemma 4.31. *For every $v\in\mathtt{T}$ and every $T^\square\in\mathrm{Herm}_{2r}^\circ(F_v)$, there exists a unique element $\mathsf{W}_{T^\square,v}\in\mathbb{Z}[X]$ such that*

$$\mathsf{W}_{T^\square,v}(\chi_v(\varpi_v)) = b_{2r,v}(\chi)\cdot W_{T^\square}(f_{\chi_v})$$

holds for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$, where $f_{\chi_v}\in\mathrm{I}_{r,v}^\square(\chi_v)$ is the unique section that satisfies $f_{\chi_v}|_{K_{2r,v}} = f^{\mathrm{SW}}_{\mathbf{1}_{\Lambda_v^{2r}}}|_{K_{2r,v}}$ and ϖ_v is an arbitrary uniformizer of F_v.

Proof. When $v \in \mathtt{V}_F^{\mathrm{ram}}$, this follows from [LL22, Remark 2.18 & Lemma 2.19]. When $v \in \mathsf{S}_\pi$, this follows from the discussion in [LZ22, Section 9]. The remaining cases have been settled in Lemma 3.5(1) as in these cases $f_{\chi_v} = f_{\chi_v}^{\mathrm{sph}}$ (Notation 3.4(2)). □

Notation 4.32. For every $T^\square \in \mathrm{Herm}_{2r}^\circ(F)^+$, put $\mathrm{Diff}(T^\square, V) := \{v \in \mathtt{V}_F^{\mathrm{fin}} \mid (V_v^{2r})_{T^\square} = \emptyset\}$, which is a finite subset of $\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{V}_F^{\mathrm{spl}}$ of odd cardinality. We define $\mathrm{Herm}_{2r}^\circ(F)_V^+$ to be the subset of $\mathrm{Herm}_{2r}^\circ(F)^+$ consisting of $T^\square$ such that $\mathrm{Diff}(T^\square, V)$ is a singleton, whose unique element we denote by $v_{T^\square}$.

Proposition 4.33. *There exists a pair* $(\mathrm{t}_1, \mathrm{t}_2) \in \mathbb{S}_{\mathbb{L}}^\diamond \times \mathbb{S}_{\mathbb{L}}^\diamond$ *satisfying* $\chi_{\hat\pi}^\diamond(\mathrm{t}_1)\chi_\pi^\diamond(\mathrm{t}_2) \neq 0$, *such that for every* $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$, *every* $(\mathrm{s}_1, \mathrm{s}_2) \in \mathbb{S}_{\pi,L}^\diamond \times \mathbb{S}_{\hat\pi,L}^\diamond$ *and every* $(e_1, e_2) \in \mathbb{N}^{\mathtt{P}} \times \mathbb{N}^{\mathtt{P}}$, *we have*

$$\begin{aligned}
&\mathrm{Nm}_{E/F}\left(\mathrm{vol}^\natural(L) \sum_{u \nmid \infty p} \langle Z_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u}\right) \\
&= W_{2r}\left(\sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)_V^+ \\ \partial_{r,r}T^\square = (T_1,T_2)}} \mathsf{W}'_{T^\square, v_{T^\square}}(1) \cdot I_{T^\square}((\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]} \otimes \mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})^{v_{T^\square}}) \cdot [v_{T^\square}]\right) \\
&\quad + W_{2r}\sum_{v \in \mathsf{S}_\pi} \frac{2}{q_v^{2r} - 1}\left(\sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)^+ \\ \partial_{r,r}T^\square = (T_1,T_2)}} \mathsf{W}^{\mathrm{sph}}_{T^\square, v}(1) \cdot I_{T^\square}((\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]} \otimes \mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})^{v})\right) \cdot [v]
\end{aligned}$$

in $\Gamma_{F,p} \otimes_{\mathbb{Z}_p} \mathbb{L}$, *where* W_{2r} *is the rational constant in Lemma 3.2,* $\mathsf{W}^{\mathrm{sph}}_{T^\square, v} \in \mathbb{Z}[X]$ *is the polynomial in Lemma 3.5(1), and* $I_{T^\square}$ *is (the product of) the functional in §4.1(H9).*

Proof. We first note that by Proposition A.7, the local p-adic height at $u \nmid \infty p$ coincides with Beilinson's local index. To compute the local indices at different u, we have four cases:

Suppose that u lies over $\mathtt{V}_F^{\mathrm{spl}}$. By [LL22, Proposition 4.20] in which we may take $\mathtt{R}'$ to be $\mathtt{R} \cup \mathtt{V}_F^{(p)}$ which has cardinality at least 2 (see also Remark 4.9), we can find a pair $(\mathrm{t}_1^u, \mathrm{t}_2^u) \in \mathbb{S}_{\mathbb{L}}^\diamond \times \mathbb{S}_{\mathbb{L}}^\diamond$ satisfying $\chi_{\hat\pi}^\diamond(\mathrm{t}_1^u)\chi_\pi^\diamond(\mathrm{t}_2^u) \neq 0$ such that $\langle Z_{T_1}(\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u} = 0$. Moreover, we may take $\mathrm{t}_1^u = \mathrm{t}_1^u = 1$ for all but finitely many u.

Suppose that u lies over an element $v \in \mathtt{V}_F^{\mathrm{unr}} \setminus \mathsf{S}_\pi$. By [LL21, Proposition 8.1] and Remark A.6, we have

$$\begin{aligned}
&\mathrm{vol}^\natural(L) \cdot \langle Z_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u} \\
&= -W_{2r}\left(\sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)_V^+ \\ \partial_{r,r}T^\square = (T_1,T_2) \\ v_{T^\square} = v}} \frac{b_{2r,v}(\mathbf{1})}{\log q_v^2} \cdot W'_{T^\square}(0, 1_{4r}, \mathbf{1}_{\Lambda_v^{2r}}) \cdot I_{T^\square}((\mathrm{s}_1\phi_1^{[e_1]} \otimes \mathrm{s}_2\phi_2^{[e_2]})^{v})\right) \cdot [u],
\end{aligned}$$

where $W_{T^\square}(s, 1_{4r}, \mathbf{1}_{\Lambda_v^{2r}})$ denotes the usual Siegel–Whittaker function with complex variable s (see [LL21, (3.3)] for example). In our case, the character χ_v plays the role as $|\ |_{F_v}^s$, which implies that

$$\mathsf{W}_{T^\square, v}(q_v^{-s}) = \prod_{i=1}^{n} L(s + i, \eta_{E/F,v}^{n-i}) \cdot W_{T^\square}(s, 1_{4r}, \mathbf{1}_{\Lambda_v^{2r}}). \tag{4.12}$$

Together with the relation $\mathrm{Nm}_{E/F}[u] = 2[v]$, we obtain

(4.13)

$$\mathrm{Nm}_{E/F}\left(\mathrm{vol}^\natural(L) \cdot \langle Z_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u}\right) = W_{2r}\left(\sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)_V^+ \\ \partial_{r,r}T^\square = (T_1,T_2) \\ v_{T^\square} = v}} \mathsf{W}'_{T^\square}(1) \cdot I_{T^\square}((\mathrm{s}_1\phi_1^{[e_1]} \otimes \mathrm{s}_2\phi_2^{[e_2]})^{v})\right) \cdot [v].$$

Suppose that u lies over an element $v \in \mathtt{V}_F^{\mathrm{ram}}$. By [LL22, Proposition 4.28] and Remark A.6, we have

$$\begin{aligned}&\mathrm{vol}^\natural(L) \cdot \langle Z_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u} \\ &= -W_{2r}\left(\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)_V^+\\ \partial_{r,r}T^\square=(T_1,T_2)\\ v_{T^\square}=v}} \frac{b_{2r,v}(\mathbf{1})}{\log q_v}\cdot W'_{T^\square}(0,1_{4r},\mathbf{1}_{\Lambda_v^{2r}})\cdot I_{T^\square}((\mathrm{s}_1\phi_1^{[e_1]}\otimes \mathrm{s}_2\phi_2^{[e_2]})^v)\right)\cdot[u].\end{aligned}$$

Now we have (4.12) again but $\mathrm{Nm}_{E/F}[u]=[v]$, which imply (4.13) as well.

Suppose that u lies over an element $v \in \mathtt{S}_\pi$. By [LL21, Proposition 9.1] (see also Remark 4.9) and Remark A.6, we can find a pair $(\mathrm{t}_1^u,\mathrm{t}_2^u)\in \mathbb{S}_{\mathbb{L}}^\diamond\times\mathbb{S}_{\mathbb{L}}^\diamond$ satisfying $\chi_{\hat\pi}^\diamond(\mathrm{t}_1^u)\chi_\pi^\diamond(\mathrm{t}_2^u)\neq 0$ such that

$$\begin{aligned}&\mathrm{vol}^\natural(L) \cdot \langle Z_{T_1}(\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u} \\ &= -W_{2r}\left(\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)_V^+\\ \partial_{r,r}T^\square=(T_1,T_2)\\ v_{T^\square}=v}} \frac{b_{2r,v}(\mathbf{1})}{\log q_v^2}\cdot W'_{T^\square}(0,1_{4r},\mathbf{1}_{\Lambda_v^{2r}})\cdot I_{T^\square}((\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]}\otimes \mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})^v)\right)\cdot[u] \\ &\quad + W_{2r}\frac{1}{q_v^{2r}-1}\left(\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}} \mathsf{W}^{\mathrm{sph}}_{T^\square,v}(1)\cdot I_{T^\square}((\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]}\otimes \mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})^v)\right)\cdot[u].\end{aligned}$$

Now we have (4.12) and $\mathrm{Nm}_{E/F}[u]=2[v]$, which imply

$$\begin{aligned}&\mathrm{Nm}_{E/F}\left(\mathrm{vol}^\natural(L) \cdot \langle Z_{T_1}(\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u}\right) \\ &= W_{2r}\left(\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)_V^+\\ \partial_{r,r}T^\square=(T_1,T_2)\\ v_{T^\square}=v}} \mathsf{W}'_{T^\square,v}(1)\cdot I_{T^\square}((\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]}\otimes \mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})^v)\right)\cdot[v] \\ &\quad + W_{2r}\frac{2}{q_v^{2r}-1}\left(\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}} \mathsf{W}^{\mathrm{sph}}_{T^\square,v}(1)\cdot I_{T^\square}((\mathrm{t}_1^u\mathrm{s}_1\phi_1^{[e_1]}\otimes \mathrm{t}_2^u\mathrm{s}_2\phi_2^{[e_2]})^v)\right)\cdot[v].\end{aligned}$$

Finally, for $i=1,2$, we take $\mathrm{t}_i=\prod_u \mathrm{t}_i^u$ to be the (finite) product of the above auxiliary Hecke operators. The proposition follows by taking the sum over all $u\nmid\infty p$, which is a finite sum. □

4.7. **Local height above p.**

Take an element $u\in\mathtt{P}$ with $v\in\mathtt{V}_F^{(p)}$ its underlying place. For technical purposes, we fix an E-linear isomorphism $\overline{E_u}\xrightarrow{\sim}\mathbb{C}$.

Lemma 4.34. *Suppose that $n<p$. There exists a pair* $(\mathrm{t}_1,\mathrm{t}_2)\in\mathbb{S}_{\mathbb{L}}^\diamond\times\mathbb{S}_{\mathbb{L}}^\diamond$ *satisfying* $\chi_{\hat\pi}^\diamond(\mathrm{t}_1)\chi_\pi^\diamond(\mathrm{t}_2)\neq0$*, such that for every* $(T_1,T_2)\in\mathrm{Herm}_r^\circ(F)^+\times\mathrm{Herm}_r^\circ(F)^+$*, every* $(\mathrm{s}_1,\mathrm{s}_2)\in\mathbb{S}_{\pi,L}^\diamond\times\mathbb{S}_{\hat\pi,L}^\diamond$ *and every* $(e_1,e_2)\in\mathbb{N}^{\mathtt{P}}\times\mathbb{N}^{\mathtt{P}}$*, we have*

$$\langle Z_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u}\in(O_{E_u}^\times)^{\mathrm{fr}}\otimes_{\mathbb{Z}_p}\mathbb{L}.$$

Proof. In view of Remark A.15, we would like to apply Theorem A.8, for which we need an integral model of X_L over O_{E_u}. For this, we need an auxiliary Shimura variety that admits such a model via moduli interpretation. Choose a CM type $\mathbf{\Phi}$ of E such that the p-adic places of E induced by $\mathbf{\Phi}$ via the fixed isomorphism $\overline{E_u}\xrightarrow{\sim}\mathbb{C}$ form a subset $\mathtt{P}_{\mathbf{\Phi}}$ of $\mathtt{P}$ of cardinality $[F:\mathbb{Q}]$ that contains u. Then the reflex field $E_{\mathbf{\Phi}}\subseteq\mathbb{C}$ of $\mathbf{\Phi}$ is contained in E_u. Recall that we have the $\mathbb{Q}$-torus T from §2.2 and fix a neat open compact subgroup K_T of $T(\mathbb{A}^\infty)$ that is maximal at primes not in $\Diamond \setminus \{p\}$. We have the Shimura variety Y_{K_T} of T with respect to the CM type $\mathbf{\Phi}$ at level K_T, which is finite étale over Spec $E_{\mathbf{\Phi}}$. Put $X := (X_L \otimes_E E_u) \otimes_{E_{\mathbf{\Phi}}} Y_{K_T}$, which is a finite étale cover of $X_L \otimes_E E_u$ and hence a smooth projective scheme over E_u of pure dimension $n-1$. The ring $\mathbb{S}^{\Diamond}$ extends naturally to a ring of finite étale correspondences (see §A.1) of X. For every $x \in V^r \otimes_F \mathbb{A}_F^{\infty}$, we denote by $Z(x)$ the pullback of $Z(x)_L$ to X.

Now for the lemma, it suffices to find elements $(\mathrm{t}_1, \mathrm{t}_2) \in \mathbb{S}^{\Diamond}_{\mathbb{L}} \times \mathbb{S}^{\Diamond}_{\mathbb{L}}$ satisfying $\chi^{\Diamond}_{\tilde{\pi}}(\mathrm{t}_1)\chi^{\Diamond}_{\pi}(\mathrm{t}_2) \neq 0$, such that for every $x_1, x_2 \in V^r \otimes_F \mathbb{A}_F^{\infty}$ satisfying

$$T(x_i) \in \mathrm{Herm}_r^{\circ}(F)^+, \quad x_{i,v} \in \bigcup_{e \in \mathbb{N}^{\mathtt{P}_v}} \Lambda_{v,i}^{[e]}, \quad i = 1, 2, \tag{4.14}$$

we have $\langle \mathrm{t}_1^* Z(x_1), \mathrm{t}_2^* Z(x_2) \rangle_{X, E_u} \in O_{E_u}^{\times} \otimes_{\mathbb{Z}_p} \mathbb{L}$.

Put $K := E_u$ with the residue field κ. The K-scheme X admits an integral model $\mathcal{X}$ over O_K such that for every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{X}(S)$ is the set of equivalence classes (given by p-principal isogenies) of tuples $(A_0, \lambda_0, \eta_0; A, \lambda, \eta; G_{u^{\mathtt{c}}} \to G_{A,u^{\mathtt{c}}})$ where

- A_0 is an abelian scheme over S with an action of O_E of signature type $\mathbf{\Phi}$, together with a compatible p-principal polarization λ_0 and a level structure η_0 away from p,
- A is an abelian scheme over S with an action of O_E of signature type $n\mathbf{\Phi} - \mathrm{inc} + \mathrm{inc}^{\mathtt{c}}$ (inc being the inclusion $E \hookrightarrow \mathbb{C}$), together with a compatible p-principal polarization λ, so that $G_{A,u^{\mathtt{c}}} := A[(u^{\mathtt{c}})^{\infty}]$ is an O_{F_v}-divisible module of dimension 1 and relative height n,
- η is an L^v-level structure for the hermitian space $\mathrm{Hom}_{O_E}(A_0, A) \otimes_F \mathbb{A}_F^{\infty v}$,
- $G_{u^{\mathtt{c}}} \to G_{A,u^{\mathtt{c}}}$ is an isogeny of O_{F_v}-divisible modules over S whose kernel is contained in $G_{u^{\mathtt{c}}}[\varpi_v]$ and has degree q_v^r.

The reader may consult [LL21, Section 7] for more details about the first three items, which are not so related to our argument below. By the same argument for [TY07, Proposition 3.4], we know that $\mathcal{X}$ is a projective strictly semistable scheme over O_K to which finite étale correspondences in $\mathbb{S}^{\Diamond}_{\mathbb{L}}$ naturally extend. Moreover, if we put $\mathsf{X} := \mathcal{X} \otimes_{O_K} \kappa$ and let X_1 (resp. X_2) be the closed locus of X on which the kernel of $G_{u^{\mathtt{c}}} \to G_{A,u^{\mathtt{c}}}$ (resp. $G_{A,u^{\mathtt{c}}} \to G_{u^{\mathtt{c}}}/G_{u^{\mathtt{c}}}[\varpi_v]$) is not étale, then under the notation of §A.5,

$$\mathsf{X}^{(1)} = \mathsf{X}_1 \coprod \mathsf{X}_2, \quad \mathsf{X}^{(2)} = \mathsf{X}_1 \bigcap \mathsf{X}_2, \quad \mathsf{X}^{(3)} = \mathsf{X}^{(4)} = \cdots = \emptyset.$$

We then would like to apply Theorem A.8 with $\mathbb{T} = \mathbb{S}^{\Diamond}_{\mathbb{L}}$, $\mathfrak{m} = \mathrm{Ker}\,\chi^{\Diamond}_{\tilde{\pi}}$ and $\mathfrak{m}' = \mathrm{Ker}\,\chi^{\Diamond}_{\pi}$. To check (A.3), we realize that both $\chi^{\Diamond}_{\tilde{\pi}}$ and $\chi^{\Diamond}_{\pi}$ can be defined over a number field $\mathbb{E}$ contained in $\mathbb{L}$. Thus, by [KM74, Theorem 2], it suffices to show that

$$\bigoplus_{q \geqslant 0} \mathrm{H}^q_{\text{ét}}(\mathsf{X}^{(2)} \otimes_{\kappa} \overline{\kappa}, \mathbb{E}_{\ell})_{\mathfrak{m}} = \bigoplus_{q \geqslant 0} \mathrm{H}^q_{\text{ét}}(\mathsf{X}^{(2)} \otimes_{\kappa} \overline{\kappa}, \mathbb{E}_{\ell})_{\mathfrak{m}'} = 0 \tag{4.15}$$

where ℓ is an arbitrary prime of $\mathbb{E}$ not above p. Indeed, there is a finite flat morphism $\mathcal{X}_1 \to \mathcal{X}$ to which finite étale correspondences in $\mathbb{S}^{\Diamond}_{\mathbb{L}}$ naturally extend, in which $\mathcal{X}_1$ is the integral model with a Drinfeld level-1 structure at v as the one used in [LL21, Section 7]. Then (4.15) follows from claim (2) in the proof of [LL21, Lemma 7.3] with $m = j = 1$.

Denote by $\mathcal{Z}(x)$ the Zariski closure of $Z(x)$ in $\mathcal{X}$. By Theorem A.8 and Remark A.15, it suffices to show the following two claims for the lemma.

1. For every $x_1, x_2 \in V^r \otimes_F \mathbb{A}_F^{\infty}$ satisfying (4.14) and every $\mathrm{t}_1, \mathrm{t}_2 \in \mathbb{S}^{\Diamond}_{\mathbb{L}}$, we have $\mathrm{t}_1^* \mathcal{Z}(x_1) \cap \mathrm{t}_2^* \mathcal{Z}(x_2) = \emptyset$.
2. For every $x \in V^r \otimes_F \mathbb{A}_F^{\infty}$ with $T(x) \in \mathrm{Herm}_r^{\circ}(F)^+$, the dimension of $\mathcal{Z}(x) \cap \mathsf{X}^{(h)}$ is at most $r - h$ for $h = 1, 2$.

Part (1) follows from (the same argument for) [LL21, Lemma 7.2].

For (2), since $\mathcal{Z}(x)$ remains the same if we scale x by an element in $F^{\times}$, we may assume that $x_v \in (\Lambda_{v,1} \oplus \Lambda_{v,2})^r$ for every $v \in \mathtt{V}_F^{(p)}$. Up to a Hecke translation away from p, which does not affect the conclusion of (2), we may also assume that $x \in V^r$. We have a moduli scheme $\mathcal{Y}(x)$ finite over $\mathcal{X}$, such that for every object $S = (A_0, \lambda_0, \eta_0; A, \lambda, \eta; G_{u^{\mathtt{c}}} \to G_{A,u^{\mathtt{c}}})$ of $\mathcal{X}$, the set $\mathcal{Y}(x)(S)$ consists of elements $y \in \mathrm{Hom}_{O_E}(A_0^{\oplus r}, A) \otimes \mathbb{Z}_{(p)}$ satisfying $T(y) = T(x)$ and $y^p \in \eta(L^p x)$. By [LL21, Lemma 5.4], $\mathcal{Z}(x)$ is contained in the image of $\mathcal{Y}(x)$ in $\mathcal{X}$. Thus, it suffices to show that the dimension of $\mathsf{Y}(x)^{(h)}$ is at most $r - h$ for $h = 1, 2$, where $\mathsf{Y}(x)^{(h)} := \mathcal{Y}(x) \times_{\mathcal{X}} \mathsf{X}^{(h)}$.

Let $V_x \subseteq V$ be the hermitian subspace (of dimension r) that is the orthogonal complement of the subspace spanned by x. Put $H_x := \mathrm{U}(V_x)$ which is naturally a subgroup of H, and put $L_x := L \cap H_x(\mathbb{A}_F^{\infty})$. We have a similar moduli scheme $\mathcal{X}_x$ over O_K for V_x similar to the one for V but with the hyperspecial level structure at p. More precisely, for every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{X}_x(S)$ is the set of equivalence classes (given by p-principal isogenies) of tuples $(A_0, \lambda_0, \eta_0; A_1, \lambda_1, \eta_1)$ where

- (A_0, λ_0, η_0) is like the one in the definition of $\mathcal{X}$,
- A_1 is an abelian scheme over S with an action of O_E of signature type $r\mathbf{\Phi} - \mathrm{inc} + \mathrm{inc}^{\mathtt{c}}$, together with a compatible p-principal polarization λ_1,
- η_1 is an L^p_x-level structure for the hermitian space $\mathrm{Hom}_{O_E}(A_0, A_1) \otimes_F \mathbb{A}_F^{\infty p}$.

In particular, $\mathcal{X}_x$ is a projective smooth scheme over O_K of pure relative dimension $r-1$. Put $\mathsf{X}_x := \mathcal{X}_x \otimes_{O_K} \kappa$; and for $h \geqslant 1$, denote by $\mathsf{X}_x^{[h]}$ the closed locus of X_x where the height of the connected part of $G_{A_1,u^{\mathtt{c}}} := A_1[(u^{\mathtt{c}})^\infty]$ is at least h. It is known that $\mathsf{X}_x^{[h]}$ has pure dimension $r-h$. Claim (2) will follow if there is a finite morphism $f\colon \mathcal{Y}(x) \to \mathcal{X}_x$ that sends $\mathsf{Y}(x)^{(h)}$ into $\mathsf{X}_x^{[h]}$ for $h = 1, 2$, which we now construct.

Take a point $P = (A_0, \lambda_0, \eta_0; A, \lambda, \eta; G_{u^{\mathtt{c}}} \to G_{A,u^{\mathtt{c}}}; y)$ of $\mathcal{Y}(x)(S)$. Put $A' := (A^\vee/(\lambda \circ y)A_0^{\oplus r})^\vee$, which inherits an action of O_E which has signature type $r\mathbf{\Phi} - \mathrm{inc} + \mathrm{inc}^{\mathtt{c}}$ and admits a natural map to A. Since $T(x) \in \mathrm{Herm}_r^\circ(F)$, the induced map $\lambda'\colon A' \to A \xrightarrow{\lambda} A^\vee \to A'^\vee$ is a quasi-polarization such that $\lambda'[p^\infty]$ is an isogeny. For every $\tilde{u} \in \mathsf{P}$, we have the induced isogeny $\lambda'_{\tilde{u}^{\mathtt{c}}}\colon G_{A',\tilde{u}^{\mathtt{c}}} \to G^\vee_{A',\tilde{u}}$. Put

$$A_1 := A' \Big/ \bigoplus_{\tilde{u} \in \mathsf{P}_{\mathbf{\Phi}}} \mathrm{Ker}\, \lambda'_{\tilde{u}^{\mathtt{c}}}$$

and let $\lambda_1\colon A_1 \to A_1^\vee$ be the quasi-polarization induced from λ', which is in fact p-principal from the construction. We can also define a natural L^p_x-level structure η_1 for A_1 whose details we leave to the reader. Then we define $f(P)$ to be $(A_0, \lambda_0, \eta_0; A_1, \lambda_1, \eta_1)$. Since the O_{F_v}-divisible module $G_{A_0,u^{\mathtt{c}}}$ is étale, the height of the connected part of $G_{A_1,u^{\mathtt{c}}}$ equals to that of $G_{A,u^{\mathtt{c}}}$. In particular, f sends $\mathsf{Y}(x)^{(h)}$ into $\mathsf{X}_x^{[h]}$ for $h = 1, 2$. It remains to show that f is finite. Since $\mathcal{Y}(x)$ is proper over O_K, it suffices to show that the fiber of f over an arbitrary $\overline{\kappa}$-point is finite. Indeed, when $S = \mathrm{Spec}\, \overline{\kappa}$, $G_{A',\tilde{u}^{\mathtt{c}}}$ has dimension 1 (resp. is étale) if $\tilde{u} = u$ (resp. $\tilde{u} \in \mathsf{P}_{\mathbf{\Phi}} \setminus \{u\}$), and the degree of $\lambda'_{\tilde{u}^{\mathtt{c}}}$ is bounded by the moment matrix $T(x)$. It follows that up to isomorphism, there are only finitely many such isogenies $G_{A',\tilde{u}^{\mathtt{c}}} \to G_{A_1,\tilde{u}}$ with fixed $G_{A_1,\tilde{u}}$ for every $\tilde{u} \in \mathsf{P}_{\mathbf{\Phi}}$. Thus, f is finite and claim (2) is confirmed.

The lemma is finally proved. □

Proposition 4.35. *Suppose that $n < p$. There exist an integer $M \geqslant 0$ and a pair $(\mathrm{t}_1, \mathrm{t}_2) \in \mathbb{S}^\diamond_{O_{\mathbb{L}}} \times \mathbb{S}^\diamond_{O_{\mathbb{L}}}$ satisfying $\chi^\diamond_{\hat{\pi}}(\mathrm{t}_1)\chi^\diamond_{\pi}(\mathrm{t}_2) \neq 0$, such that for every $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$, every $(\mathrm{s}_1, \mathrm{s}_2) \in (\mathbb{S}^\diamond_{\pi,L} \cap \mathbb{S}^\diamond_{O_{\mathbb{L}}}) \times (\mathbb{S}^\diamond_{\hat{\pi},L} \cap \mathbb{S}^\diamond_{O_{\mathbb{L}}})$ and every $(e_1, e_2) \in \mathbb{N}^{\mathsf{P}} \times \mathbb{N}^{\mathsf{P}}$, we have*

$$\langle Z_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L, Z_{T_2}(\mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_{E_u} \in (O_{E_u}^\times)^{\mathrm{fr}} \otimes_{\mathbb{Z}_p} (p^{|e_{2,v}|-M} O_{\mathbb{L}}).$$

The rest of this subsection is devoted to the proof of this proposition. We may assume that $\mathrm{V}_{\pi,L} \neq 0$ and $\mathrm{V}_{\hat{\pi},L} \neq 0$ since otherwise the proposition is trivial.

Let S be the kernel of the norm map $\mathrm{Nm}_{E/F}\colon \mathrm{Res}_{O_E/O_F}\mathbf{G} \to \mathbf{G}_{O_F}$. Consider the reciprocity map

$$\mathrm{rec}\colon \mathrm{Aut}(\mathbb{C}/E) \to E^\times \backslash \mathbb{A}_E^{\infty,\times} \to S(F)\backslash S(\mathbb{A}_F^\infty) \tag{4.16}$$

in which the first one is from the global class field theory and the second (surjective) one sends a to $a^{\mathtt{c}}/a$. For $d \in \mathbb{N}$, we

- put $L_{S,v}^{[d]} := S(O_{F_v}) \cap (1 + \mathfrak{p}_v^d O_{E_v})$,
- let $E^{[d]} \subseteq \mathbb{C}$ be the finite abelian extension of E such that the map rec (4.16) induces an isomorphism
$$\mathrm{Gal}(E^{[d]}/E) \simeq S(F)\backslash S(\mathbb{A}_F^\infty) \Big/ L_{S,v}^{[d]} \prod_{\tilde{v} \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{v\}} S(O_{F_{\tilde{v}}}),$$
- denote by $\mathrm{Z}^r_{[d]}(X_L)$ the image of the norm map
$$\mathrm{Nm}_{E^{[d]}/E}\colon \mathrm{Z}^r(X_L \otimes_E E^{[d]}) \to \mathrm{Z}^r(X_L).$$

Lemma 4.36. *For every $x \in V^r \otimes_F \mathbb{A}_F^\infty$ satisfying $T(x) \in \mathrm{Herm}_r^\circ(F)^+$ and $x_v \in \Lambda_{v,2}^{[e]}$ for some $e \in \mathbb{N}^{\mathsf{P}_v}$, we have*

$$Z(x)_L \in \mathrm{Z}^r_{[|e|]}(X_L).$$

Proof. Up to a Hecke translation away from p, which does not affect the conclusion of the lemma, we may assume that $x \in V^r$. Let $V_x \subseteq V$ be the hermitian subspace (of dimension r) that is the orthogonal complement of the subspace spanned by x. Put $H_x := \mathrm{U}(V_x)$ which is naturally a subgroup of H, and put $L_x := L \cap H_x(\mathbb{A}_F^\infty)$. We have the Shimura variety X_{x,L_x} for H_x with level L_x, similar to X_L. By definition, $Z(x)_L$ is the fundamental cycle of the finite unramified morphism $X_{x,L_x} \to X_L$ defined over E.

We have the determinant map $\det\colon H_x \to S \otimes_{O_F} F$ which identifies $S \otimes_{O_F} F$ with the maximal abelian quotient of H_x. Then the set of connected components of $X_{x,L_x} \otimes_E \mathbb{C}$ is canonically parameterized by the set $S(F)\backslash S(\mathbb{A}_F^\infty)/\det L_x$. For every $s \in S(F)\backslash S(\mathbb{A}_F^\infty)/\det L_x$, we denote by X_{x,L_x}^s the corresponding connected component. The definition of canonical models of Shimura varieties implies that $\gamma X_{x,L_x}^s = X_{x,L_x}^{\mathrm{rec}(\gamma)s}$ for every $\gamma \in \mathrm{Aut}(\mathbb{C}/E)$, where rec is the map (4.16).

We claim that $\det L_{x,v} \subseteq L_{S,v}^{[|e|]}$. Then we have the quotient map

$$S(F)\backslash S(\mathbb{A}_F^\infty)/\det L_x \to S(F)\backslash S(\mathbb{A}_F^\infty) \Big/ L_{S,v}^{[|e|]} \prod_{\tilde{v} \in \mathtt{V}_F^{\mathrm{fin}} \backslash \{v\}} S(O_{F_{\tilde{v}}})$$

Let $\mathfrak{S}$ be the fiber of 1 in the above map. Then $\sum_{s \in \mathfrak{S}} X_{x,L_x}^s$ is defined over $E^{[|e|]}$; and $\mathrm{Nm}_{E^{[|e|]}/E} \sum_{s \in \mathfrak{S}} X_{x,L_x}^s = X_{x,L_x}$. The lemma then follows.

It remains to show the claim, which is an exercise in linear algebra. We assume $e \neq 0$ as the case for $e = 0$ is trivial. By definition, $L_{x,v}$ is simply the subgroup of L_v that fixes x_v, or equivalently, $x'_v := \varpi_v^e \cdot x_v$. By the definition of $\Lambda_{v,2}^{[e]}$ in §4.5(S3), x'_v belongs to $(\Lambda_{v,1} \oplus \Lambda_{v,2})^r$ such that $T(x'_v) \in \varpi_v^{|e|}\mathrm{Herm}_r(O_{F_v})$ and that $x'_v \bmod \Lambda_{v,1}$ generates $\Lambda_{v,2}$. It follows that the image of x'_v in $(\Lambda_{v,1} \oplus \Lambda_{v,2})^r \otimes_{O_{F_v}} O_{F_v}/\mathfrak{p}_v^{|e|}$ generates a Lagrangian $O_{E_v} \otimes_{O_{F_v}} O_{F_v}/\mathfrak{p}_v^{|e|}$-submodule of $(\Lambda_{v,1} \oplus \Lambda_{v,2}) \otimes_{O_{F_v}} O_{F_v}/\mathfrak{p}_v^{|e|}$. In particular, every element in $L_{x,v}$, which stabilizes $\Lambda_{v,1} \oplus \Lambda_{v,2}$, has determinant 1 modulo $\mathfrak{p}_v^{|e|}$. The claim follows. □

For $d \in \mathbb{N}$, let u_d be the place of $E^{[d]}$ induced from the fixed isomorphism $\overline{E_u} \xrightarrow{\sim} \mathbb{C}$, which is above u. Put $K := E_u$, $K_d := (E^{[d]})_{u_d}$ for $d \in \mathbb{N}$ and $K_\infty := \bigcup_{d \geqslant 0} K_d$. Then K_0/K is unramified and K_d/K_0 is totally ramified of degree $(q_v - 1)q_v^{d-1}/|U_E|$ for $d > 0$, where U_E is the torsion subgroup of $O_E^\times$.

Recall that $\mathrm{V}_{\pi,L}$ and $\mathrm{V}_{\hat{\pi},L}$ are subspaces of $\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, \mathbb{L}(r))$. Put

$$\mathrm{T}_{\pi,L} := \mathrm{V}_{\pi,L} \cap \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, O_{\mathbb{L}}(r))^{\mathrm{fr}}, \quad \mathrm{T}_{\hat{\pi},L} := \mathrm{V}_{\hat{\pi},L} \cap \mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L, O_{\mathbb{L}}(r))^{\mathrm{fr}},$$

both being $O_{\mathbb{L}}[\mathrm{Gal}(\overline{E}/E)]$-modules. For $d \in \mathbb{N}$, we put

$$\mathrm{N}_\infty \mathrm{H}^1_f(K_d, \mathrm{T}_{\pi,L}) := \bigcap_{d' \geqslant d} \mathrm{Im}\left(\mathrm{Cor}_{K_{d'}/K_d}\colon \mathrm{H}^1_f(K_{d'}, \mathrm{T}_{\pi,L}) \to \mathrm{H}^1_f(K_d, \mathrm{T}_{\pi,L})\right),$$

in which $\mathrm{Cor}_{K_{d'}/K_d}$ denotes the corresponding corestriction map.

Lemma 4.37. *There exists an integer $M \geqslant 0$ such that p^M annihilates $\mathrm{H}^1_f(K_d, \mathrm{T}_{\pi,L})/\mathrm{N}_\infty\mathrm{H}^1_f(K_d, \mathrm{T}_{\pi,L})$ for every $d \in \mathbb{N}$.*

Proof. By Lemma 4.15, we know that $\mathrm{V}_{\pi,L}|_{K_d}$ satisfies the Panchishkin condition (Definition A.12) for every $d \in \mathbb{N}$. By Lemma 4.10, we may apply [Nek93, Theorem 6.9] to $\mathrm{V}_{\pi,L}|_{K_d}$.[17] Thus, by the same argument at the end of the proof of [Nek95, Proposition II.5.10], it suffices to show that $\mathrm{H}^0(K_\infty, \mathrm{V}_{\pi,L}) = \mathrm{H}^0(K_\infty, \mathrm{V}_{\hat{\pi},L}) = 0$.

We follow the strategy in [Shn16, Section 8]. We may choose an element $\xi \in S(F)$ such that $\xi = (\varpi_v^{1_u - 1_{u^c}})^{[K_0:K]}$ in $\mathrm{Gal}(E^{[0]}/E)$. Then by the same argument for [Shn16, Proposition 8.3], K_∞ is contained in K_ξ – the field attached to the Lubin–Tate group relative to the extension K_0/K with parameter ξ. Let $\chi_\xi\colon \mathrm{Gal}(K_\xi/K_0) \to O_K^\times$ be the character given by the Galois action on the torsion points of this relative Lubin–Tate group; and let $K(\chi_\xi)$ be the corresponding one-dimensional representation. Let L be the maximal subfield of K that is unramified over $\mathbb{Q}_p$. By the same argument for [Shn16, Proposition 8.4], $K(\chi_\xi)$ is crystalline, and that the q_v-Frobenius map (which is L-linear) acts on $\mathbb{D}_{\mathrm{cris}}(K(\chi_\xi))$, which is a free $K \otimes_{\mathbb{Q}_p} L$-module of rank 1, by multiplication by ξ^{-1}.[18] Note that $\mathbb{L}$

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[17] Though our extension K_∞/K_d is in general not a $\mathbb{Z}_p$-extension as assumed in [Nek93, 6.2], the argument for [Nek93, Theorem 6.9] works without change.

[18] Note that in this article, we always use the covariant version for $\mathbb{D}_{\mathrm{dR}}$ and $\mathbb{D}_{\mathrm{cris}}$.

is a subfield of $\mathbb{C}$ and hence $\overline{K}$ via the fixed isomorphism $\overline{K}=\overline{E_u}\xrightarrow{\sim}\mathbb{C}$. Let V be either $\mathrm{V}_{\pi,L}\otimes_{\mathbb{L}}\overline{K}$ or $\mathrm{V}_{\hat{\pi},L}\otimes_{\mathbb{L}}\overline{K}$. Repeating the argument in [Shn16, Proposition 8.9] (which followed an approachp in [Nek95]) to V, we obtain $\mathrm{H}^0(K_\xi,\mathrm{V})=0$ since V is crystalline of pure weight -1 by Lemma 4.15.

The lemma is proved. ☐

Proof of Proposition 4.35. Let $M\geqslant 0$ be the integer in Lemma 4.37 and $(\mathrm{t}_1,\mathrm{t}_2)\in\mathbb{S}^{\diamond}_{O_{\mathbb{L}}}\times\mathbb{S}^{\diamond}_{O_{\mathbb{L}}}$ the pair in Lemma 4.34. We first note that $Z_{T_i}(\mathrm{t}_i\mathrm{s}_i\phi_i^{[e_i]})_L\in\mathrm{Z}^r(X_L)\otimes O_{\mathbb{L}}$ for $i=1,2$. By Lemma 4.36, we may find an element $Z\in\mathrm{Z}^r(X_L\otimes_E E^{[|e_{2,v}|]})\otimes O_{\mathbb{L}}$ such that $\mathrm{Nm}_{E^{[|e_{2,v}|]}/E}Z=Z_{T_2}(\mathrm{t}_2\phi_2^{[e_2]})_L$. We may also assume that the support of Z is contained in the support of $Z_{T_2}(\mathrm{t}_2\phi_2^{[e_2]})_L$. Put

$$Z_2:=\mathrm{Nm}_{E^{[|e_{2,v}|]}\otimes_E K/K_{|e_{2,v}|}}Z\otimes_E K\in\mathrm{Z}^r(X_L\otimes_E K_{|e_{2,v}|})\otimes O_{\mathbb{L}},$$

so that $\mathrm{Nm}_{K_{|e_{2,v}|}/K}Z_2=Z_{T_2}(\mathrm{t}_2\phi_2^{[e_2]})_L\otimes_E K$. Since the natural map

$$\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L,O_{\mathbb{L}}(r))^{\mathrm{fr}}/\mathrm{T}_{\pi,L}\to\mathrm{H}^{2r-1}_{\text{ét}}(\overline{X}_L,\mathbb{L}(r))/\mathrm{V}_{\pi,L}$$

is injective, the class $Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L=\mathrm{s}_1^*Z^{\text{ét}}_{T_1}(\mathrm{s}_1\phi_1^{[e_1]})_L$ sits in $\mathrm{H}^1_f(K,\mathrm{T}_{\pi,L})$. Similarly, the cycle class of $\mathrm{s}_2^*Z_2$ sits in $\mathrm{H}^1_f(K_{|e_{2,v}|},\mathrm{T}_{\hat{\pi},L})$. By [Nek95, II.(1.9.1)], we have

$$\langle Z_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L,Z_{T_2}(\mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_K=\mathrm{Nm}_{K_{|e_{2,v}|}/K}\langle\mathrm{s}_1^*Z_{T_1}(\mathrm{t}_1\phi_1^{[e_1]})_L\otimes_E K_{|e_{2,v}|},\mathrm{s}_2^*Z_2\rangle_{K_{|e_{2,v}|}}.$$

By Lemma 4.34, we have

$$\langle\mathrm{s}_1^*Z_{T_1}(\mathrm{t}_1\phi_1^{[e_1]})_L\otimes_E K_{|e_{2,v}|},\mathrm{s}_2^*Z_2\rangle_{K_{|e_{2,v}|}}\in(O^\times_{K_{|e_{2,v}|}})^{\mathrm{fr}}\otimes_{\mathbb{Z}_p}\mathbb{L}.$$

In other words, the corresponding bi-extension is crystalline (Remark A.15). By the argument for [Nek95, Proposition II.1.11], we have

$$\langle\mathrm{s}_1^*Z_{T_1}(\mathrm{t}_1\phi_1^{[e_1]})_L\otimes_E K_{|e_{2,v}|},\mathrm{s}_2^*Z_2\rangle_{K_{|e_{2,v}|}}\in(O^\times_{K_{|e_{2,v}|}})^{\mathrm{fr}}\otimes_{\mathbb{Z}_p}(p^{-M}O_{\mathbb{L}}).$$

Finally, since the image of the norm map $\mathrm{Nm}_{K_d/K}\colon(O^\times_{K_d})^{\mathrm{fr}}\to(O^\times_K)^{\mathrm{fr}}$ is precisely $p^d(O^\times_K)^{\mathrm{fr}}$ for $d\in\mathbb{N}$, we have

$$\langle Z_{T_1}(\mathrm{t}_1\mathrm{s}_1\phi_1^{[e_1]})_L,Z_{T_2}(\mathrm{t}_2\mathrm{s}_2\phi_2^{[e_2]})_L\rangle_K\in(O^\times_K)^{\mathrm{fr}}\otimes_{\mathbb{Z}_p}(p^{|e_{2,v}|-M}O_{\mathbb{L}}).$$

The proposition is proved. ☐

4.8. **Proof of Theorem 4.21.** In this subsection, we prove Theorem 4.21.

Take $e\in\mathbb{N}$, which is regarded as a constant tuple according to the context.

For every $v\in\mathtt{R}$, we

- choose a pair $(\phi_{v,1},\phi_{v,2})\in\mathscr{R}_v$ (4.8) and put $\Phi_v:=\phi_{v,1}\otimes\phi_{v,2}$,
- let $\mathfrak{f}_v\in\mathscr{S}(\mathrm{Herm}_{2r}(F_v),\mathbb{Z}_{(p)})$ be the unique element such that $\mathfrak{f}_v^{\mathbf{1}_v}=f^{\mathrm{SW}}_{\Phi_v}$,
- put $f_{\chi_v}:=\mathfrak{f}_v^{\chi_v}\in\mathrm{I}^{\square}_{r,v}(\chi_v)$ (3.4) for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$.

For every $v\in\mathtt{T}$, we

- put $\Phi_v:=\phi_{v,1}\otimes\phi_{v,2}=\mathbf{1}_{\Lambda_v^{2r}}$ (§4.5(S2)),
- let $f_{\chi_v}\in\mathrm{I}^{\square}_{r,v}(\chi_v)$ be the section from Lemma 4.31 for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$, that is, the standard section such that $f_{\mathbf{1}_v}=f^{\mathrm{SW}}_{\Phi_v}$.

For every $v\in\mathtt{V}_F^{(p)}$, we

- put $\Phi_v^{[e]}:=\phi_{v,1}^{[0]}\otimes\phi_{v,2}^{[e]}$ (§4.5(S3)),
- put $f^{[e]}_{\chi_v}:=b_{2r,v}(\mathbf{1})^{-1}\cdot\mathrm{vol}(L_v,\mathrm{d}h_v)\cdot(\mathfrak{f}^{[e+\varepsilon^{\mathtt{c}}]}_{\chi_v})^{\chi_v}\in\mathrm{I}^{\square}_{r,v}(\chi_v)$ for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$, so that $f^{[e]}_{\mathbf{1}_v}=f^{\mathrm{SW}}_{\Phi_v^{[e]}}$ by Lemma 4.29(3).

Then we

- put
$$\phi_1 := \left(\bigotimes_{v\in\mathtt{V}_F^{(p)}} \phi_{v,1}^{[0]}\right)\otimes\left(\bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} \phi_{v,1}\right),\quad \phi_2^{[e]} := \left(\bigotimes_{v\in\mathtt{V}_F^{(p)}} \phi_{v,2}^{[e]}\right)\otimes\left(\bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} \phi_{v,2}\right) \tag{4.17}$$
in $\mathscr{S}(V^r\otimes_F\mathbb{A}_F^\infty,\mathbb{Z}_{(p)})^{K_r^\diamond\times L}$,
- put
$$\Phi^{[e]} := \left(\bigotimes_{v\in\mathtt{V}_F^{(p)}} \Phi_v^{[e]}\right)\otimes\left(\bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} \Phi_v\right)\in\mathscr{S}(V^r\otimes_F\mathbb{A}_F^\infty,\mathbb{Z}_{(p)})^{K_{2r}^\diamond\times L},$$
- put
$$f_{\chi^\infty}^{[e]} := \left(\bigotimes_{v\in\mathtt{V}_F^{(p)}} f_{\chi_v}^{[e]}\right)\otimes\left(\bigotimes_{v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{V}_F^{(p)}} f_{\chi_v}\right)\in\mathrm{I}_r^\square(\chi)^\infty$$
for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$.

Finally, we

- let $M\in\mathbb{N}$ be the smallest element such that all of the following elements
$$p^M\,\mathrm{vol}^\natural(L),\qquad \frac{p^M W_{2r}^{\diamond\setminus\{p\}}}{q_{v'}^{2r}-1}\prod_{v\in\mathtt{V}_F^{(p)}}\mathrm{vol}(L_v,\,\mathrm{d}h_v),\forall v'\in\mathsf{S}_\pi$$
belong to $\mathbb{Z}_{(p)}$, where $W_{2r}^{\diamond\setminus\{p\}}$ is defined in (3.7);
- fix an open compact subgroup $K^\dagger$ of $G_{r,r}(\mathbb{A}_F^\infty)=G_r(\mathbb{A}_F^\infty)\times G_r(\mathbb{A}_F^\infty)$ of the form
$$K^\dagger_{\diamond\setminus\{\infty,p\}}\times\left(\prod_{v\in\mathtt{V}_F^{(p)}}\mathscr{G}_{r,r}(O_{F_v})\times_{\mathscr{G}_{r,r}(O_{F_v}/\varpi_v)}\mathscr{P}_{r,r}(O_{F_v}/\varpi_v)\right)\times\left(K_r^\diamond\times K_r^\diamond\right)$$
(Definition 2.9) in which $K^\dagger_{\diamond\setminus\{\infty,p\}}$ contains
$$\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}\setminus\mathtt{R}}(K_{r,v}\cap M_r(F_v))\times(K_{r,v}\cap M_r(F_v))$$
and fixes $\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}}f_{\chi_v}$ for every finite character $\chi\colon\Gamma_{F,p}\to\mathbb{C}^\times$;
- fix $(\mathrm{t}_1,\mathrm{t}_2)\in\mathbb{S}^\diamond_{O_\mathbb{L}}\times\mathbb{S}^\diamond_{O_\mathbb{L}}$ that is the product of those pairs from Proposition 4.33 and Proposition 4.35 for every $u\in\mathtt{P}$ (and a suitable scalar), which satisfies $\chi^\diamond_{\hat\pi}(\mathrm{t}_1)\chi^\diamond_\pi(\mathrm{t}_2)\neq0$;
- fix $(\mathrm{s}_1,\mathrm{s}_2)\in(\mathbb{S}^\diamond_{\pi,L}\cap\mathbb{S}^\diamond_{O_\mathbb{L}})\times(\mathbb{S}^\diamond_{\hat\pi,L}\cap\mathbb{S}^\diamond_{O_\mathbb{L}})$ such that $\chi^\diamond_{\hat\pi}(\mathrm{s}_1)\chi^\diamond_\pi(\mathrm{s}_2)\neq0$, which is possible by Lemma 4.8(2).

Remark 4.38. For every $v\in\mathtt{V}_F^{\mathrm{spl}}\setminus\mathtt{V}_F^{(\diamond)}$, we have a canonical isomorphism
$$\mathbb{Z}[L_v\backslash H(F_v)/L_v]\simeq\mathbb{Z}[K_{r,v}\backslash G_r(F_v)/K_{r,v}]$$
of rings via Satake isomorphisms. By [Liu11a, Proposition A.5], we know that the action of $\mathrm{s}\in\mathbb{Z}[L_v\backslash H(F_v)/L_v]$ on $\mathscr{S}(V_v^r)^{K_{r,v}\times L_v}$ via the Weil representation $\omega_{r,v}$ coincides with that of $\hat{\mathrm{s}}\in\mathbb{Z}[K_{r,v}\backslash G_r(F_v)/K_{r,v}]$, where $\hat{\mathrm{s}}$ denotes the adjoint of s.

Motivated by Proposition 4.33, for every $e\in\mathbb{N}$, every pairs $(g_1,g_2)\in M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})$, we define following elements in $\Gamma_{F,p}\otimes_{\mathbb{Z}_p}\mathbb{L}$:
$$(\mathcal{E}^{[e]}_{(g_1,g_2)})_{T^\square} := p^M W_{2r}\cdot\mathsf{W}'_{T^\square,v_{T^\square}}(1)\cdot I_{T^\square}((\mathrm{t}_1\mathrm{s}_1g_1\phi_1\otimes\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[e]})^{v_{T^\square}})\cdot[v_{T^\square}]$$
for every $T^\square\in\mathrm{Herm}^\circ_{2r}(F)^+_V$ (Notation 4.33), and
$$({}^v\mathcal{E}^{[e]}_{(g_1,g_2)})_{T^\square} := p^M W_{2r}\frac{2}{q_v^{2r}-1}\cdot\mathsf{W}^{\mathrm{sph}}_{T^\square,v}(1)\cdot I_{T^\square}((\mathrm{t}_1\mathrm{s}_1g_1\phi_1\otimes\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[e]})^v)\cdot[v]$$

for every $T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+$ and every $v \in \mathtt{S}_\pi$.

For every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$, we denote by $\mathbb{L}_\chi$ the finite (normal) extension of $\mathbb{L}$ generated by values of χ, which is a subfield of $\overline{\mathbb{Q}}_p$.

Lemma 4.39. *We have*

1. *There exists a (module-)finite $\mathbb{Z}_{(p)}$-ring $\mathbb{O}$ contained in $\mathbb{C}$ such that for every $e \in \mathbb{N}$, every pairs $(g_1, g_2) \in M_r(F_\mathtt{R}) \times M_r(F_\mathtt{R})$, and every $T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+$, there is a unique integral $\mathbb{L} \otimes_{\mathbb{Z}_{(p)}} \mathbb{O}$-valued p-adic measure (Definition 3.41) $\mathrm{d}(\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}$ on $\Gamma_{F,p}$, such that for every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ and every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,*

$$\iota(\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}(\chi) = p^M W_{2r}^\diamond \cdot b_{2r}^\diamond(\iota\chi) \cdot W_{T^\Box}((g_1, g_2) \cdot \iota(\hat{\mathtt{s}}_1\hat{\mathtt{t}}_1, \hat{\mathtt{s}}_2\hat{\mathtt{t}}_2) f^{[e]}_{\iota\chi^\infty}), \tag{4.18}$$

   where $W_{2r}^\diamond$ is from (3.7).

2. *The measure $\mathrm{d}(\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}$ in (1) satisfies $(\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}(\mathbf{1}) = 0$ and*

$$\partial(\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}(\mathbf{1}) = \begin{cases} (\mathcal{E}^{[e]}_{(g_1,g_2)})_{T^\Box} & \text{if } T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+_V, \\ 0 & \text{if } T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+ \setminus \mathrm{Herm}_{2r}^\circ(F)^+_V. \end{cases}$$

3. *For every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$, the assignment*

$$\mathscr{E}^{[e]}_{-}(\chi)\colon (g_1, g_2) \mapsto \mathscr{E}^{[e]}_{(g_1,g_2)}(\chi) \coloneqq \sum_{T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+} (\mathscr{E}^{[e]}_{(g_1,g_2)})_{T^\Box}(\chi) q^{T^\Box}$$

   belongs to $O_{\mathbb{L}_\chi} \otimes_{\mathbb{Z}_{(p)}} \mathrm{SF}_{2r}(\mathbb{O})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$.[19] *For every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{L})$, we have $\mathscr{E}^{[e]}_{(g_1,g_2)}(\sigma\chi) = \sigma\mathscr{E}^{[e]}_{(g_1,g_2)}(\chi)$, where σ acts on $O_{\mathbb{L}_\chi} \otimes_{\mathbb{Z}_{(p)}} \mathrm{SF}_{2r}(\mathbb{O})$ via the first factor.*

4. *For every $v \in \mathtt{S}_\pi$, the assignment*

$${}^v\mathcal{E}^{[e]}_{-}\colon (g_1, g_2) \mapsto {}^v\mathcal{E}^{[e]}_{(g_1,g_2)} \coloneqq \sum_{T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+} ({}^v\mathcal{E}^{[e]}_{(g_1,g_2)})_{T^\Box} q^{T^\Box}$$

   belongs to $O_\mathbb{L} \otimes_{\mathbb{Z}_{(p)}} \mathrm{SF}_{2r}(\mathbb{Z}_{(p)})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$.

Proof. By Lemma 4.29, the right-hand side of (4.18) equals

$$\left(p^M W_{2r}^\diamond \prod_{v\in\mathtt{V}_F^{(p)}} b_{2r,v}(\mathbf{1})^{-1} \operatorname{vol}(L_v,\, \mathrm{d}h_v)\right) \times \left(\iota\chi_p(\mathrm{Nm}_{E_p/F_p} \det T^\Box_{12}) \mathbf{1}_{\mathfrak{T}_p^{[e+\varepsilon^\mathtt{c}]}}(T^\Box)\right)$$
$$\times \left(\prod_{v\in\mathtt{R}} W_{T^\Box}((g_{1,v}, g_{2,v}) f_{\iota\chi_v})\right) \times \left(\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}\setminus\mathtt{R}} W_{T^\Box}(f_{\iota\chi_v})\right) \times \left(b_{2r}^\diamond(\iota\chi) \cdot W_{T^\Box}(\iota(\hat{\mathtt{s}}_1\hat{\mathtt{t}}_1, \hat{\mathtt{s}}_2\hat{\mathtt{t}}_2) f_{\iota\chi^\diamond})\right).$$

We have

- $C \coloneqq p^M W_{2r}^\diamond \prod_{v\in\mathtt{V}_F^{(p)}} b_{2r,v}(\mathbf{1})^{-1} \operatorname{vol}(L_v,\, \mathrm{d}h_v) = p^M W_{2r}^{\diamond\setminus\{p\}} \prod_{v\in\mathtt{V}_F^{(p)}} \operatorname{vol}(L_v,\, \mathrm{d}h_v)$, which belongs to $\mathbb{Z}_{(p)}$;
- there is an element $\mathscr{W}_{T^\Box,p} \in \mathbb{Z}[\Gamma_{F,p}]$ satisfying

$$\iota\mathscr{W}_{T^\Box,p}(\chi) = \iota\chi_p(\mathrm{Nm}_{E_p/F_p} \det T^\Box_{12}) \mathbf{1}_{\mathfrak{T}_p^{[e+\varepsilon^\mathtt{c}]}}(T^\Box)$$

  for every χ, ι as above;
- for every $v \in \mathtt{R}$, there is a constant $C_{T^\Box,(g_{1,v},g_{2,v})} \in \mathbb{Z}_{(p)}$ that equals $W_{T^\Box}((g_{1,v}, g_{2,v}) f_{\iota\chi_v})$ for every χ as above;
- by Lemma 4.31, for every $v \in \mathtt{T}$, there is an element $\mathscr{W}_{T^\Box,v} \in \mathbb{Z}[\Gamma_{F,p}]$ satisfying $\iota\mathscr{W}_{T^\Box,v}(\chi) = b_{2r,v}(\iota\chi) \cdot W_{T^\Box}(f_{\iota\chi_v})$ for every χ, ι as above;
- it is easy to see that for every $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})} \setminus \mathtt{R}$, there is an element $\mathscr{B}_v \in \mathbb{Z}[\Gamma_{F,p}]$ satisfying $\iota\mathscr{B}_v(\chi) = b_{2r,v}(\iota\chi)^{-1}$ for every χ, ι as above;

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[19] Once again, we have strict inclusions $O_{\mathbb{L}_\chi} \otimes_{\mathbb{Z}_{(p)}} \mathrm{SF}_{2r}(\mathbb{O})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})} \subsetneq (O_{\mathbb{L}_\chi} \otimes_{\mathbb{Z}_{(p)}} \mathrm{SF}_{2r}(\mathbb{O}))^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})} \subsetneq \mathrm{SF}_{2r}(O_{\mathbb{L}_\chi} \otimes_{\mathbb{Z}_{(p)}} \mathbb{O})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$.

- by Lemma 3.5(1), there exist $\mathbb{O}$ as in the statement of (1) and finitely many elements $c_1, \dots, c_t \in O_{\mathbb{L}}$ such that for every $T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+$, there are elements $\mathscr{W}_{T^\Box,1}^\diamondsuit, \dots, \mathscr{W}_{T^\Box,t}^\diamondsuit$ in $\mathbb{O}[\Gamma_{F,p}]$ satisfying

$$\iota \sum_{i=1}^{t} c_i \otimes \mathscr{W}_{T^\Box,i}^\diamondsuit(\chi) = b_{2r}^\diamondsuit(\iota\chi) \cdot W_{T^\Box}(\iota(\hat{\mathrm{s}}_1\hat{\mathrm{t}}_1, \hat{\mathrm{s}}_2\hat{\mathrm{t}}_2) f_{\iota\chi^\diamond})$$

  for every χ, ι as above.

For each i, put

$$\mathscr{W}_{T^\Box,\mathtt{T},i} := \left(\prod_{v \in \mathtt{V}_F^{(\diamondsuit \setminus \{\infty,p\})} \setminus \mathtt{R}} \mathscr{W}_{T^\Box,v} \right) \cdot \mathscr{W}_{T^\Box,i}^\diamondsuit \in \mathbb{O}[\Gamma_{F,p}].$$

Moreover, for every $v \in \mathtt{T} \setminus \mathtt{V}_F^{\mathrm{spl}}$, we can write $\mathscr{W}_{T^\Box,\mathtt{T},i} = \mathscr{W}_{T^\Box,v} \cdot \mathscr{W}_{T^\Box,\mathtt{T},i}^{v}$ for a unique element $\mathscr{W}_{T^\Box,\mathtt{T},i}^{v} \in \mathbb{O}[\Gamma_{F,p}]$.

For (1), we may take

$$(\mathscr{E}_{(g_1,g_2)}^{[e]})_{T^\Box} = \sum_{i=1}^{t} c_i \otimes \left(C \prod_{v \in \mathtt{R}} C_{T^\Box,(g_{1,v},g_{2,v})} \right) \cdot \left(\prod_{v \in \mathtt{V}_F^{(\diamondsuit \setminus \{\infty,p\})} \setminus \mathtt{R}} \mathscr{B}_v \right) \cdot \mathscr{W}_{T^\Box,p} \cdot \mathscr{W}_{T^\Box,\mathtt{T},i} \in (O_{\mathbb{L}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{O})[\Gamma_{F,p}].$$

The uniqueness is automatic.

Part (3) is obvious from the construction in (1).

The proof of (4) is similar by realizing that $\sum_{i=1}^{t} c_i \otimes \mathscr{W}_{T^\Box,\mathtt{T},i}^{v}(\mathbf{1}) \in \mathbb{Z}_{(p)}$.

For (2), note that $W_{T^\Box}(f_{\mathbf{1}_v}) = 0$ for every $v \in \mathrm{Diff}(T^\Box, V)$, which implies that $(\mathscr{E}_{(g_1,g_2)}^{[e]})_{T^\Box}(\mathbf{1}) = 0$, and for $T^\Box \in \mathrm{Herm}_{2r}^\circ(F)^+ \setminus \mathrm{Herm}_{2r}^\circ(F)_V^+$ that $\partial(\mathscr{E}_{(g_1,g_2)}^{[e]})_{T^\Box}(\mathbf{1}) = 0$. For $T^\Box \in \mathrm{Herm}_{2r}^\circ(F)_V^+$, we have $\mathscr{E}_{T^\Box,v_{T^\Box}}(\mathbf{1}) = 0$. Then by the p-adic Leibniz rule, we have

$$\partial(\mathscr{E}_{(g_1,g_2)}^{[e]})_{T^\Box}(\mathbf{1}) = p^M W_{2r} \cdot b_{2r}^\infty(\mathbf{1}) b_{2r,v_{T^\Box}}(\mathbf{1})^{-1} W_{T^\Box}((g_1,g_2) \cdot \iota(\hat{\mathrm{s}}_1\hat{\mathrm{t}}_1, \hat{\mathrm{s}}_2\hat{\mathrm{t}}_2) f_{\iota\chi^{\infty v_{T^\Box}}}^{[e]}) \cdot \partial\mathscr{W}_{T^\Box,v_{T^\Box}}(\mathbf{1}).$$

By Remark 4.38 and §4.1(H9), we have

$$b_{2r}^\infty(\mathbf{1}) b_{2r,v_{T^\Box}}(\mathbf{1})^{-1} W_{T^\Box}((g_1,g_2) \cdot \iota(\hat{\mathrm{s}}_1\hat{\mathrm{t}}_1, \hat{\mathrm{s}}_2\hat{\mathrm{t}}_2) f_{\iota\chi^{\infty v_{T^\Box}}}^{[e]}) = I_{T^\Box}((\mathrm{t}_1\mathrm{s}_1 g_1 \phi_1 \otimes \mathrm{t}_2\mathrm{s}_2 g_2 \phi_2^{[e]})^{v_{T^\Box}}).$$

This, it remains to show that

$$\partial\mathscr{W}_{T^\Box,v_{T^\Box}}(\mathbf{1}) = \mathsf{W}'_{T^\Box,v_{T^\Box}}(1) \cdot [v_{T^\Box}],$$

which is tautological as $\mathscr{W}_{T^\Box,v_{T^\Box}} = \mathsf{W}_{T^\Box,v_{T^\Box}}([v_{T^\Box}])$ from Lemma 4.31.

The lemma is proved. □

We search for Eisenstein series whose q-expansions are given by $\mathscr{E}_-^{[e]}(\chi)$ and ${}^{v}\mathcal{E}_-^{[e]}$. We refer to §3.2 for the notation concerning Eisenstein series. For every $e \in \mathbb{N}$, every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ and every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$, define an Eisenstein series

$$E_{\iota\chi}^{[e]} := p^M \cdot b_{2r}^\diamondsuit(\mathbf{1})^{-1} \cdot b_{2r}^\diamondsuit(\iota\chi) \cdot E(-, f_\infty^{[r]} \otimes \iota(\hat{\mathrm{s}}_1\hat{\mathrm{t}}_1, \hat{\mathrm{s}}_2\hat{\mathrm{t}}_2) f_{\iota\chi^\infty}^{[e]}) \in \mathcal{A}_{2r,\mathrm{hol}}^{[r]}.$$

and, for every $v \in \mathsf{S}_\pi$, an Eisenstein series

$${}^{v}E_{\iota}^{[e]} := p^M \frac{2}{q_v^{2r} - 1} \cdot E(-, f_\infty^{[r]} \otimes \iota(\hat{\mathrm{s}}_1\hat{\mathrm{t}}_1, \hat{\mathrm{s}}_2\hat{\mathrm{t}}_2)\, {}^{v}f_{\mathbf{1}^\infty}^{[e]}) \in \mathcal{A}_{2r,\mathrm{hol}}^{[r]},$$

where ${}^{v}f_{\mathbf{1}^\infty}^{[e]}$ is obtained from $f_{\mathbf{1}^\infty}^{[e]}$ after replacing the component $f_{\mathbf{1}_v}$ by $f_{\mathbf{1}_v}^{\mathrm{sph}}$ from Notation 3.4.

Lemma 4.40. *We have*

(1) For every finite character $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ and every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\boldsymbol{q}_{2r}^{\mathrm{an}}((g_1,g_2) \cdot E_{\iota\chi}^{[e]}) = \iota\mathscr{E}_{(g_1,g_2)}^{[e]}(\chi)$$

holds for every $(g_1,g_2) \in M_r(F_{\mathtt{R}}) \times M_r(F_{\mathtt{R}})$.

(2) *For every* $v \in \mathtt{S}_\pi$ *and every embedding* $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\boldsymbol{q}^{\mathrm{an}}_{2r}((g_1,g_2)\cdot {}^vE^{[e]}_\iota) = \iota\, {}^v\mathcal{E}^{[e]}_{(g_1,g_2)}$$

holds for every $(g_1,g_2)\in M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})$.

Proof. Since for $v\in\mathtt{R}$ (which is nonempty), $g_{1,v}\phi_{1,v}\otimes g_{2,v}\phi_{2,v}$ again belongs to $\mathscr{R}_v$, both cases follow from the discussion in [Liu11b, Section 2B] and Lemma 3.2. □

Definition 4.41. For every open compact subgroup $K\subseteq G_r(\mathbb{A}_F^\infty)$ and every subring $\mathbb{M}$ of $\mathbb{C}$, we define $\mathcal{A}^K_\mathbb{M}$ the $\mathbb{M}$-submodule of $\mathcal{A}^{[r]}_{r,r,\mathrm{hol}}$ consisting of all φ that are fixed by K and satisfy $\boldsymbol{q}^{\mathrm{an}}_{r,r}((g_1,g_2)\cdot\varphi)\in\mathrm{SF}_{r,r}(\mathbb{M})$ for every $(g_1,g_2)\in M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})$.

Lemma 4.42. *Suppose that K contains*

$$\prod_{v\in\mathtt{V}_F^{\diamondsuit\setminus\{\infty\}}\setminus\mathtt{R}}(K_{r,v}\cap M_r(F_v))\times(K_{r,v}\cap M_r(F_v))$$

so that the tautological map $\mathcal{A}^K_\mathbb{M}\to\mathrm{SF}_{r,r}(\mathbb{M})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$ *sending* φ *to the assignment* $(g_1,g_2)\mapsto\boldsymbol{q}^{\mathrm{an}}_{r,r}((g_1,g_2)\cdot\varphi)$ *is injective.*

(1) *For rings* $\mathbb{Z}_{(p)}\subseteq\mathbb{M}\subseteq\mathbb{M}'\subseteq\mathbb{C}$, *the natural diagram*

$$\begin{tikzcd}
O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^K_\mathbb{M} \arrow[r] \arrow[d] & O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^K_{\mathbb{M}'} \arrow[d] \\
O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{M})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})} \arrow[r] & O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{M}')^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}
\end{tikzcd}$$

is Cartesian.

(2) *For a ring* $\mathbb{Z}_{(p)}\subseteq\mathbb{M}\subseteq\mathbb{C}$, *the natural diagram*

$$\begin{tikzcd}
O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^K_\mathbb{M} \arrow[r] \arrow[d] & \prod_{\iota\colon\mathbb{L}\to\mathbb{C}}\mathcal{A}^K_\mathbb{C} \arrow[d] \\
O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{M})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})} \arrow[r] & \prod_{\iota\colon\mathbb{L}\to\mathbb{C}}\mathrm{SF}_{r,r}(\mathbb{C})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}
\end{tikzcd}$$

is Cartesian.

Proof. Part (1) follows from Definition 4.41 and the fact that $O_\mathbb{L}$ is flat over $\mathbb{Z}_{(p)}$.

For (2), consider an element $x=\sum_{j=1}^s c_j\otimes x_j$ of $O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{M})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$ in which $c_1,\dots,c_s$ are $\mathbb{Z}_{(p)}$-linearly independent elements of $O_\mathbb{L}$, satisfying that for every $\iota\colon\mathbb{L}\to\mathbb{C}$, its image in $\mathrm{SF}_{r,r}(\mathbb{C})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$ comes from $\mathcal{A}^K_\mathbb{C}$. Since $\mathbb{L}$ has characteristic zero, we may find embeddings $\iota_1,\dots,\iota_s$ such that $A\coloneqq(\iota_ic_j)_{1\leqslant i,j\leqslant s}$ is invertible. If we write $\iota_ix=y_i$ for $y_i\in\mathcal{A}^K_\mathbb{C}$, then ${}^{\mathrm{t}}(x_1,\dots,x_s)=A^{-1}\,{}^{\mathrm{t}}(y_1,\dots,y_s)$. In particular, x belongs to $O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{C})^{M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})}$. Applying (1) with $\mathbb{M}'=\mathbb{C}$, we obtain (2). □

Lemma 4.43. *Recall the map* $\varrho_{r,r}$ *from Definition 2.5.*

(1) *For every finite character* $\chi\colon\Gamma_{F,p}\to\overline{\mathbb{Q}}_p^\times$, *there exists* $e_\chi\in\mathbb{N}$ *such that for every* $e\geqslant e_\chi$, *there exists a (unique) element* $D^{[e]}_\chi\in O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_\mathbb{O}$ *satisfying*

$$(1\times\boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot D^{[e]}_\chi)=\varrho_{r,r}\mathscr{E}^{[e]}_{(g_1,g_2)}(\chi)$$

for every $(g_1,g_2)\in M_r(F_\mathtt{R})\times M_r(F_\mathtt{R})$. *Moreover, the sequence* $\{D^{[N!]}_\chi\}$ *converges in* $O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_\mathbb{O}$ *when* $N\to\infty$.

(2) *For every* $v\in\mathtt{S}_\pi$ *and* $e\in\mathbb{N}$, *there exists a (unique) element* ${}^vD^{[e]}\in O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{Z}_{(p)}}$ *satisfying*

$$(1\times\boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot{}^vD^{[e]}=\varrho_{r,r}\mathcal{E}^{[e]}_{(g_1,g_2)}$$

for every $(g_1,g_2)\in M_r(F_\mathbb{R})\times M_r(F_\mathbb{R})$*. Moreover, the sequence* $\{{}^{\nu}D^{[N!]}\}$ *converges in* $O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{Z}_{(p)}}$ *when* $N\to\infty$.

Proof. For (1), note that by Lemma 4.39(3),

$$\varrho_{r,r}\mathscr{E}^{[e]}_{(g_1,g_2)}(\chi)\in O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{O})^{M_r(F_\mathbb{R})\times M_r(F_\mathbb{R})}.$$

Then by Lemma 4.40(1) and Lemma 4.42(2), for every $e\in\mathbb{N}$, there is a (unique) element $D^{[e]}_\chi\in O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^{[e]}}_{\mathbb{O}}$ for some subgroup $K^{[e]}\subseteq K^\dagger$ of finite index such that

$$(1\times\boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot D^{[e]}_\chi)=\varrho_{r,r}\mathscr{E}^{[e]}_{(g_1,g_2)}(\chi)$$

holds for every $(g_1,g_2)\in M_r(F_\mathbb{R})\times M_r(F_\mathbb{R})$. Now Lemma 3.38 tells us that we may take $K^{[e]}=K^\dagger$ when $e\geqslant e_\chi$ in that lemma.

For the convergence, we have a natural inclusion

$$O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{O}}\hookrightarrow\mathbb{L}_\chi\otimes_{\mathbb{Q}_p}\left(\mathcal{H}^{[r]}_{r,r}(K^\dagger)\otimes_\mathbb{Q}\mathbb{M}_{K^\dagger}\right)\tag{4.19}$$

(Definition 2.3) for some number field $\mathbb{M}_{K^\dagger}\subseteq\mathbb{C}$ containing $\mathbb{O}$ depending on $K^\dagger$. It is well-known that the limit of the operators $\{\mathrm{U}_p^{N!/2}\times\mathrm{U}_p^{N!/2}\}_{N\geqslant2}$ exists in $\mathrm{End}_{\mathbb{Q}_p}\left(\mathcal{H}^{[r]}_{r,r}(K^\dagger)\right)$, which is the projection to the (Siegel-)ordinary part (see, for example, [Hid98, Page 685]). Thus, by Lemma 3.11, $\{D^{[N!]}_\chi\}$ converges in $\mathbb{L}_\chi\otimes_{\mathbb{Q}_p}\left(\mathcal{H}^{[r]}_{r,r}(K^\dagger)\otimes_\mathbb{Q}\mathbb{M}_{K^\dagger}\right)$. Since the inclusion (4.19) is closed, the limit belongs to the source.

The proof for (2) is similar, by using Lemma 4.39(4) and Lemma 4.40(2). □

Lemma 4.44. *For every* $g_2\in G_r(F_\mathbb{R})$ *and* $T_2\in\mathrm{Herm}^\circ_r(F)^+$*, the sequence* $\{Z^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[N!]})_L\}_N$ *converges in* $\mathrm{H}^1_f(E,\mathrm{V}_{\tilde\pi,L})$.

Proof. Fix an embedding $\mathbb{Q}\langle p\rangle\hookrightarrow\overline{\mathbb{Q}}_p$ and all representations will have coefficients in $\overline{\mathbb{Q}}_p$. The assignment

$$\boldsymbol{\phi}_p\mapsto\mathscr{S}(V^r\otimes_F F_p,\overline{\mathbb{Q}}_p)^{L_p}\to Z^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^p\boldsymbol{\phi}_p)_L$$

factors through $\theta(\tilde\pi_p)^L$ (§4.1(H10) but with $\mathbb{C}$ replaced by $\overline{\mathbb{Q}}_p$) as a $\overline{\mathbb{Q}}_p[L_p\backslash H(F_p)/L_p]$-module, by the influence of s_2. Write $(\overline{\mathbb{Q}}_p)_{T_2}$ the character of $N_r(F_p)$ such that for every $b\in\mathrm{Herm}_r(F_p)$, $n(b)$ acts by $\psi_{F,p}(\mathrm{tr}\,T_2b)$. Then, by Lemma 4.1, there exists an element $w\in\mathrm{Hom}_{N_r(F_p)}(\tilde\pi_p,(\overline{\mathbb{Q}}_p)_{T_2})$ such that the assignment $Z^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^p-)_L$ factors through the composition

$$\mathscr{S}(V^r\otimes_F F_p,\overline{\mathbb{Q}}_p)^{L_p}\to\tilde\pi_p\otimes_{\overline{\mathbb{Q}}_p}\theta(\tilde\pi_p)^L\xrightarrow{w\otimes1}\theta(\tilde\pi_p)^L.$$

By Lemma 4.29(1), $\phi^{[e]}_{p,2}=\mathrm{U}^e_p\phi^{[0]}_{p,2}$ hence is invariant under $I_p:=\prod_{v\in\mathrm{V}^{(p)}_F}I_v$ (3.13). Since $\{\mathrm{U}^{N!}_p\}$ is convergent as a sequence of endomorphisms on $\tilde\pi_p^{I_p}$ and $w|_{\tilde\pi_p^{I_p}}$ is continuous, the lemma follows. □

In what follows, we put

$$D_\chi:=\lim_{N\to\infty}D^{[N!]}_\chi\in O_{\mathbb{L}_\chi}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{O}},$$

$${}^{\nu}D:=\lim_{N\to\infty}{}^{\nu}D^{[N!]}\in O_\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{Z}_{(p)}},$$

$$\zeta_{g_2,T_2}:=p^M\lim_{N\to\infty}Z^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[N!]})_L\in\mathrm{H}^1_f(E,\mathrm{V}_{\tilde\pi,L}).$$

For every integer $d\geqslant1$, we recall the subgroup U_d of $\Gamma_{F,p}$ and the set of representative Γ_d of $\Gamma_{F,p}/U_d$ fixed from §3.6. For every $\lambda\in\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p},\mathbb{Z}_p)$, define

$$I_{\lambda,d}:=\left(\sum_{x\in\Gamma_d}\lambda(x)\frac{\chi(x)^{-1}}{|\Gamma_{F,p}/U_d|}\sum_{\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times}D_\chi\right)+\left(\sum_{\nu\in\mathrm{S}_\pi}{}^{\nu}D\right).$$

Proposition 4.45. *Suppose that* $n<p$*. Take an element* $\lambda\in\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p},\mathbb{Z}_p)$ *and put* $\lambda_E:=\lambda\circ\mathrm{Nm}_{E/F}$.

(1) *The sequence* $\{I_{\lambda,d}\}_{d\geqslant1}$ *is a convergent sequence in* $\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}^{K^\dagger}_{\mathbb{O}}$.

(2) *Put* $I_\lambda := \lim_{d\to\infty} I_{\lambda,d}$. *Then*

$$(1\times \boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot I_\lambda) = \mathrm{vol}^\natural(L) \sum_{T_1,T_2\in \mathrm{Herm}_r^\circ(F)^+\times\mathrm{Herm}_r^\circ(F)^+} \lambda_E \left\langle Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1 g_1\phi_1)_L, \zeta_{g_2,T_2}\right\rangle_E \cdot q^{T_1,T_2}$$

holds for every $(g_1,g_2)\in M_r(F_{\mathrm{R}})\times M_r(F_{\mathrm{R}})$.

(3) *The limit* I_λ *belongs to* $O_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathscr{A}^{K^\dagger}_{\mathbb{Z}_{(p)}}$.

Proof. By Lemma 4.39, we have $D^{[e]}_{\sigma\chi}=\sigma D^{[e]}_{\chi}$ hence $D_{\sigma\chi}=\sigma D_\chi$ for every $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{L})$. Thus, $D_{\lambda,d}$ belongs to $O_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathscr{A}^{K^\dagger}_{\mathbb{O}}$.

For every $(g_1,g_2)\in M_r(F_{\mathrm{R}})\times M_r(\mathrm{R})$ and $(T_1,T_2)\in\mathrm{Herm}_r^\circ(F)^+\times\mathrm{Herm}_r^\circ(F)^+$, denote by $(\mathscr{D}_{(g_1,g_2)})_{T_1,T_2}(\chi)$ the q^{T_1,T_2}-th coefficient of $(1\times\boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot D_\chi)$ and ${}^v(\mathcal{D}_{(g_1,g_2)})_{T_1,T_2}$ the q^{T_1,T_2}-th coefficient of $(1\times\boldsymbol{q}^{\mathrm{an}}_{r,r})((g_1,g_2)\cdot {}^vD)$. Since $\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathscr{A}^{K^\dagger}_{\mathbb{O}}$ is a finite dimensional $\mathbb{L}$-vectors space, for the remaining statements of (1,2), it suffices to show that for every $(g_1,g_2)\in M_r(F_{\mathrm{R}})\times M_r(\mathrm{R})$ and $(T_1,T_2)\in\mathrm{Herm}_r^\circ(F)^+\times\mathrm{Herm}_r^\circ(F)^+$, the sequence

$$\left(\sum_{x\in\Gamma_d}\lambda(x)\frac{\chi(x)^{-1}}{|\Gamma_{F,p}/U_d|}\sum_{\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times}(\mathscr{D}_{(g_1,g_2)})_{T_1,T_2}(\chi)\right)+\left(\sum_{v\in\mathrm{S}_\pi}{}^v(\mathcal{D}_{(g_1,g_2)})_{T_1,T_2}\right)\tag{4.20}$$

in $\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}$, converges to $\lambda_E\left\langle Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,\zeta_{g_2,T_2}\right\rangle_E$ when $d\to\infty$.

Without loss of generality, we assume $(g_1,g_2)=(1_r,1_r)$ and suppress it (together with redundant parentheses) from the notation. For every $d\geqslant1$, we may find an element $N=N_d\in\mathbb{N}$ satisfying:

(a) $N!\geqslant e_\chi$ (Lemma 4.43(1)) for every $\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times$;

(b) we have

$$\sum_{x\in\Gamma_d}\lambda(x)\frac{\chi(x)^{-1}}{|\Gamma_{F,p}/U_d|}\sum_{\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times}\left(\mathscr{D}_{T_1,T_2}(\chi)-\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\mathscr{E}^{[N!]}_{T^\square}(\chi)\right)\in p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O};$$

(c) for every $v\in\mathrm{S}_\pi$,

$${}^v\mathcal{D}_{T_1,T_2}-\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}{}^v\mathcal{E}^{[N!]}_{T^\square}\in p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O};$$

(d) $N!\geqslant d+M_u$ for every $u\in\mathrm{P}$, where M_u is the integer from Proposition 4.35 (for u);

(e) we have

$$\mathrm{vol}^\natural(L)\cdot\lambda_E\left\langle Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,\zeta_{g_2,T_2}-p^MZ^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[N!]})_L\right\rangle_E\in p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}.$$

We claim that

$$(4.20)-\mathrm{vol}^\natural(L)\cdot\lambda_E\left\langle Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,\zeta_{g_2,T_2}\right\rangle_E\in p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}.\tag{4.21}$$

Indeed, the left-hand side can be written as the sum of four differences:

$$(4.20)-\left(\sum_{x\in\Gamma_d}\lambda(x)\frac{\chi(x)^{-1}}{|\Gamma_{F,p}/U_d|}\sum_{\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times}\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\mathscr{E}^{[N!]}_{T^\square}(\chi)\right)-\left(\sum_{v\in\mathrm{S}_\pi}\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}{}^v\mathcal{E}^{[N!]}_{T^\square}\right),\tag{4.22}$$

$$\left(\sum_{x\in\Gamma_d}\lambda(x)\frac{\chi(x)^{-1}}{|\Gamma_{F,p}/U_d|}\sum_{\chi\colon\Gamma_{F,p}/U_d\to\overline{\mathbb{Q}}_p^\times}\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\mathscr{E}^{[N!]}_{T^\square}(\chi)\right)-\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\partial\mathscr{E}^{[N!]}_{T^\square}(\mathbf{1}),\tag{4.23}$$

$$\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\left(\partial\mathscr{E}_{T^\square}^{[N!]}(\mathbf{1})+\sum_{v\in\mathtt{S}_\pi}{}^v\mathcal{E}_{T^\square}^{[N!]}\right)-\mathrm{vol}^\natural(L)\cdot\lambda_E\left\langle Z_{T_1}^{\text{ét}}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,p^MZ_{T_2}^{\text{ét}}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[N!]})_L\right\rangle_E, \tag{4.24}$$

$$\mathrm{vol}^\natural(L)\cdot\lambda_E\left\langle Z_{T_1}^{\text{ét}}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,p^MZ_{T_2}^{\text{ét}}(\mathrm{t}_2\mathrm{s}_2g_2\phi_2^{[N!]})_L\right\rangle_E-\mathrm{vol}^\natural(L)\cdot\lambda_E\left\langle Z_{T_1}^{\text{ét}}(\mathrm{t}_1\mathrm{s}_1g_1\phi_1)_L,\zeta_{g_2,T_2}\right\rangle_E. \tag{4.25}$$

By (b) and (c), (4.22) belongs to $p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}$. By Lemma 4.39(1,2) and Lemma 3.42, (4.23) belongs to $p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}$. By (4.11), Proposition 4.33 and Proposition 4.35 (which is applicable by (d)), (4.24) belongs to $p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}$. Finally, by (e), (4.25) belongs to $p^dO_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{O}$. Together, (4.21) holds. Thus, (1) and (2) are proved.

For (3), we see from the above discussion that the q^{T_1,T_2}-th coefficient of $(1\times\boldsymbol{q}_{r,r}^{\mathrm{an}})((g_1,g_2)\cdot I_\lambda)$ is also the limit of

$$\sum_{\substack{T^\square\in\mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\left(\partial(\mathscr{E}_{(g_1,g_2)}^{[N!]})_{T^\square}(\mathbf{1})+\sum_{v\in\mathtt{S}_\pi}{}^v(\mathcal{E}_{(g_1,g_2)}^{[N!]})_{T^\square}\right),$$

which belongs to $O_{\mathbb{L}}$. In other words, the assignment

$$(g_1,g_2)\mapsto(1\times\boldsymbol{q}_{r,r}^{\mathrm{an}})((g_1,g_2)\cdot I_\lambda)$$

belongs to $\mathrm{SF}_{r,r}(O_{\mathbb{L}})^{M_r(F_{\mathtt{R}})\times M_r(F_{\mathtt{R}})}$. It is straightforward to check that

$$\mathbb{L}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{O})^{M_r(F_{\mathtt{R}})\times M_r(F_{\mathtt{R}})}\cap\mathrm{SF}_{r,r}(O_{\mathbb{L}})^{M_r(F_{\mathtt{R}})\times M_r(F_{\mathtt{R}})}=O_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathrm{SF}_{r,r}(\mathbb{Z}_{(p)})^{M_r(F_{\mathtt{R}})\times M_r(F_{\mathtt{R}})}.$$

Then (3) follows from (1) and Lemma 4.42(1) (with $\mathbb{M}=\mathbb{Z}_{(p)}$ and $\mathbb{M}'=\mathbb{O}$). □

Proposition 4.46. *Suppose that*

(a) $n<p$;
(b) $\partial\mathscr{L}_p^\diamondsuit(\pi)(\mathbf{1})\neq 0$;
(c) for every $v\in\mathtt{R}$, there exist $\varphi_v^\vee\in\pi_v^\vee$, $\varphi_v\in\pi_v$ such that $Z(\varphi_v^\vee\otimes\varphi_v,f_{\Phi_v}^{\mathrm{SW}})\neq 0$ (Lemma 3.30).

Then there exists $\lambda\in\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p},\mathbb{Z}_p)$ such that $I_\lambda\neq 0$.

The proof of the above proposition will be given in the next subsection. Now we move to the proof of Theorem 4.21.

Proof of Theorem 4.21. By [LL21, Proposition 3.13], for every $v\in\mathtt{R}$, we may choose a pair $(\phi_{v,1},\phi_{v,2})\in\mathscr{R}_v$ (§4.5(S1)) such that condition (c) in Proposition 4.46 holds. Choose $\lambda\in\mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p},\mathbb{Z}_p)$ such that $I_\lambda\neq 0$ by this proposition. In particular, we may choose some $g_2\in G_r(F_{\mathtt{R}})$ and $T_2\in\mathrm{Herm}_r^\circ(F)^+$, such that the q^{T_2}-th coefficient of $(1,g_2)\cdot I_\lambda$, which we denote by $\varphi_{g_2,T_2,\lambda}$, is nonzero. Since I_λ belongs to $O_{\mathbb{L}}\otimes_{\mathbb{Z}_{(p)}}\mathcal{A}_{\mathbb{Z}_{(p)}}^K$ by Proposition 4.45(3), $\varphi_{g_2,T_2,\lambda}$ is a strongly nonzero element in $\mathbb{L}\otimes_{\mathbb{Q}}\mathcal{A}_{r,\mathrm{hol}}^{[r]}$ (Definition 4.25), which satisfies

$$\begin{aligned}(1\times\boldsymbol{q}_r^{\mathrm{an}})(g_1\cdot\varphi_{g_2,T_2,\lambda})&=\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+}\lambda_E\left\langle Z_{T_1}^{\text{ét}}(g_1\mathrm{t}_1\mathrm{s}_1\phi_1)_L,\zeta_{g_2,T_2}\right\rangle_E\cdot q^{T_1}\\&=\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+}\lambda_E\left\langle\wp_\pi(Z_{T_1}^{\text{ét}}(g_1\mathrm{t}_1\mathrm{s}_1\phi_1)_L),\zeta_{g_2,T_2}\right\rangle_E\cdot q^{T_1}\end{aligned}$$

for every $g_1\in G_r(F_{\mathtt{R}})$ by Proposition 4.45(2). By Lemma 4.27, the above identity indeed holds for every $g_1\in G_r(\mathbb{A}_F^\infty)$. Thus, we may apply Lemma 4.26 with L, $\mathrm{t}_1\mathrm{s}_1\phi_1$, ζ_{g_2,T_2}, λ_E and $\varphi_{g_2,T_2,\lambda}$, hence Theorem 4.21 follows. □

Remark 4.47. Unfortunately, the strategy for proving Theorem 4.21 hence giving an unconditional construction of the Selmer theta lifts can not be applied to give an unconditional construction of the arithmetic theta lifts (on the level of Chow groups) appeared in [LL21,LL22], since our strategy replies on the fact that $\mathrm{H}^1(E,\mathrm{H}^{2r-1}(\overline{X}_L,\mathbb{Q}_p(r)))$ as a $\mathbb{Q}_p[L\backslash H(\mathbb{A}_F^\infty)/L]$-module is semisimple and automorphic – this is not known for $\mathrm{CH}^r(X_L)$.

4.9. **Proof of Theorem 4.22.** In this subsection, we prove Proposition 4.46 and Theorem 4.22. Both proofs require choices of vectors from $\hat{\pi}$ and π, which we do now. Choose decomposable elements $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathcal{V}_{\hat{\pi}}$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \mathcal{V}_\pi$ satisfying

- (T2) $\varphi^\dagger_{1,v} \in (\pi^\vee_v)^-$, $\varphi_{2,v} \in \pi^-_v$ and $\langle \pi^\vee_v(\mathtt{w}_r)\varphi^\dagger_{1,v}, \varphi_{2,v}\rangle_{\pi_v} = q_v^{-d_v r^2}$ for $v \in \mathtt{V}^{(p)}_F$,
- (T3) $\varphi^\dagger_{1,v} \in (\pi^\vee_v)^{K_{r,v}}$, $\varphi_{2,v} \in \pi_v^{K_{r,v}}$ and $\langle \varphi^\dagger_{1,v}, \varphi_{2,v}\rangle_{\pi_v} = 1$ for $v \in \mathtt{T} \setminus \mathtt{S}_\pi$,
- (T4) $\varphi^\dagger_{1,v}, \varphi_{2,v}$ are new vectors[20] with respect to $K_{r,v}$ and $\langle \varphi^\dagger_{1,v}, \varphi_{2,v}\rangle_{\pi_v} = 1$ for $v \in \mathtt{S}_\pi$.

Proposition 4.48. *Suppose that $n < p$. Take an element $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ and regard I_λ as an element of $\mathbb{L} \otimes_{\mathbb{Q}_p} \left(\mathcal{H}^{[r]}_{r,r}(K^\dagger) \otimes_{\mathbb{Q}} \mathbb{C}\right)$ (Definition 2.3). Then*

$$\langle \varphi_1 \otimes \varphi_2, I_\lambda \rangle_{\pi,\hat{\pi}} = p^M \cdot \chi^\diamond_{\hat{\pi}}(\mathrm{t}_1\mathrm{s}_1)\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2) \cdot \partial_\lambda \mathscr{L}^\diamond_p(\pi)(\mathbf{1}) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}),$$

where $\langle\ ,\ \rangle_{\pi,\hat{\pi}}$ is introduced in Notation 3.35 and $Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v})$ is from Lemma 3.30.

Proof. We first compute $\left\langle \varphi_1 \otimes \varphi_2, D_\chi \right\rangle_{\pi,\hat{\pi}}$ and $\langle \varphi_1 \otimes \varphi_2, {}^v D\rangle_{\pi,\hat{\pi}}$.

Let $\chi\colon \Gamma_{F,p} \to \overline{\mathbb{Q}}_p^\times$ be a finite character. By definition, we have

$$\left\langle \varphi_1 \otimes \varphi_2, D_\chi \right\rangle_{\pi,\hat{\pi}} = \lim_{N\to\infty} \left\langle \varphi_1 \otimes \varphi_2, D_\chi^{[N!]} \right\rangle_{\pi,\hat{\pi}}.$$

For the right-hand side, we perform a computation similar to the one in the proof of Theorem 3.37. For every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$, we have

$$\begin{aligned} \iota \left\langle \varphi_1 \otimes \varphi_2, D_\chi^{[N!]} \right\rangle_{\pi,\hat{\pi}} &= \frac{1}{(\mathrm{P}^\iota_\pi)^2} \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} \varphi^\iota_1(g_1^\dagger)\varphi^\iota_2(g_2^\dagger) E^{[N!]}_{\iota\chi}((g_1,g_2))\,\mathrm{d}g_1\,\mathrm{d}g_2 \\ &= \frac{1}{(\mathrm{P}^\iota_\pi)^2} \iint\limits_{(G_r(F)\backslash G_r(\mathbb{A}_F))^2} (\varphi^\dagger_1)^\iota(g_1)\varphi^\iota_2(g_2) E^{[N!]}_{\iota\chi}(\imath(g_1,g_2))\,\mathrm{d}g_1\,\mathrm{d}g_2 \end{aligned} \tag{4.26}$$

by Lemma 4.43(1) and Lemma 4.40(1). By the doubling integral expansion and Lemma 3.31,

$$\begin{aligned} (4.26) = p^M \cdot \iota\chi^\diamond_{\hat{\pi}}(\mathrm{t}_1\mathrm{s}_1)\iota\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2) \cdot \frac{1}{\mathrm{P}^\iota_\pi} \cdot \frac{Z_r^{[F:\mathbb{Q}]}}{b^\diamond_{2r}(\mathbf{1})} \\ \times L(\tfrac{1}{2}, \mathrm{BC}(\iota\pi^\diamond) \otimes (\iota\chi^\diamond \circ \mathrm{Nm}_{E/F})) \cdot \prod_{v\in\mathtt{V}_F^{(p)}} Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, (\mathfrak{f}^{[N!]}_{\iota\chi_v})^{\iota\chi_v}) \prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\iota\chi_v}). \end{aligned}$$

By (T2) and Lemma 3.11, for every $v \in \mathtt{V}^{(p)}_F$,

$$Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, (\mathfrak{f}^{[N!]}_{\iota\chi_v})^{\iota\chi_v}) = \left(\iota \prod_{u \in \mathrm{P}_v} \alpha(\pi_u)\right)^{-N!} Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, (\mathfrak{f}^{[0]}_{\iota\chi_v})^{\iota\chi_v}).$$

By Proposition 3.32 and (T2), for every $v \in \mathtt{V}^{(p)}_F$,

$$Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, (\mathfrak{f}^{[0]}_{\iota\chi_v})^{\iota\chi_v}) = \prod_{u\in\mathrm{P}_v} \gamma(\tfrac{1+r}{2}, \iota\underline{\pi_u} \otimes \chi_v, \psi_{F,v})^{-1}.$$

---

[20] A new vector in an almost unramified representation of $G_r(F_v)$ is a vector in the (one-dimensional) space in [Liu22, Definition 5.3(2)].

Together, we have

$$(4.26) = \frac{1}{\mathrm{P}^\iota_\pi} \cdot \frac{Z_r^{[F:\mathbb{Q}]}}{b^\diamond_{2r}(\mathbf{1})} \cdot \prod_{v\in \mathtt{V}_F^{(p)}} \prod_{u \in \mathrm{P}_v} \gamma(\tfrac{1+r}{2}, \iota(\underline{\pi_u} \otimes \chi_v), \psi_{F,v})^{-1} \cdot L(\tfrac{1}{2}, \mathrm{BC}(\iota\pi^\diamond) \otimes (\iota\chi^\diamond \circ \mathrm{Nm}_{E/F}))$$
$$\times p^M \cdot \iota\chi^\diamond_{\hat\pi}(\mathrm{t}_1\mathrm{s}_1)\iota\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2) \cdot \left(\iota \prod_{u\in\mathrm{P}} \alpha(\pi_u)\right)^{-N!} \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z^\iota(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\iota\chi_v}),$$

which, by Theorem 3.37 and Lemma 3.30, equals

$$p^M \cdot \iota\left(\chi^\diamond_{\hat\pi}(\mathrm{t}_1\mathrm{s}_1)\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2)\right) \cdot \iota\mathscr{L}^\diamond_p(\pi)(\chi) \cdot \iota\left(\prod_{u\in\mathrm{P}} \alpha(\pi_u)^{-N!}\right) \cdot \iota\left(\prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\chi_v})\right).$$

As a consequence, we have

$$\left\langle \varphi_1 \otimes \varphi_2, D^{[N!]}_\chi \right\rangle_{\pi,\hat\pi} = p^M \cdot \chi^\diamond_{\hat\pi}(\mathrm{t}_1\mathrm{s}_1)\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2) \cdot \mathscr{L}^\diamond_p(\pi)(\chi) \cdot \left(\prod_{u\in\mathrm{P}} \alpha(\pi_u)^{-N!}\right) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\chi_v}),$$

hence

$$\left\langle \varphi_1 \otimes \varphi_2, D_\chi \right\rangle_{\pi,\hat\pi} = p^M \cdot \chi^\diamond_{\hat\pi}(\mathrm{t}_1\mathrm{s}_1)\chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2) \cdot \mathscr{L}^\diamond_p(\pi)(\chi) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\chi_v}). \tag{4.27}$$

By a similar argument, for every $v \in \mathtt{S}_\pi$, we have

$$\left\langle \varphi_1 \otimes \varphi_2, {}^vD^{[N!]} \right\rangle_{\pi,\hat\pi} = 0$$

since $Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f^{\mathrm{sph}}_{\mathbf{1}_v}) = 0$. Thus, $\langle \varphi_1 \otimes \varphi_2, {}^vD \rangle_{\pi,\hat\pi} = 0$.

Now the proposition follows from (4.27), (3.19), and the p-adic Leibniz rule. □

Lemma 4.49. *For every $v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})} \setminus \mathtt{R}$, we have $Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}) \neq 0$.*

Proof. By [Liu22, Proposition 5.6] and (T4) when $v \in \mathtt{S}_\pi$, [LL22, Proposition 3.6] and (T3) when $v \in \mathtt{V}_F^{\mathrm{ram}}$, Lemma 3.31 and (T3) when $v \in \mathtt{T} \setminus (\mathtt{S}_\pi \cup \mathtt{V}_F^{\mathrm{ram}})$, we have

$$Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}) = C_v \cdot \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_v))}{b_{2r,v}(\mathbf{1})}$$

for a constant $C_v \in \mathbb{Q}^\times$. Then the nonvanishing is clear. □

Proof of Proposition 4.46. We would like to apply Proposition 4.48. By condition (c), for every $v \in \mathtt{R}$, we may find $\varphi^\vee_v \in \pi^\vee_v$, $\varphi_v \in \pi_v$ such that $Z(\varphi^\vee_v \otimes \varphi_v, f^{\mathrm{SW}}_{\Phi_v}) \neq 0$, that is, $Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}) \neq 0$. Together with Lemma 4.49, we have $\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}) \neq 0$. By condition (b), there exists $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ such that $\partial_\lambda \mathscr{L}^\diamond_p(\pi)(\mathbf{1}) \neq 0$. Thus, by Proposition 4.48, $I_\lambda \neq 0$. The proposition is proved. □

Proof of Theorem 4.22. For (1), we may apply Theorem 4.21 so that we have elements $\mathcal{Z}^\pi_{\phi_1,L}$ and $\mathcal{Z}^{\hat\pi}_{\phi_2,L}$ from Proposition 4.19. By Remark 4.23(2), it suffices to show (4.6) for a single choice of data $(\varphi_1, \varphi_2, \phi_1, \phi_2)$ (as in the statement of Theorem 4.22) satisfying $\prod_{v\in\mathtt{V}_F^{(\diamond\setminus\{\infty\})}} Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}) \neq 0$. Thus, by Lemma 4.49, it suffices to show (4.6) for our particular choices of $(\phi_1, \phi_2 \coloneqq \phi_2^{[0]})$ as in (4.17) and (φ_1, φ_2) from (T2–T4), together satisfying the following extra requirement

(T1) $Z(\varphi^\dagger_{1,v} \otimes \varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}) \neq 0$ for $v \in \mathtt{R}$.

This is possible by [LL21, Proposition 3.13].

By Remark 4.38 and Lemma 4.29(1),

$$\Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1)_L = \chi^\diamond_{\hat\pi}(\mathrm{t}_1\mathrm{s}_1)^{-1}\Theta^{\mathrm{Sel}}_{\mathrm{t}_1\mathrm{s}_1\phi_1}(\varphi_1)_L, \quad \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)_L = \chi^\diamond_\pi(\mathrm{t}_2\mathrm{s}_2)^{-1}\left(\prod_{u\in\mathrm{P}} \alpha(\pi_u)^{-e}\right)\Theta^{\mathrm{Sel}}_{\mathrm{t}_2\mathrm{s}_2\phi_2^{[e]}}(\varphi_2)_L$$

hold for every $e \in \mathbb{N}$. By Definition 4.20,

$$\Theta^{\mathrm{Sel}}_{\mathrm{t}_1\mathrm{s}_1\phi_1}(\varphi_1)_L = \langle \varphi_1^{\dagger}, \mathcal{Z}^{\pi}_{\mathrm{t}_1\mathrm{s}_1\phi_1,L}\rangle_{\pi}, \quad \Theta^{\mathrm{Sel}}_{\mathrm{t}_2\mathrm{s}_2\phi_2^{[e]}}(\varphi_2)_L = \langle \varphi_2^{\dagger}, \mathcal{Z}^{\hat{\pi}}_{\mathrm{t}_2\mathrm{s}_2\phi_2^{[e]},L}\rangle_{\hat{\pi}},$$

in which

$$\mathcal{Z}^{\pi}_{\mathrm{t}_1\mathrm{s}_1\phi_1,L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\hat{\pi},L}) \otimes_{\mathbb{L}} \left(\mathcal{V}_{\hat{\pi}} \otimes_{\mathbb{Q}} \mathbb{M}\right), \quad \mathcal{Z}^{\hat{\pi}}_{\mathrm{t}_2\mathrm{s}_2\phi_2^{[e]},L} \in \mathrm{H}^1_f(E, \mathrm{V}_{\hat{\pi},L}) \otimes_{\mathbb{L}} \left(\mathcal{V}_{\hat{\pi}} \otimes_{\mathbb{Q}} \mathbb{M}\right)$$

for some field $\mathbb{M} \subseteq \mathbb{C}$. Indeed, for given $\mathtt{R}$-components of ϕ_1 and ϕ_2, we can shrink $\mathbb{M}$ to a number field, which is in particular independent of e. Again by Lemma 4.29(1), the sequence $\{\mathcal{Z}^{\hat{\pi}}_{\mathrm{t}_2\mathrm{s}_2\phi_2^{[N!]},L}\}$ converges when $N \to \infty$, whose limit we simply denote by $\mathcal{Z}_2$. Then

$$\Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)_L = \chi^{\diamond}_{\pi}(\mathrm{t}_2\mathrm{s}_2)^{-1}\langle \varphi_2^{\dagger}, \mathcal{Z}_2\rangle_{\hat{\pi}}$$

Therefore, for every element $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ with $\lambda_E \coloneqq \lambda \circ \mathrm{Nm}_{F/F}$,

$$\begin{aligned} \lambda\langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^{\natural}_{\pi,F} &= \mathrm{vol}^{\natural}(L) \cdot \lambda_E \langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1)_L, \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)_L\rangle_E \\ &= \chi^{\diamond}_{\hat{\pi}}(\mathrm{t}_1\mathrm{s}_1)^{-1}\chi^{\diamond}_{\pi}(\mathrm{t}_2\mathrm{s}_2)^{-1} \cdot \left\langle \varphi_1 \otimes \varphi_2, \mathrm{vol}^{\natural}(L) \cdot \lambda_E \left\langle \mathcal{Z}^{\pi}_{\mathrm{t}_1\mathrm{s}_1\phi_1,L}, \mathcal{Z}_2 \right\rangle_E \right\rangle_{\pi,\hat{\pi}} \end{aligned} \tag{4.28}$$

By Proposition 4.19,

$$\begin{aligned} \boldsymbol{q}_{r,r}(g_1 \cdot \mathcal{Z}^{\pi}_{\mathrm{t}_1\mathrm{s}_1\phi_1,L}) &= \sum_{T_1 \in \mathrm{Herm}^{\circ}_r(F)^+} Z^{\text{ét}}_{T_1}(\mathrm{t}_1\mathrm{s}_1 g_1 \phi_1)_L q^{T_1}, \\ \boldsymbol{q}_{r,r}(g_2 \cdot \mathcal{Z}_2) &= p^{-M} \sum_{T_2 \in \mathrm{Herm}^{\circ}_r(F)^+} \zeta_{g_2,T_2} \end{aligned}$$

hold for every pair $(g_1, g_2) \in M_r(F_{\mathtt{R}}) \times M_r(F_{\mathtt{R}})$, where we recall that $\zeta_{g_2,T_2} = p^M \lim\limits_{N\to\infty} Z^{\text{ét}}_{T_2}(\mathrm{t}_2\mathrm{s}_2 g_2 \phi_2^{[N!]})_L$.

Thus, by Proposition 4.45(2), we have

$$\mathrm{vol}^{\natural}(L) \cdot \lambda_E \left\langle \mathcal{Z}^{\pi}_{\mathrm{t}_1\mathrm{s}_1\phi_1,L}, \mathcal{Z}_2 \right\rangle_E = p^{-M} I_{\lambda}.$$

By Proposition 4.48,

$$(4.28) = \chi^{\diamond}_{\hat{\pi}}(\mathrm{t}_1\mathrm{s}_1)^{-1}\chi^{\diamond}_{\pi}(\mathrm{t}_2\mathrm{s}_2)^{-1} \cdot \left\langle \varphi_1 \otimes \varphi_2, p^{-M} I_{\lambda} \right\rangle_{\pi,\hat{\pi}} = \partial_{\lambda}\mathscr{L}^{\diamond}_p(\pi)(\mathbf{1}) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^{\dagger}_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}).$$

In other words,

$$\begin{aligned} \langle \Theta^{\mathrm{Sel}}_{\phi_1}(\varphi_1), \Theta^{\mathrm{Sel}}_{\phi_2}(\varphi_2)\rangle^{\natural}_{\pi,F} &= \partial\mathscr{L}^{\diamond}_p(\pi)(\mathbf{1}) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^{\dagger}_{1,v} \otimes \varphi_{2,v}, f_{\mathbf{1}_v}) \\ &= \partial\mathscr{L}^{\diamond}_p(\pi)(\mathbf{1}) \cdot \prod_{v \in \mathtt{V}_F^{(\diamond\setminus\{\infty,p\})}} Z(\varphi^{\dagger}_{1,v} \otimes \varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}). \end{aligned}$$

Finally, by Proposition 3.32 and (T2),

$$Z(\varphi^{\dagger}_{1,v} \otimes \varphi_{2,v}, f^{\mathrm{SW}}_{\phi_{1,v}\otimes\phi_{2,v}}) = \prod_{u \in \mathrm{P}_v} \gamma(\tfrac{1+r}{2}, \underline{\pi_u}, \psi_{F,v})^{-1}$$

for every $v \in \mathtt{V}_F^{(p)}$. Together, we obtain (4.6). Part (1) is proved.

For (2), it suffices to show the vanishing under every embedding $\iota\colon \mathbb{L} \to \mathbb{C}$. Thus, we may regard $\mathbb{L}$ as a subfield of $\mathbb{C}$ and π as defined over $\mathbb{C}$. For every $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\Gamma_{F,p}, \mathbb{Z}_p)$ with $\lambda_E \coloneqq \lambda \circ \mathrm{Nm}_{E/F}$, we have a map

$$\boldsymbol{i}^{\lambda}_{r,r}\colon \mathscr{S}(V^{2r} \otimes_F \mathbb{A}_F^{\infty}) = \mathscr{S}(V^r \otimes_F \mathbb{A}_F^{\infty}) \otimes_{\mathbb{C}} \mathscr{S}(V^r \otimes_F \mathbb{A}_F^{\infty}) \to \mathcal{SF}_{r,r}(\mathbb{C})$$

(Definition 2.6) of $\mathbb{C}[G_{r,r}(\mathbb{A}_F^{\infty})]$-modules sending (ϕ_1, ϕ_2) to the assignment

$$(g_1, g_2) \mapsto \sum_{(T_1,T_2) \in \mathrm{Herm}_r(F)^+ \times \mathrm{Herm}_r(F)^+} \lambda_E \left\langle \wp_{\pi}\left(Z^{\text{ét}}_{T_1}(\omega_r(g_1)\phi_1)_L\right), \wp_{\hat{\pi}}\left(Z^{\text{ét}}_{T_2}(\omega_r(g_2)\phi_2)_L\right)\right\rangle_E \cdot q^{T_1,T_2}.$$

We prove (2) by contradiction. Assume the opposite hence $\boldsymbol{i}^{\lambda}_{r,r}$ is nontrivial for some λ. Then it is clear from the construction that $\boldsymbol{i}^{\lambda}_{r,r}$ factors through successive $G_{r,r}(\mathbb{A}^{\infty}_F)$-equivariant quotient maps

$$\mathscr{S}(V^{2r}\otimes_F \mathbb{A}^{\infty}_F)\to \mathrm{I}^{\square}_r(\mathbf{1})=\prod_{v\in \mathtt{V}^{\infty}_F}\mathrm{I}^{\square}_{r,v}(\mathbf{1})\to \pi\boxtimes\hat{\pi}.$$

We claim that the image of $\boldsymbol{i}^{\lambda}_{r,r}$ is contained in $\boldsymbol{q}^{\infty}_{r,r}\mathcal{A}^{[r]}_{r,r,\mathrm{hol}}$ (Definition 2.6). By [LL22, Proposition 4.8(1)], it suffices to show that $\boldsymbol{i}^{\lambda}_{r,r}(\phi_1,\phi_2)\in \boldsymbol{q}^{\infty}_{r,r}\mathcal{A}^{[r]}_{r,r,\mathrm{hol}}$ for one choice of pair (ϕ_1,ϕ_2) such that $\phi_1\otimes\phi_2$ has nonzero image under the unique nontrivial map in $\mathrm{Hom}_{G_r(\mathbb{A}^{\infty}_F)\times G_r(\mathbb{A}^{\infty}_F)}(\mathrm{I}^{\square}_r(\mathbf{1}),\pi\boxtimes\hat{\pi})$. Indeed, we choose the pair to be $(\mathrm{t}_1\mathrm{s}_1\phi_1,\mathrm{t}_2\mathrm{s}_2\phi'_2)$ in which ϕ_1 and $(\phi'_2)^p$ (away-from-p part) are from the proof of (1), and $\phi'_{2,v}$ for $v\in\mathtt{V}^{(p)}_F$ is an arbitrary element in $\mathscr{S}(V^r_v)$ whose image in the quotient $\hat{\pi}_v\boxtimes\theta(\hat{\pi}_v)$ (Lemma 4.1) is the limit of the images of $\phi^{[N!]}_{v,2}$ in that quotient when $N\to\infty$ (which exists by Lemma 4.29(1)). By Lemma 4.27 (applied to both variables), it suffices to show that there exists $J_\lambda\in\mathcal{A}^{[r]}_{r,r,\mathrm{hol}}$ such that

$$\begin{aligned}\boldsymbol{q}^{\mathrm{an}}_{r,r}((g_1,g_2)\cdot J_\lambda)&=\sum_{(T_1,T_2)\in\mathrm{Herm}_r(F)^+\times\mathrm{Herm}_r(F)^+}\lambda_E\left\langle\wp_\pi\left(Z^{\text{ét}}_{T_1}(\omega_r(g_1)\mathrm{t}_1\mathrm{s}_1\phi_1)_L\right),\wp_{\hat{\pi}}\left(Z^{\text{ét}}_{T_2}(\omega_r(g_2)\mathrm{t}_2\mathrm{s}_2\phi'_2)_L\right)\right\rangle_E\cdot q^{T_1,T_2}\\&=\sum_{(T_1,T_2)\in\mathrm{Herm}_r(F)^+\times\mathrm{Herm}_r(F)^+}\lambda_E\left\langle Z^{\text{ét}}_{T_1}(\omega_r(g_1)\mathrm{t}_1\mathrm{s}_1\phi_1)_L,Z^{\text{ét}}_{T_2}(\omega_r(g_2)\mathrm{t}_2\mathrm{s}_2\phi'_2)_L\right\rangle_E\cdot q^{T_1,T_2}\end{aligned}$$

for every pair $(g_1,g_2)\in M_r(F_{\mathrm{R}})\times M_r(F_{\mathrm{R}})$. Then by Proposition 4.45(2), we may take J_λ to be $\mathrm{vol}^{\natural}(L)^{-1}p^{-M}\cdot I_\lambda$ (regarded as an element of $\mathcal{A}^{[r]}_{r,r,\mathrm{hol}}$). It remains to show that I_λ vanishes hence the map $\boldsymbol{i}^{\lambda}_{r,r}$ vanishes, resulting in a contradiction. Once again, since $\boldsymbol{i}^{\lambda}_{r,r}$ factors through $\pi\boxtimes\hat{\pi}$, it suffices to show that $\langle\varphi_1\otimes\varphi_2,I_\lambda\rangle_{\pi,\hat{\pi}}=0$ for a single (decomposable) pair (φ_1,φ_2) such that $Z(\varphi^{\dagger}_{1,v}\otimes\varphi_{2,v},f_{\mathbf{1}_v})\neq 0$ for every $v\in\mathtt{V}^{\mathrm{fin}}_F$. Indeed, we can just take (φ_1,φ_2) to be the pair from the proof of (1). Then the vanishing of $\langle\varphi_1\otimes\varphi_2,I_\lambda\rangle_{\pi,\hat{\pi}}$ follows from Proposition 4.48, since we have assumed that the vanishing order of $\mathscr{L}^{\diamondsuit}_p(\pi)$ at $\mathbf{1}$ is at least one. Part (2) is proved. □

APPENDIX A. BI-EXTENSIONS AND p-ADIC HEIGHT PAIRINGS

In this appendix, we develop further the theory of p-adic heights on general varieties. We fix a prime number p and an integer $n\geqslant 2$. Moreover, $\mathbb{W}$ denotes a finite flat local extension of $\mathbb{Z}_p$ and $\mathbb{L}$ denotes a finite product of finite extensions of $\mathrm{Frac}(\mathbb{W})$.

A.1. **Étale correspondences.**

Let X be a scheme. An *étale correspondence* on X is a diagram

$$t\colon X\xleftarrow{f}X'\xrightarrow{g}X$$

in which both f and g are finite étale morphisms.

The collection of all étale correspondences on X forms a monoidal category $\text{ÉtCor}(X)$. See [Liu19, Definition 2.11] for more details. We denote by $\mathrm{EC}(X)$ the (unital) $\mathbb{Z}$-algebra generated by the underlying monoid of isomorphism classes of objects of $\text{ÉtCor}(X)$. For every ring R, an *R-ring of étale correspondences* on X is an R-ring $\mathbb{T}$ together with a homomorphism $\mathbb{T}\to\mathrm{EC}(X)_R$ that is R-linear and unital. Usually, we only write $\mathbb{T}$ for the notation when the homomorphism $\mathbb{T}\to\mathrm{EC}(X)_R$ is clear.

Notation A.1. Let S be a subset of X. For an étale correspondence t as above, we put $S^t:=f(g^{-1}(S))$. For a finite linear combination $t=\sum c_it_i$ with $c_i\neq 0$, we put $S^t:=\bigcup_i S^{t_i}$.

A.2. **Remarks on sheaves.**

Let S be a site. For a diagram of rings $R_\bullet$, we denote by

- $\mathbf{M}(S,R_\bullet)$ the abelian category of sheaves of $R_\bullet$-modules on S,
- $\mathbf{C}^+(S,R_\bullet)=\mathrm{C}^+(\mathbf{M}(S,R_\bullet))$ the abelian category of bounded below complexes in $\mathbf{M}(S,R_\bullet)$,
- $\mathbf{D}^+(S,R_\bullet)=\mathrm{D}^+(\mathbf{M}(S,R_\bullet))$ the derived category of $\mathbf{M}(S,R_\bullet)$ with bounded below cohomology.

In this article, we mainly use two kinds of diagrams of rings. The first is a singleton valued in a ring R so that $\mathbf{M}(S,R_\bullet)=\mathbf{M}(S,R)$ is the usual category of sheaves of R-modules on S. The second is the diagram

$$\mathbb{W}_\bullet\colon\ \cdots\to\mathbb{W}/p^3\to\mathbb{W}/p^2\to\mathbb{W}/p$$

so that $\mathbf{M}(S,\mathbb{W}_\bullet)$ consists of sequences $F_\bullet=(\cdots\to F_3\to F_2\to F_1)$ of sheaves of $\mathbb{W}$-modules on S in which F_l is annihilated by p^l. Then we have the left-exact functor

$$\varprojlim\colon \mathbf{M}(S,\mathbb{W}_\bullet)\to\mathbf{M}(S,\mathbb{W})$$

and the exact restriction functor

$$-_l\colon \mathbf{M}(S,\mathbb{W}_\bullet)\to\mathbf{M}(S,\mathbb{W}/p^l)$$

for every $l\geqslant 1$.

In what follows, we suppress S (together with the comma after it) in the notation when S is a point. We have the global section functor $\Gamma(S,-)\colon \mathbf{M}(S,R_\bullet)\to\mathbf{M}(R_\bullet)$. For every $q\in\mathbb{Z}$, denote by $\mathbf{H}^q(S,-)$ the q-th cohomology of $\mathrm{R}\Gamma(S,-)$.

Example A.2. For a scheme X and an integer r, we have the object $\mu_{p^\bullet}^{\otimes r}\in\mathbf{M}(X_{\text{ét}},\mathbb{Z}_{p^\bullet})$ and put

$$\mathbb{L}(r)_X\coloneqq\left(\mathrm{R}\varprojlim\left(\mu_{p^\bullet}^{\otimes r}\right)_X\right)\otimes_{\mathbb{Z}_p}\mathbb{L}\in\mathbf{D}^+(X_{\text{ét}},\mathbb{L}).$$

Then $\mathbf{H}^q(X_{\text{ét}},\mathbb{L}(r)_X)$ coincides with Jannsen's continuous étale cohomology [Jan88] of X (with coefficients in $\mathbb{L}$), usually denoted by $\mathrm{H}^q(X,\mathbb{L}(r))$. For a relative scheme $\pi\colon X\to S$ of finite type, $\mathbf{H}^q(S_{\text{ét}},\mathrm{R}\pi_!\mathbb{L}(r)_X)$ coincides with the continuous cohomology of X/S with proper support, usually denoted by $\mathrm{H}^q_c(X,\mathbb{L}(r))$ when the base S is clear.

Now let G be a *profinite* group that acts on S. We similarly define three categories $\mathbf{M}_{\mathrm{G}}(S,R_\bullet)$, $\mathbf{C}^+_{\mathrm{G}}(S,R_\bullet)$, and $\mathbf{D}^+_{\mathrm{G}}(S,R_\bullet)$ for compatible G-equivariant $R_\bullet$-modules on S. We have the global section functor $\Gamma(S,-)\colon\mathbf{M}_{\mathrm{G}}(S,R_\bullet)\to\mathbf{M}_{\mathrm{G}}(R_\bullet)$ and the G-invariants functor $\Gamma_{\mathrm{G}}\colon\mathbf{M}_{\mathrm{G}}(R_\bullet)\to\mathbf{M}(R_\bullet)$. For every $q\in\mathbb{Z}$, denote by $\mathbf{H}^q_{\mathrm{G}}(S,-)$ the q-th cohomology of $\mathrm{R}\Gamma_{\mathrm{G}}\circ\mathrm{R}\Gamma(S,-)$. The natural transformation from Γ_{G} to the forgetful functor (forgetting the G-action) induces, for every $q\in\mathbb{Z}$, a functor $\mathbf{H}^q_{\mathrm{G}}(S,-)\to\mathbf{H}^q(S,-)$, whose kernel we denote by $\mathbf{H}^q_{\mathrm{G}}(S,-)^0$.

For an object $\mathscr{C}\in\mathbf{D}^+_{\mathrm{G}}(S,\mathbb{W}_\bullet)$, we put

$$\mathscr{C}_{\mathbb{L}}\coloneqq\left(\mathrm{R}\varprojlim\mathscr{C}\right)\otimes_{\mathbb{W}}\mathbb{L}\in\mathbf{D}^+_{\mathrm{G}}(S,\mathbb{L}).\tag{A.1}$$

Definition A.3. We say that an object $\mathscr{C}\in\mathbf{D}^+_{\mathrm{G}}(S,\mathbb{W}_\bullet)$ is *admissible* if

(1) $\mathbf{H}^q(S,\mathscr{C}_l)$ is a discrete G-module for every $q\in\mathbb{Z}$ and $l\geqslant 1$, that is, every element of $\mathbf{H}^q(S,\mathscr{C}_l)$ has an open stabilizer;
(2) $\mathrm{R}^1\varprojlim_l\mathbf{H}^q(S,\mathscr{C}_l)$ has finite exponent for every $q\in\mathbb{Z}$.

It is clear that if $\mathscr{C}$ is admissible, then the natural map $\mathbf{H}^q(S,\mathscr{C}_{\mathbb{L}})\to\left(\varprojlim_l\mathbf{H}^q(S,\mathscr{C}_l)\right)\otimes_{\mathbb{W}}\mathbb{L}$ is an isomorphism of G-modules, through which we view $\mathbf{H}^q(S,\mathscr{C}_{\mathbb{L}})$ as a topological G-module. The following lemma slightly generalizes [Jan88, Corollary 3.4] in the case of rational coefficients.

Lemma A.4. *Let $\mathscr{C}\in\mathbf{D}^+_{\mathrm{G}}(S,\mathbb{Z}_{p^\bullet})$ be an admissible object. Then there is a spectral sequence*

$$\mathrm{H}^p_{\mathrm{cont}}(\mathrm{G},\mathbf{H}^q(S,\mathscr{C}_{\mathbb{L}}))\Rightarrow\mathbf{H}^{p+q}_{\mathrm{G}}(S,\mathscr{C}_{\mathbb{L}}).$$

In particular, we have the edge map

$$\mathbf{H}^q_{\mathrm{G}}(S,\mathscr{C}_{\mathbb{L}})^0\to\mathrm{H}^1_{\mathrm{cont}}(\mathrm{G},\mathbf{H}^{q-1}(S,\mathscr{C}_{\mathbb{L}})).$$

Proof. We first note that $(\Gamma_{\mathrm{G}}\circ\Gamma(S,-))\circ\varprojlim=(\Gamma_{\mathrm{G}}\circ\varprojlim)\circ\Gamma(S,-)$ and all of the three functors preserve injectives. By the same argument for [Jan88, Theorem 3.3], there is a spectral sequence

$$\mathrm{H}^p(\mathrm{G},(\mathbf{H}^q(S,\mathscr{C}_l)))\Rightarrow\mathbf{H}^{p+q}_{\mathrm{G}}(S,\mathrm{R}\varprojlim\mathscr{C})$$

which is simply the Grothendieck spectral sequence for the composition $\mathrm{R}(\Gamma_{\mathrm{G}}\circ\varprojlim)\circ\mathrm{R}\Gamma(S,-)$. By the similar argument for [Jan88, Theorem 2.2] and Definition A.3, there is a canonical isomorphism

$$\mathrm{H}^p(\mathrm{G},(\mathbf{H}^q(S,\mathscr{C}_l)))\otimes_{\mathbb{W}}\mathbb{L}\simeq\mathrm{H}^p_{\mathrm{cont}}(\mathrm{G},\mathbf{H}^q(S,\mathscr{C}_{\mathbb{L}})).$$

The lemma then follows as $\mathbf{H}^{p+q}_{\mathrm{G}}(S,\mathrm{R}\varprojlim\mathscr{C})\otimes_{\mathbb{W}}\mathbb{L}=\mathbf{H}^{p+q}_{\mathrm{G}}(S,\mathscr{C}_{\mathbb{L}})$. □

The lemma below will only be used in §B.6.

Lemma A.5. *Let S be a site with an action of* G *and q an integer. Consider a distinguished triangle*

$$\mathscr{A} \xrightarrow{\mu} \mathscr{B} \xrightarrow{\nu} \mathscr{C} \xrightarrow{+1}$$

of admissible objects in $\mathbf{D}^+_{\mathrm{G}}(S, \mathbb{W}_\bullet)$*, inducing the following commutative diagram*

$$\begin{tikzcd}
\mathbf{H}^{q-1}_{\mathrm{G}}(S, \mathscr{C}_{\mathbb{L}}) \arrow[r, "\lambda"] \arrow[d] & \mathbf{H}^{q}_{\mathrm{G}}(S, \mathscr{A}_{\mathbb{L}}) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \mathbf{H}^{q}_{\mathrm{G}}(S, \mathscr{B}_{\mathbb{L}}) \arrow[r, "\nu"] \arrow[d, "\beta"] & \mathbf{H}^{q}_{\mathrm{G}}(S, \mathscr{C}_{\mathbb{L}}) \arrow[d, "\gamma"] \\
\mathbf{H}^{q-1}(S, \mathscr{C}_{\mathbb{L}}) \arrow[r, "\overline{\lambda}"] & \mathbf{H}^{q}(S, \mathscr{A}_{\mathbb{L}}) \arrow[r, "\overline{\mu}"] & \mathbf{H}^{q}(S, \mathscr{B}_{\mathbb{L}}) \arrow[r, "\overline{\nu}"] & \mathbf{H}^{q}(S, \mathscr{C}_{\mathbb{L}})
\end{tikzcd}$$

in which all vertical maps are induced from forgetting G*-actions. Then for every element $b \in \mathbf{H}^q_{\mathrm{G}}(S, \mathscr{B}_{\mathbb{L}})$ satisfying $\gamma(\nu(b)) = 0$, the image of $\nu(b)$ under the composite map*

$$\mathbf{H}^q_{\mathrm{G}}(S, \mathscr{C}_{\mathbb{L}})^0 \to \mathrm{H}^1_{\mathrm{cont}}(\mathrm{G}, \mathbf{H}^{q-1}(S, \mathscr{C}_{\mathbb{L}})) \to \mathrm{H}^1_{\mathrm{cont}}\left(\mathrm{G}, \frac{\mathbf{H}^{q-1}(S, \mathscr{C}_{\mathbb{L}})}{\mathbf{H}^{q-1}(S, \mathscr{B}_{\mathbb{L}})}\right)$$

can be represented by the (continuous) 1*-cocycle $g \mapsto g\overline{a} - \overline{a}$ for $g \in \mathrm{G}$, where $\overline{a}$ is an arbitrary element in $\mathbf{H}^q(S, \mathscr{A}_{\mathbb{L}})$ satisfying $\overline{\mu}(\overline{a}) = \beta(b)$.*

Proof. It is easy to check that the 1-cocycle does not depend on the choice of $\overline{x}$. Thus, it suffices to check the statement for one such element.

Take an injective resolution $0 \to A^\bullet_\bullet \xrightarrow{\mu^\bullet} B^\bullet_\bullet \xrightarrow{\nu^\bullet} C^\bullet_\bullet \to 0$ in $\mathbf{C}^+_{\mathrm{G}}(\mathbb{W}_\bullet)$ (the bullet in the superscript denotes the cohomological degree) of the exact triangle

$$\mathrm{R}\Gamma(S, \mathscr{A}) \to \mathrm{R}\Gamma(S, \mathscr{B}) \to \mathrm{R}\Gamma(S, \mathscr{C}) \xrightarrow{+1} .$$

Put $X^\bullet_{\mathbb{L}} := \left(\varprojlim_l X^\bullet_l\right) \otimes_{\mathbb{W}} \mathbb{L} \in \mathbf{C}^+_{\mathrm{G}}(\mathbb{L})$ for $X = A, B, C$. Then the diagram in the statement can be replaced by

$$\begin{tikzcd}
\mathrm{H}^{q-1}\left((C^\bullet_{\mathbb{L}})^{\mathrm{G}}\right) \arrow[r, "\lambda"] \arrow[d] & \mathrm{H}^{q}\left((A^\bullet_{\mathbb{L}})^{\mathrm{G}}\right) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \mathrm{H}^{q}\left((B^\bullet_{\mathbb{L}})^{\mathrm{G}}\right) \arrow[r, "\nu"] \arrow[d, "\beta"] & \mathrm{H}^{q}\left((C^\bullet_{\mathbb{L}})^{\mathrm{G}}\right) \arrow[d, "\gamma"] \\
\mathrm{H}^{q-1}\left(C^\bullet_{\mathbb{L}}\right) \arrow[r, "\overline{\lambda}"] & \mathrm{H}^{q}\left(A^\bullet_{\mathbb{L}}\right) \arrow[r, "\overline{\mu}"] & \mathrm{H}^{q}\left(B^\bullet_{\mathbb{L}}\right) \arrow[r, "\overline{\nu}"] & \mathrm{H}^{q}\left(C^\bullet_{\mathbb{L}}\right)
\end{tikzcd}$$

in which all vertical arrows are induced by natural inclusions. By definition, we have

$$\mathbf{H}^q_{\mathrm{G}}(S, \mathscr{C}_{\mathbb{L}})^0 = \mathrm{H}^q\left((C^\bullet_{\mathbb{L}})^{\mathrm{G}}\right)^0 = \frac{\mathrm{Ker}\left((E^q_{\mathbb{L}})^{\mathrm{G}} \xrightarrow{\mathrm{d}} (C^{q+1}_{\mathbb{L}})^{\mathrm{G}}\right) \cap \mathrm{Im}\left(C^{q-1}_{\mathbb{L}} \xrightarrow{\mathrm{d}} E^q_{\mathbb{L}}\right)}{\mathrm{Im}\left((C^{q-1}_{\mathbb{L}})^{\mathrm{G}} \xrightarrow{\mathrm{d}} (E^q_{\mathbb{L}})^{\mathrm{G}}\right)}.$$

It follows from the formation of the spectral sequence in Lemma A.4 that for $c \in \mathrm{H}^q\left((C^\bullet_{\mathbb{L}})^{\mathrm{G}}\right)^0$, its image in $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{G}, \mathrm{H}^{q-1}(C^\bullet_{\mathbb{L}}))$ under the edge map can be represented by the 1-cocycle that sends $g \in \mathrm{G}$ to the (cohomology class in $\mathrm{H}^{q-1}(C^\bullet_{\mathbb{L}})$) of $gc^{q-1} - c^{q-1}$, where $c^{q-1} \in C^{q-1}_{\mathbb{L}}$ is an arbitrary element whose differential $\mathrm{d}c^{q-1}$ in $C^q_{\mathbb{L}}$ represents c (so that $gc^{q-1} - c^{q-1}$ is closed).

To prove the lemma, let $b^q \in (B^q_{\mathbb{L}})^{\mathrm{G}}$ be a closed element that represents $b \in \mathrm{H}^q\left((B^\bullet_{\mathbb{L}})^{\mathrm{G}}\right)$. Since $\nu^q(y^q)$ induces an element in $\mathrm{H}^q\left((C^\bullet_{\mathbb{L}})^{\mathrm{G}}\right)^0$, we may choose an element $c^{q-1} \in C^{q-1}_{\mathbb{L}}$ whose differential $\mathrm{d}c^{q-1}$ in $C^q_{\mathbb{L}}$ equals $\nu^q(b^q)$. To construct an element $\overline{a}$ that lifts $\beta(b)$, we take an element $b^{q-1} \in B^{q-1}_{\mathbb{L}}$ such that $\mu^{q-1}(b^{q-1}) = c^{q-1}$. Then $\nu^q(b^q - \mathrm{d}b^{q-1}) = 0$, which implies that $a^q := b^q - \mathrm{d}b^{q-1}$ belongs to $A^q_{\mathbb{L}}$ and is closed. Then we may take $\overline{a} \in \mathrm{H}^q\left(A^\bullet_{\mathbb{L}}\right)$ to be the cohomology class of a^q. Since b^q is fixed by G, we have for every $g \in \mathrm{G}$, $ga^q - a^q = g\mathrm{d}b^{q-1} - \mathrm{d}b^{q-1} = \mathrm{d}(gb^{q-1} - b^{q-1})$, which is a closed element in $A^q_{\mathbb{L}}$. However, $\mathrm{d}(gb^{q-1} - b^{q-1})$ exactly represents the class of $gc^{q-1} - c^{q-1}$ under the coboundary map $\overline{\lambda}\colon \mathrm{H}^{q-1}\left(C^\bullet_{\mathbb{L}}\right) \to \mathrm{H}^q\left(A^\bullet_{\mathbb{L}}\right)$. The lemma is proved. □

In what follows, G will be the absolute Galois group of a field K of characteristic different from p, which we denote by G_K. As for common practice, we simply write $\mathrm{H}^q(K,-)$ for $\mathrm{H}^q_{\mathrm{cont}}(\mathrm{G}_K,-)$. We also introduce the additive category $\mathbf{M}_K(\mathbb{L})$ of topological $\mathbb{L}$-vector spaces with continuous actions by G_K.

A.3. **Bi-extensions of cycles.** Let K be a field of characteristic different from p with a fixed algebraic closure $\overline{K}$. Let X be a projective smooth scheme over K of pure dimension $n-1$. For every integer d, put

$$\mathrm{Z}^d(X)^0_{\mathbb{L}} := \mathrm{Ker}\left(\mathrm{Z}^d(X)_{\mathbb{L}} \to \mathrm{H}^{2d}(X_{\overline{K}}, \mathbb{L}(d))\right).$$

Now we consider two elements $c \in \mathrm{Z}^d(X)^0_{\mathbb{L}}$ and $c' \in \mathrm{Z}^{d'}(X)^0_{\mathbb{L}}$ with $d + d' = n$, such that c and c' have disjoint supports. Choose disjoint nonempty closed subsets Z and Z' of X of pure codimension d and d' containing the supports of c and c', respectively. Denote by $i\colon Z \to X$ and $i'\colon Z' \to X$ the closed immersions and put $U := X \setminus Z$ and $U' := X \setminus Z'$. We have the following diagram

$$\begin{tikzcd}
U \cap U' \arrow[r, "j'"] \arrow[d, "j"'] & U \arrow[d, "j"] \\
U' \arrow[r, "j'"] & X
\end{tikzcd}$$

of open immersions. We have the following induced diagram

(A.2)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{H}^{2d-2}(X_{\overline{K}}, \mathbb{L}(d)) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^{2d-2}(Z'_{\overline{K}}, \mathbb{L}(d)) \arrow[r] \arrow[d, equal] & \mathrm{H}^{2d-1}(X_{\overline{K}}, j'_!\mathbb{L}(d)) \arrow[r] \arrow[d] & \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)) \arrow[r] \arrow[d] & 0 \\
\mathrm{H}^{2d-2}(U_{\overline{K}}, \mathbb{L}(d)) \arrow[r] & \mathrm{H}^{2d-2}(Z'_{\overline{K}}, \mathbb{L}(d)) \arrow[r] & \mathrm{H}^{2d-1}(U_{\overline{K}}, j'_!\mathbb{L}(d)) \arrow[r] \arrow[d] & \mathrm{H}^{2d-1}(U_{\overline{K}}, \mathbb{L}(d)) \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{H}^{2d}(Z_{\overline{K}}, i^!\mathbb{L}(d)) \arrow[r, equal] \arrow[d] & \mathrm{H}^{2d}(Z_{\overline{K}}, i^!\mathbb{L}(d)) \arrow[d] & \\
 & & \mathrm{H}^{2d}(X_{\overline{K}}, j'_!\mathbb{L}(d)) \arrow[r, "\sim"] & \mathrm{H}^{2d}(X_{\overline{K}}, \mathbb{L}(d)) &
\end{tikzcd}$$

in $\mathbf{M}_K(\mathbb{L})$.

The element c gives rise to a map $\kappa^c\colon \mathbb{L} \to \mathrm{H}^{2d}(Z_{\overline{K}}, i^!\mathbb{L}(d))$ whose image is contained in the kernel of the map $\mathrm{H}^{2d}(Z_{\overline{K}}, i^!\mathbb{L}(d)) \to \mathrm{H}^{2d}(X_{\overline{K}}, \mathbb{L}(d))$. The element c' gives rise to a map $\kappa_{c'}\colon \mathrm{H}^{2d-2}(Z'_{\overline{K}}, \mathbb{L}(d-1)) \to \mathbb{L}$ that vanishes on the image of the map $\mathrm{H}^{2d-2}(X_{\overline{K}}, \mathbb{L}(d)) \to \mathrm{H}^{2d-2}(Z'_{\overline{K}}, \mathbb{L}(d))$. Applying the pullback along κ^c and the pushforward along $\kappa_{c'}(1)$ to the diagram (A.2), we obtain the following bi-extension diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{L}(1) \arrow[r] \arrow[d, equal] & \mathrm{E}_{c'} \arrow[r] \arrow[d] & \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{L}(1) \arrow[r] & \mathrm{E}^c_{c'} \arrow[r] \arrow[d] & \mathrm{E}^c \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{L} \arrow[r, equal] \arrow[d] & \mathbb{L} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

in $\mathbf{M}_K(\mathbb{L})$. It is easy to see that the above diagram does not depend on the choices of Z and Z'. We have three induced extension classes

- $[\mathrm{E}^c] \in \mathrm{H}^1(K, \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)))$,
- $[\mathrm{E}^\vee_{c'}(1)] = [\mathrm{E}^{c'}] \in \mathrm{H}^1(K, \mathrm{H}^{2d'-1}(X_{\overline{K}}, \mathbb{L}(d')))$,
- $[\mathrm{E}^c_{c'}] \in \mathrm{H}^1(K, \mathrm{E}_{c'})$.

A.4. **Relation with Beilinson's local index.** In this subsection, we assume that K is a non-archimedean local field whose residue field has characteristic different from p. We study the relation between Nekovář's local p-adic height and Beilinson's local index.

Assume that the cycle classes of c and c' in $\mathrm{H}^{2d}(X, \mathbb{L}(d))$ and $\mathrm{H}^{2d'}(X, \mathbb{L}(d'))$ vanish, respectively.[21] Then the image of $[\mathrm{E}^c_{c'}]$ in $\mathrm{H}^1(K, \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)))$ vanishes, hence $[\mathrm{E}^c_{c'}]$ belongs to the image of the map $\mathrm{H}^1(K, \mathbb{L}(1)) \to \mathrm{H}^1(K, \mathrm{E}_{c'})$ which is injective. Following Nekovář [Nek93], we denote by $\langle c, c' \rangle^{\mathrm{N}}_{X,K}$ the image of $[\mathrm{E}^c_{c'}]$ under the natural isomorphism $\mathrm{H}^1(K, \mathbb{L}(1)) = \widehat{K^\times} \otimes_{\widehat{\mathbb{Z}}} \mathbb{L}$ given by the Kummer maps.

We recall the definition of Beilinson's local index [Beĭ87, Section 2] (see also [LL21, Appendix B]). We have the refined cycle class $[c] \in \mathrm{H}^{2d}_Z(X, \mathbb{L}(d)) = \mathrm{H}^{2d}(Z, j^!\mathbb{L}(d))$, which is contained in the kernel of the map $\mathrm{H}^{2d}_Z(X, \mathbb{L}(d)) \to \mathrm{H}^{2d}(X, \mathbb{L}(d))$, hence we may choose an element $\gamma \in \mathrm{H}^{2d-1}(U, \mathbb{L}(d))$ that maps to $[c]$ under the coboundary map $\mathrm{H}^{2d-1}(U, \mathbb{L}(d)) \to \mathrm{H}^{2d}_Z(X, \mathbb{L}(d))$. Similarly, we can choose an element $\gamma' \in \mathrm{H}^{2d'-1}(U', \mathbb{L}(d'))$ for c'. Beilinson's local index, which we denote by $\langle c, c' \rangle^{\mathrm{B}}_{X,K}$, is defined as the image of $\gamma \cup \gamma'$ under the composite map

$$\mathrm{H}^{2n-2}(U \cap U', \mathbb{L}(n)) \to \mathrm{H}^{2n-1}(X, \mathbb{L}(n)) \xrightarrow{\mathrm{Tr}_{X/K}} \mathrm{H}^1(\mathrm{Spec}\, K, \mathbb{L}(1)) = \mathrm{H}^1(K, \mathbb{L}(1)) = \widehat{K^\times} \otimes_{\widehat{\mathbb{Z}}} \mathbb{L},$$

in which the first map is the coboundary map in the Mayer–Vietoris exact sequence for the open covering $X = U \cup U'$.

Remark A.6. In fact, in [Beĭ87, Section 2] and [LL21, Appendix B], the local index $\langle c, c' \rangle^{\mathrm{B}}_K$ takes value in $\mathbb{L}$ via the canonical isomorphism $\mathrm{H}^1(\mathrm{Spec}\, K, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$ that is the composition

$$\mathrm{H}^1(\mathrm{Spec}\, K, \mathbb{Q}_p(1)) \to \mathrm{H}^2_{\mathrm{Spec}\, \kappa}(\mathrm{Spec}\, O_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Spec}\, \kappa, \mathbb{Q}_p) \simeq \mathbb{Q}_p$$

in which κ is the residue field of K. By [GD77, 2.1.3] or [Nek95, II.(2.16.1)], the induced isomorphism $\widehat{K^\times} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p \to \mathbb{Q}_p$ sends a uniformizer of K to -1, rather than 1.

The following proposition is simply [Sch94, Theorem 5.3].

Proposition A.7. *Let the situation be as above. Then*

$$\langle c, c' \rangle^{\mathrm{N}}_{X,K} = \langle c, c' \rangle^{\mathrm{B}}_{X,K}.$$

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[21] This is automatic if the monodromy-weight conjecture holds for $\mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d))$.

A.5. **Crystalline property of bi-extensions.** In this subsection, we assume that K is a finite extension of $\mathbb{Q}_p$ with residue field κ. Denote by W the Witt ring of κ, and by K_0 the fraction field of W, which is canonically a subfield of K.

We assume that X admits a proper strictly semistable model $\mathcal{X}$ over O_K. Put $\mathsf{X} := \mathcal{X} \otimes_{O_K} \kappa$. For every integer $h \geqslant 1$, denote by $\mathsf{X}^{(h)}$ the disjoint union of intersections of h different irreducible components of X, which is either empty or a proper smooth scheme over κ.

Theorem A.8. *Suppose that $n < p$. Let $\mathbb{T}$ be an $\mathbb{L}$-ring of étale correspondences on $\mathcal{X}$ and $\mathfrak{m}, \mathfrak{m}'$ two maximal ideals of $\mathbb{T}$ satisfying that*

$$\bigoplus_{h>1,q\geqslant 0} \left(\mathrm{H}^q_{\mathrm{cris}}(\mathsf{X}^{(h)}/W) \otimes_{\mathbb{Z}_p} \mathbb{L}\right)_{\mathfrak{m}} = \bigoplus_{h>1,q\geqslant 0} \left(\mathrm{H}^q_{\mathrm{cris}}(\mathsf{X}^{(h)}/W) \otimes_{\mathbb{Z}_p} \mathbb{L}\right)_{\mathfrak{m}'} = 0. \tag{A.3}$$

Then there exist elements $t \in \mathbb{T} \setminus \mathfrak{m}$ and $t' \in \mathbb{T} \setminus \mathfrak{m}'$ depending only on $\mathcal{X}$ such that the following holds: For two arbitrary elements $c \in \mathrm{Z}^d(X)^0_{\mathbb{L}}$ and $c' \in \mathrm{Z}^{d'}(X)^0_{\mathbb{L}}$ with $d + d' = n$ satisfying that

1. *$(\operatorname{supp} \mathcal{C})^t \cap (\operatorname{supp} \mathcal{C}')^{t'} = \emptyset$ for every $t, t' \in \mathbb{T}$, where $\mathcal{C}$ and $\mathcal{C}'$ denote the Zariski closures of c and c' in $\mathcal{X}$, respectively,*
2. *the codimension of $\operatorname{supp} \mathcal{C}'$ in $\mathsf{X}^{(h)}$ is at least d' for every $h \geqslant 1$,*

the following

- $[\mathrm{E}^{t^*c}] \in \mathrm{H}^1_f(K, \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)))$,
- $[\mathrm{E}^{\vee}_{t'^*c'}(1)] \in \mathrm{H}^1_f(K, \mathrm{H}^{2d'-1}(X_{\overline{K}}, \mathbb{L}(d')))$,
- $[\mathrm{E}^{t^*c}_{t'^*c'}] \in \mathrm{H}^1_f(K, \mathrm{E}_{t'^*c'})$,

hold simultaneously. Here, the bi-extension $\mathrm{E}^{t^*c}_{t'^*c'}$ exists by (1).

Remark A.9. By taking $\mathbb{T} = \mathbb{L}$, Theorem A.8 asserts that $\mathrm{E}^c_{c'}$ is crystalline as long as $\operatorname{supp} \mathcal{C} \cap \operatorname{supp} \mathcal{C}' = \emptyset$ when $\mathcal{X}$ is a proper smooth model of X over O_K.[22] This confirms the (equivalent) conjecture in the remark after [Shn16, Theorem 8.7] when $n < p$.[23]

Our main strategy of proving Theorem A.8 is similar to [Sat13]. The main difficulty is to show that the bi-extension $[\mathrm{E}^{t^*c}_{t'^*c'}]$ is a crystalline class. We consider the Abel–Jacobi map from the homologically trivial part of the degree $2d$ syntomic cohomology of $\mathcal{X} \setminus (\operatorname{supp} \mathcal{C}')$ with proper support to $\mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(U'_{\overline{K}}, \mathbb{Q}_p(d)))$. After we show that $\mathrm{E}_{t'^*c'}$ is crystalline for suitable t', it suffices to show that if a syntomic class comes from a (homologically trivial) cycle, then its Abel–Jacobi image vanishes in $\mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(U'_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}})$ "after localization at $\mathfrak{m}'$" (and replace K by a finite extension in fact), under the conditions in Theorem A.8. However, the main challenge for us is that unlike the situation in [Sat13] where $U' = X$, we do not have a comparison theorem for $\mathrm{H}^{2d-1}_{\mathrm{c}}(U'_{\overline{K}}, \mathbb{Q}_p)$ with some cohomology on the special fiber in general. Also, due to the constraint of the conditions in the theorem, we can not reduce the theorem to an alteration. We solve this problem in the following way. First, we show that the kernel of the Abel–Jacobi map from the above syntomic cohomology to $\mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(U'_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}})$ contains the kernel of another map whose range is a certain space defined by log rigid cohomology (Proposition B.9). Second, we show that a cycle class will vanish in this space "after localization at $\mathfrak{m}'$". For the first step, which shall hold more generally without the conditions in the theorem, we pass to a strict semistable alteration.

The full proof of Theorem A.8 will occupy the entire Appendix B.

A.6. **Recollection on p-adic Galois representation.** Let K be as in the previous subsection. Let V be a finite-dimensional continuous representation of G_K with coefficients in $\mathbb{L}$, which we assume to be *de Rham*, that is

$$\mathbb{D}_{\mathrm{dR}}(\mathrm{V}) := \left(\mathrm{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}}\right)^{\mathrm{G}_K}$$

is a free $\mathbb{L} \otimes_{\mathbb{Q}_p} K$-module of rank $\dim_{\mathbb{L}} \mathrm{V}$. In the rest of this subsection, we assume that $\mathbb{L}$ is a subfield of $\overline{\mathbb{Q}}_p$; and one can generalize the discussion to a finite product of finite extensions of $\mathbb{Q}_p$ by considering all homomorphisms $\mathbb{L} \to \overline{\mathbb{Q}}_p$ over $\mathbb{Q}_p$.

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[22] We warn the readers that this assertion is wrong if one replaces the word *smooth* by *strictly semistable*.

[23] However, our strategy for the proof of Theorem A.8 is different from the case of (local systems over) curves in [Shn16, Theorem 8.7].

We have a decreasing filtration $\mathrm{F}^i\mathbb{D}_{\mathrm{dR}}(\mathrm{V})$ of $\mathbb{L}\otimes_{\mathbb{Q}_p}K$-submodules of $\mathbb{D}_{\mathrm{dR}}(\mathrm{V})$, known as the de Rham filtration. Moreover, for every embedding $\tau\colon K\to\overline{\mathbb{Q}}_p$, Fontaine constructed a Weil–Deligne representation $\mathrm{WD}(\mathrm{V})_\tau$ of the Weil group of K with coefficients in $\overline{\mathbb{Q}}_p$, with underlying $\overline{\mathbb{Q}}_p$-vector space $\mathbb{D}_{\mathrm{dR}}(\mathrm{V})\otimes_{\mathbb{L}\otimes_{\mathbb{Q}_p}K,1\otimes\tau}\overline{\mathbb{Q}}_p$. The isomorphism class of $\mathrm{WD}(\mathrm{V})_\tau$ is independent of τ. See, for example, [TY07, Section 1] for more details. We also recall that V is crystalline if

$$\mathbb{D}_{\mathrm{cris}}(\mathrm{V}):=\left(\mathrm{V}\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{cris}}\right)^{\mathrm{G}_K}$$

is a free $\mathbb{L}\otimes_{\mathbb{Q}_p}K_0$-module of rank $\dim_{\mathbb{L}}\mathrm{V}$.

Remark A.10. It follows from the construction of $\mathrm{WD}(\mathrm{V})_\tau$ that if V is crystalline, then the following polynomials coincide:

- the characteristic polynomial of a geometric Frobenius on $\mathrm{WD}(\mathrm{V})_\tau$, for any τ;
- the characteristic polynomial of φ on $\mathbb{D}_{\mathrm{cris}}(\mathrm{V})$, where φ denotes the $[K_0:\mathbb{Q}_p]$-th power of the crystalline Frobenius.

Definition A.11. Let μ be a real number. We say that V is *pure of weight* μ if for some (hence every) $\tau\colon K\to\overline{\mathbb{Q}}_p$, all geometric Frobenius eigenvalues of $\mathrm{gr}_i\mathrm{WD}(\mathrm{V})_\tau$ are Weil $|\kappa|^{\mu+i}$-numbers for every $i\in\mathbb{Z}$, where $\mathrm{gr}_i\mathrm{WD}(\mathrm{V})_\tau$ denotes the i-th graded piece of the monodromy filtration on $\mathrm{WD}(\mathrm{V})_\tau$.[24]

We make the following definition, after [Nek93, 6.7].

Definition A.12. We say that V satisfies the *Panchishkin condition* if there exists a necessarily unique $\mathbb{L}[\mathrm{G}_K]$-submodule $\mathrm{V}^+\subseteq\mathrm{V}$ (with $\mathrm{V}^-:=\mathrm{V}/\mathrm{V}^+$) such that

$$\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(\mathrm{V}^+)=\mathbb{D}_{\mathrm{dR}}(\mathrm{V}^-)/\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(\mathrm{V}^-)=0.$$

Lemma A.13. *For a crystalline representation* V *of* G_K *over* $\mathbb{L}$*, the following are equivalent:*

(1) V *satisfies the Panchishkin condition;*
(2) the $\mathbb{L}\otimes_{\mathbb{Q}_p}K_0$*-submodule* $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})\subset\mathbb{D}_{\mathrm{cris}}(\mathrm{V})$ *on which the crystalline Frobenius acts with negative slopes*[25] *is weakly admissible, and the natural map*

$$(\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})\otimes_{K_0}K)\oplus\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(V)\xrightarrow{\sim}\mathbb{D}_{\mathrm{dR}}(\mathrm{V})\tag{A.4}$$

is a splitting of the Hodge filtration on $\mathbb{D}_{\mathrm{dR}}(\mathrm{V})$.

Proof. Assume that V is Panchishkin with a subrepresentation V^+ as in the definition. By the weak admissibility of $\mathbb{D}_{\mathrm{cris}}(\mathrm{V}^\pm)$, the crystalline Frobenius acts on $\mathbb{D}_{\mathrm{cris}}(\mathrm{V}^+)$ with negative slopes and on $\mathbb{D}_{\mathrm{cris}}(\mathrm{V}^-)$ with non-negative slopes; it follows that $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})=\mathbb{D}_{\mathrm{cris}}(\mathrm{V}^+)$ satisfies the second condition. Conversely, that condition implies that $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})\subset\mathbb{D}_{\mathrm{cris}}(\mathrm{V})$ is weakly admissible, hence by [CF00] it arises as $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})=\mathbb{D}_{\mathrm{cris}}(\mathrm{V}^+)$ from a subrepresentation $\mathrm{V}^+\subset\mathrm{V}$, which witnesses the Panchishkin condition. □

Lemma A.14. *Let* μ *be a nonzero integer and* $0\to\mathrm{V}_1\to\mathrm{V}\to\mathrm{V}_2\to0$ *a short exact sequence of crystalline representations of* G_K*. If* V_1 *and* V_2 *are pure of weight* μ*, then so is* V*. If* V_1 *and* V_2 *satisfy the Panchishkin condition, then so does* V.

Proof. The first statement is clear. We prove the second one. The sequence

$$0\to\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(\mathrm{V}_1)\to\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(\mathrm{V})\to\mathrm{F}^0\mathbb{D}_{\mathrm{dR}}(\mathrm{V}_2)\to0$$

is exact, and by definition, so is

$$0\to\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V}_1)\to\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})\to\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V}_2)\to0.$$

This implies that (A.4) is an isomorphism. By [CF00, Proposition 3.4], the module $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})$ is weakly admissible too, so that the equivalent Panchishkin condition of Lemma A.13 is satisfied. □

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[24] In particular, $\mathbb{L}(1)$ is pure of weight -2.

[25] In other words, the submodule $\mathbb{D}^+_{\mathrm{cris}}(\mathrm{V})$ generated by the generalized φ-eigenspaces relative to the eigenvalues of negative valuation.

A.7. **Decomposition of p-adic height pairing.** In this subsection, we take K to be a number field. Let X be a proper smooth scheme over K of pure dimension $n-1$ and take two positive integers d, d' satisfying $d+d'=n$.

Consider $\mathbb{L}[\mathrm{G}_K]$-submodules V and V' of $\mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d))$ and $\mathrm{H}^{2d'-1}(X_{\overline{K}}, \mathbb{L}(d'))$, respectively, satisfying

(V1) For every nonarchimedean place u of K not above p, $\mathrm{H}^i(K_u, \mathrm{V}) = \mathrm{H}^i(K_u, \mathrm{V}') = 0$ for $i \in \mathbb{Z}$.

(V2) For every place u of K above p, both $\mathrm{V}|_{K_u}$ and $\mathrm{V}'|_{K_u}$ are semistable and pure of weight -1 (Definition A.11).

(V3) For every place u of K above p, both $\mathrm{V}|_{K_u}$ and $\mathrm{V}'|_{K_u}$ satisfy the Panchishkin condition (Definition A.12).

We have the canonical p-adic height pairing

$$\langle\ ,\ \rangle_{(\mathrm{V},\mathrm{V}'),K} \colon \mathrm{H}^1_f(K, \mathrm{V}) \times \mathrm{H}^1_f(K, \mathrm{V}') \to \Gamma_{K,p} \otimes_{\mathbb{Z}_p} \mathbb{L}$$

constructed in [Nek93], using the Hodge splitting map (A.4). The pairing is $\mathbb{L}$-bilinear.

Recall that we have the Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{Z}^d(X)^0_{\mathbb{L}} \to \mathrm{H}^1(K, \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d))).$$

We denote by $\mathrm{Z}^d_{\mathrm{V}}(X)^0_{\mathbb{L}}$ the subspace of $\mathrm{Z}^d(X)^0_{\mathbb{L}}$ that is the inverse image of $\mathrm{H}^1(K, \mathrm{V})$. By [Nek00, Theorem 3.1], the image of AJ is contained in $\mathrm{H}^1_{\mathrm{st}}(K, \mathrm{H}^{2d-1}(X_{\overline{K}}, \mathbb{L}(d)))$. Moreover, (V1) implies that $\mathrm{H}^1_{\mathrm{st}}(K, \mathrm{V}) = \mathrm{H}^1_f(K, \mathrm{V})$. Thus, we have the Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{Z}^d_{\mathrm{V}}(X)^0_{\mathbb{L}} \to \mathrm{H}^1_f(K, \mathrm{V}).$$

Similarly, we have $\mathrm{Z}^{d'}_{\mathrm{V}'}(X)^0_{\mathbb{L}}$ and the corresponding Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{Z}^{d'}_{\mathrm{V}'}(X)^0_{\mathbb{L}} \to \mathrm{H}^1_f(K, \mathrm{V}').$$

Combining with the two Abel–Jacobi maps, we obtain a pairing

$$\langle\ ,\ \rangle_{(\mathrm{V},\mathrm{V}'),K} \colon \mathrm{Z}^d_{\mathrm{V}}(X)^0_{\mathbb{L}} \times \mathrm{Z}^{d'}_{\mathrm{V}'}(X)^0_{\mathbb{L}} \to \Gamma_{K,p} \otimes_{\mathbb{Z}_p} \mathbb{L}. \tag{A.5}$$

Take two elements $c \in \mathrm{Z}^d_{\mathrm{V}}(X)^0_{\mathbb{L}}$ and $c' \in \mathrm{Z}^{d'}_{\mathrm{V}'}(X)^0_{\mathbb{L}}$ with disjoint supports. Then according to [Nek93, Section 4], we have a decomposition

$$\langle c, c'\rangle_{(\mathrm{V},\mathrm{V}'),K} = \sum_{u\nmid\infty} \langle c, c'\rangle_{(\mathrm{V},\mathrm{V}'),K_u}$$

of the pairing (A.5) into local ones $\langle c, c'\rangle_{(\mathrm{V},\mathrm{V}'),K_u} \in \widehat{K_u^\times} \otimes_{\widehat{\mathbb{Z}}} \mathbb{L}$ over all *nonarchimedean* places u of K, in which $\langle c, c'\rangle_{(\mathrm{V},\mathrm{V}'),K_u} = \langle c, c'\rangle^{\mathrm{N}}_{X_u,K_u}$ for u not above p.

Remark A.15. For a place u of K above p, the bi-extension class $[\mathrm{E}^c_{c'}] \in \mathrm{H}^1(K_u, \mathrm{E}_{c'})$ belongs to $\mathrm{H}^1_f(K_u, \mathrm{E}_{c'})$ if and only if $\langle c, c'\rangle_{(\mathrm{V},\mathrm{V}'),K_u} \in O^\times_{K_u} \otimes_{\mathbb{Z}_p} \mathbb{L}$.

Appendix B. Proof of Theorem A.8

Let K be a finite extension of $\mathbb{Q}_p$ with residue field κ. Denote by W the Witt ring of κ, and by K_0 the fraction field of W, which is canonically a subfield of K. We fix an algebraic closure $\overline{K}$ of K with the residue field $\overline{\kappa}$. For every finite extension K' of K_0 contained in $\overline{K}$, put $\mathrm{G}_{K'} := \mathrm{Gal}(\overline{K}/K')$ as a profinite group.

B.1. **Preparation.** For a scheme $\mathcal{Z}$ of finite type over $O_{K'}$ with K' a finite extension of K_0 contained in $\overline{K}$, we

- put $Z := \mathcal{Z} \otimes_{O_{K'}} K'$ for the generic fiber,
- put $\mathcal{Z}_l := \mathcal{Z} \otimes \mathbb{Z}/p^l$ for every integer $l \geqslant 1$,
- put $\mathsf{Z} := \mathcal{Z} \otimes_{O_{K'}} \kappa'$ for its special fiber, where κ' is the residue field of K',
- denote by $\mathfrak{Z}$ the formal completion of $\mathcal{Z}$ along Z,
- denote by $\mathfrak{Z}_\eta$ the generic fiber of $\mathfrak{Z}$, regarded as an analytic space over K' in the sense of Berkovich,
- put $\overline{\mathcal{Z}} := \mathcal{Z} \otimes_{O_{K'}} O_{\overline{K}}$, $\overline{Z} := Z \otimes_{K'} \overline{K}$, and $\overline{\mathsf{Z}} := \mathsf{Z} \otimes_{\kappa'} \overline{\kappa}$.

We apply the similar notational convention to morphisms over $O_{K'}$ as well.[26]

Suppose that X is a subscheme of Z, we denote by $]\mathsf{X}[_{\mathfrak{Z}_\eta}$ its tubular neighbourhood in $\mathfrak{Z}_\eta$. We have the quasi-étale site $]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}}$ [Ber94, §3] with the natural map $]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}} \to (\widehat{\mathfrak{Z}_{/\mathsf{X}}})_{\acute{e}t}$, where $\widehat{\mathfrak{Z}_{/\mathsf{X}}}$ denotes the formal completion of $\mathfrak{Z}$ along X. On the other hand, the natural map $(\widehat{\mathfrak{Z}_{/\mathsf{X}}})_{\acute{e}t} \to \mathsf{X}_{\acute{e}t}$ is an equivalence of sites [Ber96, Proposition 2.1]. Together, we obtain the *specialization* map $\mathsf{s}_{(\mathsf{X},\mathfrak{Z})}\colon]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}} \to \mathsf{X}_{\acute{e}t}$, and will simply write s when no confusion arises.

Definition B.1. Let R be a ring. In the situation above, suppose that X is a closed subscheme of Z and U an open subscheme of X, we define a functor

$$\mathsf{f}^!_{(\mathsf{U},\mathsf{X})}\colon \mathbf{M}(]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}}, R) \to \mathbf{M}(]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}}, R)$$

to be the kernel of the unit transform $\mathrm{id} \to g_* \circ g^*$, where g denotes the open immersion $]\mathsf{X} \setminus \mathsf{U}[_{\mathfrak{Z}_\eta} \to]\mathsf{X}[_{\mathfrak{Z}_\eta}$.

Remark B.2. The functors g^*, g_*, and $\mathsf{f}^!_{(\mathsf{U},\mathsf{X})}$ are all exact. Moreover, there is in general no functor $\mathsf{f}\colon \mathbf{D}^+(\mathsf{X}_{\acute{e}t}, R) \to \mathbf{D}^+(\mathsf{X}_{\acute{e}t}, R)$ such that $\mathsf{f} \circ \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \simeq \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \circ \mathsf{f}^!_{(\mathsf{U},\mathsf{X})}$, even when $\mathsf{X} = \mathsf{Z}$.

Lemma B.3. *Let the situation be as in Definition B.1. The diagram*

$$\begin{tikzcd}
\mathsf{F}_! \circ \mathsf{F}^* \circ \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \arrow[r] & \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \circ g_* \circ g^*
\end{tikzcd}$$

of functors from $\mathbf{D}^+(]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}}, R)$ to $\mathbf{D}^+(\mathsf{X}_{\acute{e}t}, R)$ commutes, where $\mathsf{F}\colon \mathsf{U} \to \mathsf{X}$ denotes the open immersion. In particular, there is a canonical natural transform

$$\mathsf{F}_! \circ \mathsf{F}^* \circ \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \to \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \circ \mathsf{f}^!_{(\mathsf{U},\mathsf{X})}\colon \mathbf{D}^+(]\mathsf{X}[_{\mathfrak{Z}_\eta,\mathrm{q\acute{e}t}}, R) \to \mathbf{D}^+(\mathsf{X}_{\acute{e}t}, R).$$

Proof. It suffices to notice that the unit transform $\mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \to \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \circ g_* \circ g^*$ factors through

$$\mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \to \mathsf{G}_* \circ \mathsf{G}^* \circ \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \to \mathsf{G}_* \circ \mathsf{Rs}_{(\mathsf{X}\setminus\mathsf{U},\mathfrak{Z})*} \circ g^* \xrightarrow{\sim} \mathsf{Rs}_{(\mathsf{X},\mathfrak{Z})*} \circ g_* \circ g^*,$$

where $\mathsf{G}\colon \mathsf{X} \setminus \mathsf{U} \to \mathsf{X}$ denotes the closed immersion. □

In what follows, we will work with log-schemes, written as (X, L) with the first variable the underlying scheme and the second variable the log structure. Since the integral model in Theorem A.8 is strictly semistable, we assume that the log structures are defined in the Zariski topology.

For a scheme X and a closed subset Y, we denote by L_X^Y the log structure $\mathscr{O}_X \cap j_* \mathscr{O}_{X\setminus Y}^\times \to \mathscr{O}_X$, where $j\colon X\setminus Y \to X$ denotes the open immersion. For a log-scheme (X, L) and a morphism $f\colon X' \to X$, we write f^*L for the pullback log structure or simply $L|_{X'}$ when f is clear from the context.

We write W^{triv} for $(\operatorname{Spec} W, W^\times)$, $W[t]^\circ$ for $(\operatorname{Spec} W[t], L)$ where L is the log structure associated with $1 \mapsto t$, W° for the fiber of $W[t]^\circ$ at $t = 0$, and κ° for the fiber of W° at $p = 0$. Note that the natural morphism $W[t]^\circ \to W^{\mathrm{triv}}$ is log-smooth. For every extension K'/K_0 contained in $\overline{K}$ with the residue field κ', we put $O_{K'}^{\mathrm{can}} := (\operatorname{Spec} O_{K'}, L_{\operatorname{Spec} O_{K'}}^{\operatorname{Spec} \kappa'})$.

Let (X, M) be a fine log-scheme over a fine base log-scheme (S, L) of finite type. Recall that an *embedding system* for $(X, M)/(S, L)$ is a projective system $\{(X^\star, M^\star) \hookrightarrow (Z^\star, N^\star)\}_{\star=0,1,\dots}$ of exact closed immersions of log-schemes over (S, L) in which $X^\star$ is a Zariski hypercovering of X, $M^\star = M|_{X^\star}$, and $(Z^\star, N^\star)$ is a fine log-scheme log-smooth over (S, L) of finite type. Note that embedding system always exists.

In the case where $(S, L) = W[t]^\circ$ and (X, M) is a strictly semistable log-scheme over κ° [GK05, §2.1] of finite type, we say that an embedding system $\{(X^\star, M^\star) \hookrightarrow (Z^\star, N^\star)\}$ for $(X, M)/(S, L)$ is *admissible* if

- $(X^0, M^0) \to (Z^0, N^0)$ induces an isomorphism $(X^0, M^0) \simeq (Z^0, N^0) \times_{W[t]^\circ} \kappa^\circ$;
- Z^0 is flat and generically smooth over $W[t]$, and is smooth over W;
- $Y^0 := Z^0 \otimes_{W[t]} W$ is a relative strict normal crossings divisor of Z^0 over W;
- $N^0 = L_{Z^0}^{Y^0}$;

---

[26] Later, we will see $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ for schemes and $\mathcal{A}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ for morphisms over $O_{K'}$.

- $(X^\star, M^\star) \to (Z^\star, N^\star)$ is (isomorphic to the one) induced from $(X^0, M^0) \to (Z^0, N^0)$ in the process described in [GK05, §5.1].

B.2. **Rigid de Rham–Witt complexes.** Let (X, L) be a fine log-scheme over κ° of finite type. Let $\mathsf{F}\colon \mathsf{U} \to \mathsf{X}$ be an open subscheme. For every Zariski hypercovering $\mathsf{X}^\star$ of X, we put $\mathsf{F}^\star\colon \mathsf{U}^\star := \mathsf{U} \times_{\mathsf{X}} \mathsf{X}^\star \to \mathsf{X}^\star$.

Choose an embedding system $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Y}^\star, M^\star)\}$ for $(\mathsf{X}, L)/W^\circ$. Denote by $\mathsf{u}\colon \mathsf{X}^\star \to \mathsf{X}$ the augmentation map for the hypercovering, and put $\mathsf{s}^\star := \mathsf{s}_{(\mathsf{X}^\star, \mathfrak{Y}^\star)}$. We define the rigid de Rham–Witt complex of (X, L) to be[27]

$$\omega_{\mathsf{X}} := \mathrm{R}\mathsf{u}_* \left(\mathrm{R}\mathsf{s}^\star_* \left(\Omega^\bullet_{(\mathcal{Y}^\star, M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right) \in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0),$$

where $\Omega^\bullet_{(\mathcal{Y}^\star, M^\star)/W^\circ}$ denotes the complex on $\mathcal{Y}^\star_{\text{ét}}$ of relative logarithmic differentials of the log-smooth morphism $(\mathcal{Y}^\star, M^\star)/W^\circ$. By (the same argument in) [GK05, Lemma 1.4], the complex ω_{X} does not depend on the choice of the embedding system for $(\mathsf{X}, M)/W^\circ$.[28]

On the other hand, choose an embedding system $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Z}^\star, N^\star)\}$ for $(\mathsf{X}, L)/W[t]^\circ$, hence for $(\mathsf{X}, L)/W^{\text{triv}}$. Then $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Y}^\star, M^\star) := (\mathcal{Z}^\star, N^\star) \times_{W[t]^\circ} W^\circ\}$ is an embedding system for $(\mathsf{X}, L)/W^\circ$. We have the short exact sequence

$$0 \to \Omega^{q-1}_{(\mathcal{Y}^\star, M^\star)/W^\circ} \to \Omega^q_{(\mathcal{Z}^\star, N^\star)/W^{\text{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{\mathcal{Y}^\star} \to \Omega^q_{(\mathcal{Y}^\star, M^\star)/W^\circ} \to 0$$

of coherent sheaves on $\mathcal{Y}^\star_{\text{ét}}$, in which the first map is given by $\wedge\, \mathrm{d}\log t$, for every $q \geqslant 0$ compatible with differentials. We put

$$\widetilde{\omega}_{\mathsf{X}} := \mathrm{R}\mathsf{u}_* \left(\mathrm{R}\mathsf{s}^\star_* \left(\Omega^\bullet_{(\mathcal{Z}^\star, N^\star)/W^{\text{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right) \in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0).$$

Then there is a distinguished triangle

$$\omega^\triangle_{\mathsf{X}}\colon \qquad \omega_{\mathsf{X}}[-1] \to \widetilde{\omega}_{\mathsf{X}} \to \omega_{\mathsf{X}} \xrightarrow{N} \omega_{\mathsf{X}} \tag{B.1}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, where N denotes the connecting map; it is independent of the choice of the embedding system for $(\mathsf{X}, L)/W[t]^\circ$. Moreover, for a morphism $f\colon (\mathsf{X}', L') \to (\mathsf{X}, L)$ of fine log-schemes over κ° of finite type, we have an induced map $\omega^\triangle_{\mathsf{X}} \to \mathrm{R}f_*\widetilde{\omega}^\triangle_{\mathsf{X}'}$ of distinguished triangles in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$.

We put

$$\omega_{(\mathsf{U},\mathsf{X})} := \mathrm{R}\mathsf{u}_* \left(\mathrm{R}\mathsf{s}^\star_* \mathsf{f}^!_{(\mathsf{U}^\star,\mathsf{X}^\star)} \left(\Omega^\bullet_{(\mathcal{Y}^\star, M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right),$$

$$\widetilde{\omega}_{(\mathsf{U},\mathsf{X})} := \mathrm{R}\mathsf{u}_* \left(\mathrm{R}\mathsf{s}^\star_* \mathsf{f}^!_{(\mathsf{U}^\star,\mathsf{X}^\star)} \left(\Omega^\bullet_{(\mathcal{Z}^\star, N^\star)/W^{\text{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right),$$

both in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$ (see Definition B.1). Then by definition, we have a distinguished triangle

$$\omega^\triangle_{(\mathsf{U},\mathsf{X})}\colon \qquad \omega_{(\mathsf{U},\mathsf{X})}[-1] \to \widetilde{\omega}_{(\mathsf{U},\mathsf{X})} \to \omega_{(\mathsf{U},\mathsf{X})} \xrightarrow{N} \omega_{(\mathsf{U},\mathsf{X})} \tag{B.2}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, and a distinguished triangle of distinguished triangles

$$\omega^\triangle_{(\mathsf{U},\mathsf{X})} \to \omega^\triangle_{\mathsf{X}} \to \mathsf{G}_*\omega^\triangle_{\mathsf{X}\setminus\mathsf{U}} \xrightarrow{+1} \tag{B.3}$$

where $\mathsf{G}\colon \mathsf{X} \setminus \mathsf{U} \to \mathsf{X}$ denotes the closed immersion.

From now on, we assume that (X, L) is strictly semistable over κ°. We recall the construction of several crystalline complexes of (X, L). For every $l \geqslant 1$, let $\mathcal{D}^\star_l$ and $\mathcal{E}^\star_l$ be the (scheme part of the) PD envelops of $\mathsf{X}^\star$ in $\mathcal{Y}^\star_l$ and $\mathcal{Z}^\star_l$ (over the base W equipped with the usual PD structure), respectively.[29] We have complexes

$$\Omega^\bullet_{(\mathcal{Y}^\star, M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{\mathcal{D}^\star_\bullet}, \quad \Omega^\bullet_{(\mathcal{Z}^\star, N^\star)/W^{\text{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{\mathcal{D}^\star_\bullet}, \quad \Omega^\bullet_{(\mathcal{Z}^\star, N^\star)/W^{\text{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{\mathcal{E}^\star_\bullet}, \quad \Omega^\bullet_{(\mathcal{Z}^\star, N^\star)/W[t]^\circ} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{\mathcal{E}^\star_l}$$

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[27] More precisely, it should be called *convergent de Rham–Witt complex* since it gives the log convergent cohomology in general. However, later we will take (X, L) to be strictly semistable and proper.

[28] Of course, ω_{X} depends on the log structure L. However, as a common practice for de Rham–Witt complexes, we will not include L in the notation.

[29] The natural morphism $\mathcal{D}^\star_l \to \mathcal{E}^\star_l \otimes_{W\langle t\rangle} W$ is an isomorphism, where $W\langle t\rangle$ denotes the PD envelop of $(W[t], (t))$ over W.

in $\mathbf{C}^+(\mathsf{X}^\star_{\text{ét}}, W_\bullet)$. Put

$$\begin{aligned}
\mathscr{C}_{(\mathsf{X},L)/W^\circ} &\coloneqq \mathsf{Ru}_*\left(\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ}\otimes_{\mathscr{O}_{\mathcal{Y}^\star}}\mathscr{O}_{\mathcal{D}^\star_l}\right),\\
\widetilde{\mathscr{C}}_{(\mathsf{X},L)/W^\circ} &\coloneqq \mathsf{Ru}_*\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\mathcal{D}^\star_l}\right),\\
\mathscr{C}_{(\mathsf{X},L)/W^{\mathrm{triv}}} &\coloneqq \mathsf{Ru}_*\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\mathcal{E}^\star_l}\right),\\
\mathscr{C}_{(\mathsf{X},L)/W[t]^\circ} &\coloneqq \mathsf{Ru}_*\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W[t]^\circ}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\mathcal{E}^\star_l}\right),
\end{aligned}$$

all in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$. It is well-known that the above objects do not depend on the choice of the embedding system for $(\mathsf{X}, L)/W[t]^\circ$. In fact, $\mathscr{C}_{(\mathsf{X},L)/W^\circ}$ is nothing but the modified de Rham–Witt complex $W\omega^\bullet_{\mathsf{X}}$ [Hyo91, HK94], and $\widetilde{\mathscr{C}}_{(\mathsf{X},L)/W^\circ}$ is simply $W\widetilde{\omega}^\bullet_{\mathsf{X}}$.

Applying the notation (A.1) (with $\mathbb{W} = W$ and $\mathbb{L} = K_0$), we obtain a distinguished triangle

$$W\omega^\bullet_{\mathsf{X},K_0}[-1] \to W\widetilde{\omega}^\bullet_{\mathsf{X},K_0} \to W\omega^\bullet_{\mathsf{X},K_0} \xrightarrow{N} W\omega^\bullet_{\mathsf{X},K_0} \tag{B.4}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, similar to (B.1), in which the first arrow is given by $\wedge\,\mathrm{d}\log t$, the second arrow is the natural one, and the last arrow is the connecting map.

We would like to compare (B.1) and (B.4). We have a canonical map $\mathscr{O}_{\widehat{\mathfrak{Y}^\star_{/\mathsf{X}^\star}}} \to \varprojlim_l \mathscr{O}_{\mathcal{D}^\star_l} = \mathrm{R}\varprojlim_l \mathscr{O}_{\mathcal{D}^\star_l}$ as in [Ber97, (1.9.2)], which induces maps

$$\begin{aligned}
\mathsf{Rs}^\star_*\left(\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ}\otimes_{\mathscr{O}_{\mathcal{Y}^\star}}\mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right) &\simeq \left(\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ}\otimes_{\mathscr{O}_{\mathcal{Y}^\star}}\mathscr{O}_{\widehat{\mathfrak{Y}^\star_{/\mathsf{X}^\star}}}\right)\otimes_W K_0\\
&\to \left(\mathrm{R}\varprojlim_l \Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ}\otimes_{\mathscr{O}_{\mathcal{Y}^\star}}\mathscr{O}_{\mathcal{D}^\star_l}\right)\otimes_W K_0,\\
\mathsf{Rs}^\star_*\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right) &\simeq \left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\widehat{\mathfrak{Y}^\star_{/\mathsf{X}^\star}}}\right)\otimes_W K_0\\
&\to \left(\mathrm{R}\varprojlim_l \Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\mathcal{D}^\star_l}\right)\otimes_W K_0,
\end{aligned}$$

in $\mathbf{D}^+(\mathsf{X}^\star_{\text{ét}}, K_0)$. These maps are in fact equivalences since $(\mathsf{X}^\star, L^\star)$ is strictly semistable over κ° by an argument similar to [Ber97, §1.9]. Applying Ru_*, we obtain equivalences

$$\omega_{\mathsf{X}} \xrightarrow{\sim} W\omega^\bullet_{\mathsf{X},K_0}, \quad \widetilde{\omega}_{\mathsf{X}} \xrightarrow{\sim} W\widetilde{\omega}^\bullet_{\mathsf{X},K_0}, \tag{B.5}$$

under which (B.1) is equivalent to (B.4). On the other hand, by Lemma B.3, we have a natural map

$$\begin{aligned}
\mathsf{F}_!\mathsf{F}^*W\widetilde{\omega}^\bullet_{\mathsf{X},K_0} &\to \mathsf{Ru}_*\mathsf{F}^\star_!(\mathsf{F}^\star)^*\left(\left(\mathrm{R}\varprojlim_l \Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{\mathcal{D}^\star_l}\right)\otimes_W K_0\right)\\
&\xrightarrow{\sim} \mathsf{Ru}_*\left(\mathsf{F}^\star_!(\mathsf{F}^\star)^*\mathsf{Rs}^\star_*\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right)\\
&\to \mathsf{Ru}_*\left(\mathsf{Rs}^\star_*\mathfrak{f}^!_{(\mathsf{U}^\star,\mathsf{X}^\star)}\left(\Omega^\bullet_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\otimes_{\mathscr{O}_{\mathcal{Z}^\star}}\mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right) = \widetilde{\omega}_{(\mathsf{U},\mathsf{X})}
\end{aligned} \tag{B.6}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$.

When the model in Theorem A.8 is not smooth, we also need a cohomological variant of the rigid de Rham–Witt complex, which we now introduce. We choose an admissible embedding system $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Z}^\star, N^\star)\}$ for $(\mathsf{X}, L)/W[t]^\circ$.

For every $q \geqslant 0$, we have a natural subsheaf $\Omega^q_{\mathcal{Z}^\star/W} \subseteq \Omega^q_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}$. Put

$$\Xi^q_{\mathcal{Z}^\star} \coloneqq \frac{\Omega^{q+1}_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}}{\Omega^{q+1}_{\mathcal{Z}^\star/W}},$$

which is an $\mathscr{O}_{\mathcal{Y}^\star}$-module. The map $\wedge \,\mathrm{d}\log t$ induces a diagram

$$\begin{tikzcd}
\Omega^q_{(\mathcal{Y}^\star, M^\star)/W^\circ} \arrow[d] \arrow[dr] & \\
\Omega^{q+1}_{(\mathcal{Z}^\star, N^\star)/W^{\mathrm{triv}}} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \mathscr{O}_{\mathcal{Y}^\star} \arrow[r] & \Xi^q_{\mathcal{Z}^\star}
\end{tikzcd} \tag{B.7}$$

of coherent sheaves on $\mathcal{Y}^\star_{\text{ét}}$, compatible with differentials. Define

$$\omega^+_{\mathsf{X}} := \mathsf{R}\mathsf{u}_* \left(\mathsf{R}\mathsf{s}^\star_* \left(\Xi^\bullet_{\mathcal{Z}^\star} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right),$$
$$\omega^+_{(\mathsf{U},\mathsf{X})} := \mathsf{R}\mathsf{u}_* \left(\mathsf{R}\mathsf{s}^\star_* \mathsf{f}^!_{(\mathsf{U}^\star,\mathsf{X}^\star)} \left(\Xi^\bullet_{\mathcal{Z}^\star} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right)\right),$$

and we have a natural diagram

$$\begin{tikzcd}
\omega_{(\mathsf{U},\mathsf{X})}[-1] \arrow[d] \arrow[dr] & \\
\widetilde{\omega}_{(\mathsf{U},\mathsf{X})} \arrow[r] & \omega^+_{(\mathsf{U},\mathsf{X})}[-1]
\end{tikzcd} \tag{B.8}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, in which the vertical arrow is same as the first arrow in the first line in B.3. Let $W\Xi^\bullet_{\mathsf{X}} \in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ be the cohomological de Rham–Witt complex defined in [Sat13, Definition 8.3]. Similar to (B.5), we have a natural equivalence

$$\omega^+_{\mathsf{X}} \simeq W\Xi^\bullet_{\mathsf{X},K_0}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$ by [Sat13, Proposition 8.4]. Similar to (B.6), we have a natural map

$$\mathsf{F}_!\mathsf{F}^* W\Xi^\bullet_{\mathsf{X},K_0} \to \omega^+_{(\mathsf{U},\mathsf{X})} \tag{B.9}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, fitting into the following diagram

$$\begin{tikzcd}
\mathsf{F}_!\mathsf{F}^* W\widetilde{\omega}^\bullet_{\mathsf{X},K_0} \arrow[r] \arrow[d, "\text{(B.6)}"'] & \mathsf{F}_!\mathsf{F}^* W\Xi^\bullet_{\mathsf{X},K_0}[-1] \arrow[d, "\text{(B.9)}"] \\
\widetilde{\omega}_{(\mathsf{U},\mathsf{X})} \arrow[r] & \omega^+_{(\mathsf{U},\mathsf{X})}[-1]
\end{tikzcd} \tag{B.10}$$

in which the upper horizontal arrow is induced by the one in [Sat13, Proposition 8.10].

B.3. **Log rigid cohomology with proper support.** Let the situation be as in the previous subsection with (X, L) strictly semistable over κ°. We also assume that X is proper of pure dimension $n-1$. For every $h \geqslant 1$, let $\mathsf{X}^{(h)}$ be the disjoint union of intersections of h different irreducible components of X, and put $\mathsf{U}^{(h)} := \mathsf{U} \times_{\mathsf{X}} \mathsf{X}^{(h)}$.

Recall from [GK05, §1.5] that for a scheme Y over X of finite type, we have the log rigid cohomology $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathsf{Y}/W^\circ)$ and the log convergent cohomology $\mathrm{H}^\bullet_{\mathrm{conv}}(\mathsf{Y}/W^\circ)$ for the log-scheme $(\mathsf{Y}, L|_{\mathsf{Y}})$, with a natural map $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathsf{Y}/W^\circ) \to \mathrm{H}^\bullet_{\mathrm{conv}}(\mathsf{Y}/W^\circ)$. In particular, we have

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \omega_{\mathsf{X}}) = \mathrm{H}^q_{\mathrm{conv}}(\mathsf{X}/W^\circ), \quad \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{G}_*\omega_{\mathsf{X}\setminus\mathsf{U}}) = \mathrm{H}^q_{\mathrm{conv}}(\mathsf{X}\setminus\mathsf{U}/W^\circ).$$

for every $q \geqslant 0$.[30] Moreover, the natural map $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathsf{X}/W^\circ) \to \mathrm{H}^\bullet_{\mathrm{conv}}(\mathsf{X}/W^\circ)$ is an isomorphism by [GK05, Theorem 5.3(ii)].

Definition B.4. We define, in an *ad hoc* way, the *log rigid cohomology of* U *with proper support* to be

$$\mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) := \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \omega_{(\mathsf{U},\mathsf{X})}),$$

which *a priori* depends on the embedding $\mathsf{U} \hookrightarrow \mathsf{X}$. For $\mathsf{U} \subseteq \mathsf{U}' \subseteq \mathsf{X}$, we have a natural pushforward map $\mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) \to \mathrm{H}^q_{\mathrm{rig}}((\mathsf{U}',\mathsf{X})/W^\circ)$ by construction.

---

[30] Here, we use the fact that computing cohomology of coherent sheaves in the quasi-étale topology of analytic spaces is the same as in the G-topology.

The distinguished triangle (B.3) induces a long exact sequence

$$\cdots \to \mathrm{H}^{q-1}_{\mathrm{conv}}(\mathsf{X}\setminus\mathsf{U}/W^\circ) \to \mathrm{H}^{q}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) \to \mathrm{H}^{q}_{\mathrm{rig}}(\mathsf{X}/W^\circ) \to \mathrm{H}^{q}_{\mathrm{conv}}(\mathsf{X}\setminus\mathsf{U}/W^\circ) \to \cdots \tag{B.11}$$

of K_0-vector spaces.

We now review the weight spectral sequence for log rigid cohomology from [GK05, §5], which is the rigid analogue of Mokrane's spectral sequence for log crystalline cohomology [Mok93]. Take an admissible embedding system $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Z}^\star, N^\star)\}$ for $(\mathsf{X}, L)/W[t]^\circ$. For $j \geqslant 0$, put

$$P_j\Omega^q_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}} := \mathrm{Im}\left(\Omega^j_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}} \otimes \Omega^{q-j}_{\mathcal{Z}^\star/W} \to \Omega^q_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}\right).$$

We have the double complex

$$A^{ij}_{\mathcal{Z}^\star} := \frac{\Omega^{i+j+1}_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}}{P_j\Omega^{i+j+1}_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}}$$

of $\mathscr{O}_{\mathcal{Y}^\star}$-modules, in which the differential $A^{ij}_{\mathcal{Z}^\star} \to A^{(i+1)j}_{\mathcal{Z}^\star}$ is given by $(-1)^j\,\mathrm{d}$ and the differential $A^{ij}_{\mathcal{Z}^\star} \to A^{i(j+1)}_{\mathcal{Z}^\star}$ is given by $\wedge\,\mathrm{d}\log t$, with the filtration

$$P_kA^{ij}_{\mathcal{Z}^\star} := \frac{P_{2j+k+1}\Omega^{i+j+1}_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}}{P_j\Omega^{i+j+1}_{(\mathcal{Z}^\star,N^\star)/W^{\mathrm{triv}}}}$$

for $k \geqslant -j$. In particular, $A^{\bullet 0}_{\mathcal{Z}^\star}$ is nothing but the complex $\Xi^\bullet_{\mathcal{Z}^\star}$ from the previous subsection. Let $A^\bullet_{\mathcal{Z}^\star}$ be the total complex of $A^{\bullet\bullet}_{\mathcal{Z}^\star}$. It is shown in [GK05, §5.2] that the augmentation map $\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \to \Xi^\bullet_{\mathcal{Z}^\star} = A^{\bullet 0}_{\mathcal{Z}^\star}$ in (B.7) induces an equivalence $\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \xrightarrow{\sim} A^\bullet_{\mathcal{Z}^\star}$ in $\mathbf{D}^+(\mathcal{Y}^\star_{\text{ét}}, K_0)$. Then the total filtration on $A^\bullet_{\mathcal{Z}^\star}$ induces spectral sequences

$$\mathrm{E}(\mathsf{X})^{-k,q+k}_1 = \bigoplus_{j\geqslant\max\{0,-k\}} \mathrm{H}^{q-2j-k}_{\mathrm{rig}}(\mathsf{X}^{(2j+k+1)}/K_0) \Rightarrow \mathrm{H}^q_{\mathrm{rig}}(\mathsf{X}/W^\circ),$$

$$\mathrm{E}(\mathsf{U})^{-k,q+k}_1 = \bigoplus_{j\geqslant\max\{0,-k\}} \mathrm{H}^{q-2j-k}_{\mathrm{rig}}(\mathsf{U}^{(2j+k+1)}/K_0) \Rightarrow \mathrm{H}^q_{\mathrm{rig}}(\mathsf{U}/W^\circ),$$

which already appeared in [GK05, (4)], and

$$\mathrm{E}(\mathsf{X}\setminus\mathsf{U})^{-k,q+k}_1 = \bigoplus_{j\geqslant\max\{0,-k\}} \mathrm{H}^{q-2j-k}_{\mathrm{rig}}(\mathsf{X}^{(2j+k+1)}\setminus\mathsf{U}^{(2j+k+1)}/K_0) \Rightarrow \mathrm{H}^q_{\mathrm{conv}}(\mathsf{X}\setminus\mathsf{U}/W^\circ), \tag{B.12}$$

$$\mathrm{E}(\mathsf{U},\mathsf{X})^{-k,q+k}_1 = \bigoplus_{j\geqslant\max\{0,-k\}} \mathrm{H}^{q-2j-k}_{\mathrm{rig}}((\mathsf{U}^{(2j+k+1)},\mathsf{X}^{(2j+k+1)})/K_0) \Rightarrow \mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ). \tag{B.13}$$

Here, $\mathrm{H}^\bullet_{\mathrm{rig}}((\mathsf{U}^{(2j+k+1)},\mathsf{X}^{(2j+k+1)})/K_0)$ is defined similarly as for $\mathrm{H}^\bullet_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)$ but without the log structure, which in fact coincides with the rigid cohomology with proper support $\mathrm{H}^\bullet_{\mathrm{c,rig}}(\mathsf{U}^{(2j+k+1)}/K_0)$ defined by Berthelot since $\mathsf{X}^{(2j+k+1)}$ is proper. In particular, the spectral sequence (B.13) can also be written as

$$\mathrm{E_c}(\mathsf{U})^{-k,q+k}_1 = \bigoplus_{j\geqslant\max\{0,-k\}} \mathrm{H}^{q-2j-k}_{\mathrm{c,rig}}(\mathsf{U}^{(2j+k+1)}/K_0) \Rightarrow \mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ).$$

The following two lemmas will be used later.

Lemma B.5. *Let $d \geqslant 1$ be an integer. Suppose that* $\dim(\mathsf{X}^{(h)}\setminus\mathsf{U}^{(h)}) \leqslant d-h$ *for every* $h \geqslant 1$.

(1) The natural map $\mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) \to \mathrm{H}^q_{\mathrm{rig}}(\mathsf{X}/W^\circ)$ *is an isomorphism for* $q \geqslant 2d$.

(2) The natural map $\mathrm{E_c}(\mathsf{U})^{-k,q+k}_1 \to \mathrm{E}(\mathsf{X})^{-k,q+k}_1$ *is an isomorphism for* $q \geqslant 2d-|k|$.

(3) For the map $\mathrm{E_c}(\mathsf{U})^{0,2d-1}_1 \to \mathrm{E}(\mathsf{X})^{0,2d-1}_1$, *the direct summand*

$$\bigoplus_{j\geqslant1} \mathrm{H}^{2d-1-2j}_{\mathrm{c,rig}}(\mathsf{U}^{(2j+1)}/K_0) \to \bigoplus_{j\geqslant1} \mathrm{H}^{2d-1-2j}_{\mathrm{rig}}(\mathsf{X}^{(2j+1)}/K_0)$$

is an isomorphism.

Proof. For (1), it suffices to show that $\mathrm{H}^q_{\mathrm{conv}}(\mathsf{X}\setminus\mathsf{U}/W^\circ)=0$ for $q\geqslant 2d-1$. By the spectral sequence (B.12), it suffices to show that $\mathrm{H}^{q-2j-k}_{\mathrm{rig}}(\mathsf{X}^{(2j+k+1)}\setminus\mathsf{U}^{(2j+k+1)}/K_0)=0$ for every j, k, and $q\geqslant 2d-1$, which follows from the fact that

$$2d-1-2j-k>2(d-2j-k-1)\geqslant 2\dim(\mathsf{X}^{(2j+k+1)}\setminus\mathsf{U}^{(2j+k+1)}).$$

For (2), it follows from the fact that

$$\mathrm{H}^{q-2j-k-1}_{\mathrm{rig}}(\mathsf{X}^{(2j+k+1)}\setminus\mathsf{U}^{(2j+k+1)}/K_0)=\mathrm{H}^{q-2j-k}_{\mathrm{rig}}(\mathsf{X}^{(2j+k+1)}\setminus\mathsf{U}^{(2j+k+1)}/K_0)=0$$

for every $j\geqslant\max\{0,-k\}$ when $q\geqslant 2d-|k|$.

For (3), it follows from the fact that

$$\mathrm{H}^{2d-1-2j-1}_{\mathrm{rig}}(\mathsf{X}^{(2j+1)}\setminus\mathsf{U}^{(2j+1)}/K_0)=\mathrm{H}^{2d-1-2j}_{\mathrm{rig}}(\mathsf{X}^{(2j+1)}\setminus\mathsf{U}^{(2j+1)}/K_0)=0$$

for $j\geqslant 1$. □

Lemma B.6. *There is a spectral sequence* $\mathrm{E}^+_{\mathrm{c}}(\mathsf{U})^{-k,q+k}_1\Rightarrow\mathbf{H}^q(\mathsf{X}_{\acute{e}t},\omega^+_{(\mathsf{U},\mathsf{X})})$ *with*

$$\mathrm{E}^+_{\mathrm{c}}(\mathsf{U})^{-k,q+k}_1=\begin{cases}\mathrm{H}^{q-k}_{\mathrm{c,rig}}(\mathsf{U}^{(k+1)}/K_0), & k\geqslant 0;\\ 0, & k<0.\end{cases}$$

Moreover, the map $\mathrm{H}^q_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)=\mathbf{H}^q(\mathsf{X}_{\acute{e}t},\omega_{(\mathsf{U},\mathsf{X})})\to\mathbf{H}^q(\mathsf{X}_{\acute{e}t},\omega^+_{(\mathsf{U},\mathsf{X})})$ *is abutted by the map* $\mathrm{E}_{\mathrm{c}}(\mathsf{U})^{-k,q+k}_1\to\mathrm{E}^+_{\mathrm{c}}(\mathsf{U})^{-k,q+k}_1$ *given by the obvious projections.*

Proof. The spectral sequence follows from the filtration $P_kA^{i0}_{\mathcal{Z}^\star}$ of $A^{i0}_{\mathcal{Z}^\star}=\Xi^i_{\mathcal{Z}^\star}$. Recall that the map $\omega_{(\mathsf{U},\mathsf{X})}\to\omega^+_{(\mathsf{U},\mathsf{X})}$ is induced by the natural projection map $A^\bullet_{\mathcal{Z}^\star}\to A^{\bullet 0}_{\mathcal{Z}^\star}$. The lemma follows since $P_kA^{i0}_{\mathcal{Z}^\star}$ is the image of $P_kA^i_{\mathcal{Z}^\star}$ under this map. □

B.4. **Abel–Jacobi map via rigid cohomology.** We start to prove Theorem A.8. We fix a uniformizer ϖ of K, and regard O_K as an $W[t]$-ring via $t\mapsto\varpi$, making O_K^{can} an exact closed log-subscheme of $W[t]^\circ$. Let $\mathbb{B}_{\mathrm{cris}}$ be the crystalline period ring and $\mathbb{B}_{\mathrm{st}}$ the semistable period ring with respect to ϖ.

We fix a proper strictly semistable scheme $\mathcal{X}$ over O_K of pure (absolute) dimension $n\geqslant 2$. Then $(\mathcal{X},L^{\mathsf{X}}_{\mathcal{X}})$ is log-smooth over O_K^{can}, and $(\mathsf{X},L:=L^{\mathsf{X}}_{\mathcal{X}}|\mathsf{X})$ is strictly semistable over κ° of finite type. Consider an open immersion $\mathcal{F}\colon\mathcal{U}\to\mathcal{X}$. The following definition will be frequently used later, which is related to condition (2) in Theorem A.8.

Definition B.7. For an integer $d\geqslant 1$, we say that $\mathcal{U}$ is *d-dense* if $\dim(\mathsf{X}^{(h)}\setminus\mathsf{U}^{(h)})\leqslant d-h$ for every $h\geqslant 1$.

Put $i\colon\mathsf{X}\to\mathcal{X}$ and $j\colon X\to\mathcal{X}$ for the special fiber and the generic fiber of $\mathcal{X}$, respectively. Take an integer d satisfying $0\leqslant d<p-1$. Let $\mathscr{S}(d)_{\mathcal{X}}\in\mathbf{D}^+(\mathsf{X}_{\acute{e}t},\mathbb{Z}_{p\bullet})$ be Kato's (log) syntomic complexes for $(\mathcal{X},L^{\mathsf{X}}_{\mathcal{X}})$.[31] We have the period map $\mathscr{S}(d)_{\mathcal{X}}\to i^*\mathrm{R}j_*(\mu_{p^\bullet}^{\otimes d})_X$ which induces equivalences $\mathscr{S}(d)_{\mathcal{X}}\xrightarrow{\sim}\tau^{\leqslant d}i^*\mathrm{R}j_*(\mu_{p^\bullet}^{\otimes d})_X$ ([Kat94, Tsu00]). Put $i_{\mathcal{U}}\colon\mathsf{U}\to\mathcal{U}$ and $j_{\mathcal{U}}\colon U\to\mathcal{U}$ for the special fiber and the generic fiber of $\mathcal{U}$, respectively. We have a sequence of maps

$$\mathsf{F}_!\mathsf{F}^*i^*\mathrm{R}j_*(\mu_{p^\bullet}^{\otimes d})_X\xrightarrow{\sim}\mathsf{F}_!i^*_{\mathcal{U}}\mathrm{R}j_{\mathcal{U}*}(\mu_{p^\bullet}^{\otimes d})_U\xrightarrow{\sim}i^*\mathcal{F}_!\mathrm{R}j_{\mathcal{U}*}(\mu_{p^\bullet}^{\otimes d})_U\to i^*\mathrm{R}j_*F_!(\mu_{p^\bullet}^{\otimes d})_U$$

in $\mathbf{D}^+(\mathsf{X}_{\acute{e}t},\mathbb{Z}_{p\bullet})$, in which the last one is given by adjunction. Then we obtain the maps

$$\begin{aligned}\mathrm{R}\Gamma\left(\mathsf{X}_{\acute{e}t},\mathsf{F}_!\mathsf{F}^*\mathrm{R}\varprojlim\mathscr{S}(d)_{\mathcal{X}}\right)&\to\mathrm{R}\Gamma\left(\mathsf{X}_{\acute{e}t},\mathrm{R}\varprojlim\mathsf{F}_!\mathsf{F}^*\mathscr{S}_l(d)_{\mathcal{X}}\right)\\&\to\mathrm{R}\Gamma\left(\mathsf{X}_{\acute{e}t},\mathrm{R}\varprojlim\mathsf{F}_!\mathsf{F}^*i^*\mathrm{R}j_*(\mu_{p^\bullet}^{\otimes d})_X\right)\\&\to\mathrm{R}\Gamma\left(\mathsf{X}_{\acute{e}t},\mathrm{R}\varprojlim i^*\mathrm{R}j_*F_!(\mu_{p^\bullet}^{\otimes d})_U\right)\\&\xrightarrow{\sim}\mathrm{R}\Gamma\left(X_{\acute{e}t},\mathrm{R}\varprojlim F_!(\mu_{p^\bullet}^{\otimes d})_U\right)\end{aligned}$$

where we have used the proper base change for the last equivalence. Put

$$\mathrm{R}\Gamma_{\mathrm{c}}(U,\mathbb{Q}_p(d)):=\mathrm{R}\Gamma\left(X_{\acute{e}t},\mathrm{R}\varprojlim F_!(\mu_{p^\bullet}^{\otimes d})_U\right)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\simeq\mathrm{R}\Gamma\left(X_{\acute{e}t},F_!\mathbb{Q}_p(d)_U\right),$$

---

[31] We will recall the construction of many syntomic and crystalline complexes in §B.6 in which $\mathscr{S}(d)_{\mathcal{X}}$ is a special case.

whose q-th cohomology gives $\mathrm{H}^q_{\mathrm{c}}(U, \mathbb{Q}_p(d))$. Then the composition of the previous sequence gives a map

$$\mathrm{R\Gamma}\left(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}\right) \to \mathrm{R\Gamma}_{\mathrm{c}}(U, \mathbb{Q}_p(d)) \tag{B.14}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Q}_p)$, where we have again applied the notation (A.1) (with $\mathbb{W} = \mathbb{Z}_p$ and $\mathbb{L} = \mathbb{Q}_p$).

The Hochschild–Serre spectral sequence (Lemma A.4) induces the edge map

$$\mathrm{H}^q_{\mathrm{c}}(U, \mathbb{Q}_p(d))^0 \to \mathrm{H}^1(K, \mathrm{H}^{q-1}_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p(d))),$$

where

$$\mathrm{H}^q_{\mathrm{c}}(U, \mathbb{Q}_p(d))^0 := \mathrm{Ker}\left(\mathrm{H}^q_{\mathrm{c}}(U, \mathbb{Q}_p(d))) \to \mathrm{H}^q_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p(d))\right).$$

Let $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit$ be the inverse image of $\mathrm{H}^q_{\mathrm{c}}(U, \mathbb{Q}_p(d))^0$ under the map (B.14) (after taking q-th cohomology). Then we have the induced composite map

$$\alpha_q\colon \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit \to \mathrm{H}^1(K, \mathrm{H}^{q-1}_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p(d))) \to \mathrm{H}^1(K, \mathrm{H}^{q-1}_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}), \tag{B.15}$$

where in the last map we use the canonical embedding $\mathbb{Q}_p(d) \hookrightarrow \mathbb{B}_{\mathrm{cris}}$ in the category $\mathbf{M}_K(\mathbb{Q}_p)$.

To study the kernel of α_{2d}, we need to use crystalline complexes for $\mathcal{X}$ rather than its special fiber. Let $\mathscr{C}_{\mathcal{X}/W^{\mathrm{triv}}}$ and $\mathscr{C}_{\mathcal{X}/W[t]^\circ}$ be the objects in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ defined similarly as $\mathscr{C}_{(\mathsf{X},L)/W^{\mathrm{triv}}}$ and $\mathscr{C}_{(\mathsf{X},L)/W[t]^\circ}$ in §B.2, respectively, for which we use an embedding system for $(\mathcal{X}, L^{\mathsf{X}}_{\mathcal{X}})/W[t]^\circ$ and just replace $\mathcal{E}^\star_l$ by the PD envelop of $\mathcal{X}_l$ in $\mathcal{Z}^\star_l$. We have the following commutative diagram

$$\begin{tikzcd}
\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \arrow[r] & \mathscr{C}_{\mathcal{X}/W^{\mathrm{triv}},K_0} \arrow[r] \arrow[d] & \mathscr{C}_{\mathcal{X}/W[t]^\circ,K_0} \arrow[d] \\
& W\widetilde{\omega}^\bullet_{\mathsf{X},K_0} \arrow[r] & W\omega^\bullet_{\mathsf{X},K_0}
\end{tikzcd} \tag{B.16}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Q}_p)$, in which the first arrow is the natural map from the syntomic complex to the crystalline complex over W^{triv}, and the vertical maps are induced by the specialization at $t = 0$.

Using (B.6), we obtain a map

$$\mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \to \widetilde{\boldsymbol{\omega}}_{(\mathsf{U},\mathsf{X})} \tag{B.17}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Q}_p)$.

Lemma B.8. *Suppose that $\mathcal{U}$ is d-dense if $d \geqslant 1$. Then the composite map*

$$\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \widetilde{\boldsymbol{\omega}}_{(\mathsf{U},\mathsf{X})}) \to \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \boldsymbol{\omega}_{(\mathsf{U},\mathsf{X})}) = \mathrm{H}^{2d}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)$$

(Definition B.4) vanishes on $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit$.

Proof. When $d = 0$, the natural map $\mathrm{H}^0_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) \to \mathrm{H}^0_{\mathrm{rig}}(\mathsf{X}/W^\circ)$ is injective. When $d \geqslant 1$, since $\mathcal{U}$ is d-dense, by Lemma B.5(1) and the long exact sequence (B.11), the natural map $\mathrm{H}^{2d}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ) \to \mathrm{H}^{2d}_{\mathrm{rig}}(\mathsf{X}/W^\circ)$ is an isomorphism, in particular, injective as well.

Thus, in both cases, we may assume $\mathcal{U} = \mathcal{X}$. Then the map $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathrm{H}^{2d}_{\mathrm{rig}}(\mathsf{X}/W^\circ)$ factors through $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{X}/W[t]^\circ,K_0})$ by (B.16). We have the commutative diagram

$$\begin{tikzcd}
\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \arrow[r] \arrow[dr, dashed] & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{X}/W^{\mathrm{triv}},K_0}) \arrow[r] \arrow[d] & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{X}/W[t]^\circ,K_0}) \arrow[d, hook] \\
& \mathbf{H}^{2d}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{X}}/W^{\mathrm{triv}},K_0}) \arrow[r, hook] & \mathbf{H}^{2d}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{X}}/W[t]^\circ,K_0})
\end{tikzcd}$$

in which the injectivity of the two arrows follows from [Sat13, Proposition A.3.1].[32] By [Sat13, Lemma 9.5], $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit$ is contained in the kernel of the dashed arrow hence contained in the kernel of the map $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{X}/W[t]^\circ,K_0})$. The lemma follows. □

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[32] We will in fact review the definition of the objects $\mathscr{C}_{\overline{\mathcal{X}}/W^{\mathrm{triv}}}$ and $\mathscr{C}_{\overline{\mathcal{X}}/W[t]^\circ}$ of $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$ in a more general setup in §B.6. Meanwhile, it suffices to note that in terms of the notation, our map $\mathbf{H}^{2d}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{X}}/W^{\mathrm{triv}},K_0}) \to \mathbf{H}^{2d}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{X}}/W[t]^\circ,K_0})$ is parallel to the map $H^{2d}_{\mathrm{crys}}((\overline{X}, \overline{M})/W)_{\mathbb{Q}_p} \to H^{2d}_{\mathrm{crys}}((\overline{X}, \overline{M})/(\mathscr{E}, M_{\mathscr{E}}))_{\mathbb{Q}_p}$ in [Sat13].

The long exact sequence induced by (B.2) gives an isomorphism

$$\frac{\mathrm{H}^{q-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)}{N\mathrm{H}^{q-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)} \xrightarrow{\sim} \mathrm{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \widetilde{\omega}_{(\mathsf{U},\mathsf{X})}) \to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \omega_{(\mathsf{U},\mathsf{X})})\right).$$

Thus, by Lemma B.8, we obtain a map

$$\rho_{2d}\colon \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit \to \frac{\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)}{N\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)}. \tag{B.18}$$

For every finite extension $K^\dagger/K$ contained in $\overline{K}$ and every object $V \in \mathbf{M}_K(\mathbb{Q}_p)$, we denote by

$$\mathrm{res}_{K^\dagger}\colon \mathrm{H}^1(K,V) \to \mathrm{H}^1(K^\dagger,V)$$

the restriction map.

The following proposition is the key to the proof of Theorem A.8, whose proof will be given in §B.6.

Proposition B.9. *Suppose that $n < p$, $1 \leqslant d < p-1$, and $\mathcal{U}$ is d-dense. There exists a finite extension K_U/K (depending on U) contained in $\overline{K}$ such that*

$$\mathrm{Ker}(\rho_{2d}) \subseteq \mathrm{Ker}(\mathrm{res}_{K_U} \circ \alpha_{2d})$$

holds. Moreover, we may take K_X to be K.

Now we bring the $\mathbb{L}$-ring $\mathbb{T}$ of étale correspondences on $\mathcal{X}$ in Theorem A.8. In what follows, we write $V_{\mathbb{L}} := V \otimes_{\mathbb{Q}_p} \mathbb{L}$ for a $\mathbb{Q}_p$-vector space V. In the situation of Theorem A.8, we may assume $1 \leqslant d \leqslant n-1$ without loss of generality.

For every $t \in \mathbb{T}$, put

$$\mathcal{U}_t := \mathcal{X} \setminus \left((\mathcal{X} \setminus \mathcal{U})^{t^\vee}\right), \quad \mathcal{F}_t\colon \mathcal{U}_t \to \mathcal{X}$$

(see Notation A.1), where $t^\vee$ denotes the transpose of t. Then we have $(\mathcal{U}_t)^t \subseteq \mathcal{U}$. The element t acts on various cohomology and spectral sequences compatibly,[33] giving a commutative diagram

(B.19)

$$\begin{tikzcd}
 & \mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}} \arrow[rrr, "t^*"] \arrow[d, "N"] & & & \mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)_{\mathbb{L}} \arrow[d, "N"] & \\
 & \mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}} \arrow[rrr, "t^*"] \arrow[dr] \arrow[dd] & & & \mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)_{\mathbb{L}} \arrow[dr] \arrow[dd] & \\
 & & \mathbf{H}^{2d-1}(\mathsf{X}_{\text{ét}}, \omega^+_{(\mathsf{U}_t,\mathsf{X})})_{\mathbb{L}} \arrow[rrr, "t^*"] & & & \mathbf{H}^{2d-1}(\mathsf{X}_{\text{ét}}, \omega^+_{(\mathsf{U},\mathsf{X})})_{\mathbb{L}} \\
 & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \widetilde{\omega}_{(\mathsf{U}_t,\mathsf{X})})_{\mathbb{L}} \arrow[ur] \arrow[rrr, "t^*"] \arrow[dd] & & & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \widetilde{\omega}_{(\mathsf{U},\mathsf{X})})_{\mathbb{L}} \arrow[ur] \arrow[dd] & \\
\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_{t!}\mathsf{F}_t^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})_{\mathbb{L}} \arrow[ur, "(B.17)"] \arrow[rrr, "t^*"] & & & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})_{\mathbb{L}} \arrow[ur, "(B.17)"] & & \\
 & \mathrm{H}^{2d}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}} \arrow[rrr, "t^*"] & & & \mathrm{H}^{2d}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)_{\mathbb{L}} &
\end{tikzcd}$$

in which the two triangles are induced from (B.8).

Let $\mathbb{I} \subseteq \mathbb{T}$ be the annihilator of

$$\bigoplus_{h>1,q\geqslant 0} \mathrm{H}^q_{\mathrm{cris}}(\mathsf{X}^{(h)}/W) \otimes_{\mathbb{Z}_p} \mathbb{L} = \bigoplus_{h>1,q\geqslant 0} \mathrm{H}^q_{\mathrm{rig}}(\mathsf{X}^{(h)}/K_0)_{\mathbb{L}}.$$

---

[33] We note that for a finite étale morphism $f\colon \mathcal{X}_0 \to \mathcal{X}$, one can choose admissible embedding systems $\{(\mathsf{X}_0^\star, L_0^\star) \hookrightarrow (\mathcal{Z}_0^\star, N_0^\star)\}$ and $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Z}^\star, N^\star)\}$ for $(\mathsf{X}_0, L_0)/W[t]^\circ$ and $(\mathsf{X}, L)/W[t]^\circ$, respectively, with an étale morphism $(\mathcal{Z}_0^\star, N_0^\star) \to (\mathcal{Z}^\star, N^\star)$ that is compatible with f. We have the similar statement for embedding systems for $(\mathcal{X}_0, L^{\mathsf{X}_0}_{\mathcal{X}_0})/W[t]^\circ$ and $(\mathcal{X}, L^{\mathsf{X}}_{\mathcal{X}})/W[t]^\circ$.

Lemma B.10. *Suppose that $\mathcal{U}$ is d-dense. Then for every $t \in \mathbb{I}^{4n-5}$, the kernel of the map*

$$\frac{\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}}}{N\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}}} \to \mathbf{H}^{2d-1}(\mathsf{X}_{\text{ét}},\omega^{+}_{(\mathsf{U}_t,\mathsf{X})})_{\mathbb{L}}$$

is annihilated by t^*.

Proof. Since $\mathcal{U}$ is d-dense, $\mathcal{U}_t$ is d-dense as well. Let

$$0 = F^{-1} \subseteq F^0 \subseteq \cdots \subseteq F^{4d-2} = \mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)$$

be the filtration induced by the spectral sequence $\mathrm{E}_{\mathrm{c}}(\mathsf{U}_t)_1^{-k,q+k}$. Let V be the kernel of the map in the lemma. For every i, we regard $F^i_{\mathbb{L}} \cap V$ as the intersection of V and the image of $F^i_{\mathbb{L}}$ in the target of the map in the lemma.

By Lemma B.5(2,3) and Lemma B.6, we know that $F^i_{\mathbb{L}} \cap V/F^{i-1}_{\mathbb{L}} \cap V$ is a subquotient of $\bigoplus_{h>1,q} \mathrm{H}^q_{\mathrm{rig}}(\mathsf{X}^{(h)}/K_0)_{\mathbb{L}}$ for every $0 \leqslant i \leqslant 4d-2$. Thus, every element in $\mathbb{I}$ annihilates $F^i_{\mathbb{L}} \cap V/F^{i-1}_{\mathbb{L}} \cap V$ for $0 \leqslant i \leqslant 4d-2$, which implies that t^* annihilates V. □

Proposition B.11. *Suppose that $n < p$ and $\mathcal{U}$ is d-dense. Let t be an element in $\mathbb{I}^{4n-5}$. Then for every $c \in \mathrm{Z}^d(X)^0_{\mathbb{L}}$ such that the Zariski closure of its support in $\mathcal{X}$ is contained in $\mathcal{U}_t$, we have*

$$\mathrm{res}_{K_U}(t^*\beta_c) \in \mathrm{H}^1_f(K_U, \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U}, \mathbb{L}(d))),$$

where $\beta_c \in \mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U_t}, \mathbb{L}(d)))$ is the image of the cycle class of c in $\mathrm{H}^{2d}_{\mathrm{c}}(U_t, \mathbb{L}(d))^0$ under the edge map, and K_U/K is the finite extension in Proposition B.9.

Proof. Let $\mathscr{T}(d)_{\mathcal{X}}$ be the object in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ defined in [Sat07, Definition 4.2.4], which fits into a distinguished triangle

$$\nu^{d-1}_{\mathsf{X}}[-d-1] \to \mathscr{T}(d)_{\mathcal{X}} \to \tau^{\leqslant d} i^* \mathrm{R} j_*(\mu^{\otimes d}_{p^\bullet})_{\mathcal{X}} \xrightarrow{+1} \nu^{d-1}_{\mathsf{X}}[-d] \tag{B.20}$$

where $\nu^{d-1}_{\mathsf{X}} \in \mathbf{M}(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ is the (projective system of) logarithmic Hodge–Witt sheaves on $\mathsf{X}_{\text{ét}}$ defined in [Sat07, §2.2].

Let C be the Zariski closure of c in $\mathcal{X}$. Let $\{\mathcal{H}_i \colon C_i \to \mathcal{X}\}$ be the (finite) set of irreducible components of $\operatorname{supp} C$, which are projective flat schemes over O_K of pure (absolute) dimension $n-d$. The construction of the refined cycle class of C_i in [Sat07, Definition 5.1.2] induces a map

$$\bigoplus_i (\mathbb{Z}_p/p^\bullet)_{\mathsf{C}_i} \to \mathsf{H}^!_i \mathscr{T}(d)_{\mathcal{X}}[2d]$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ and hence a map

$$\bigoplus_i (\mathbb{Q}_p)_{\mathsf{C}_i} \to \mathsf{H}^!_i \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p}[2d]$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_p)$. As $\operatorname{supp} C$ is contained in $\mathcal{U}_t$, the natural map $\mathsf{H}^!_i \mathsf{F}_{t!}\mathsf{F}^*_t \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p} \to \mathsf{H}^!_i \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p}$ is an equivalence. Thus, we obtain a Gysin map

$$\bigoplus_i \mathrm{H}^0(C_i, \mathbb{L}) \to \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_{t!}\mathsf{F}^*_t \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p})_{\mathbb{L}}.$$

Let $\tau_c \in \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_{t!}\mathsf{F}^*_t \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p})_{\mathbb{L}}$ be the image of the cycle C under the above map.

Since $d < n < p$, the period map induces equivalences $\mathscr{S}(d)_{\mathcal{X}} \xrightarrow{\sim} \tau^{\leqslant d} i^* \mathrm{R} j_*(\mu^{\otimes d}_{p^\bullet})_{\mathcal{X}}$. Replacing $\tau^{\leqslant d} i^* \mathrm{R} j_*(\mu^{\otimes d}_{p^\bullet})_{\mathcal{X}}$ by $\mathscr{S}(d)_{\mathcal{X}}$ in (B.20) and applying (A.1), we obtain a distinguished triangle

$$\nu^{d-1}_{\mathsf{X},\mathbb{Q}_p}[-d-1] \to \mathscr{T}(d)_{\mathcal{X},\mathbb{Q}_p} \to \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \xrightarrow{+1} \nu^{d-1}_{\mathsf{X},\mathbb{Q}_p}[-d]$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Q}_p)$. Denote by $\sigma_c \in \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_{t!}\mathsf{F}^*_t \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})_{\mathbb{L}}$ the image of τ_c under the above sequence, which then belongs to $\mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_{t!}\mathsf{F}^*_t \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^{\heartsuit}_{\mathbb{L}}$ since the cycle class of c vanishes in $\mathrm{H}^{2d}(\overline{X}, \mathbb{L}(d))$ and hence in $\mathrm{H}^{2d}_{\mathrm{c}}(\overline{U_t}, \mathbb{L}(d))$. Now we compute $\rho_{2d}(\sigma_c)$, where ρ_{2d} is defined in (B.18).

In the following diagram

$$\begin{tikzcd}
\mathsf{F}_{t!}\mathsf{F}_t^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \arrow[r, "(B.16)"] \arrow[d] & \mathsf{F}_{t!}\mathsf{F}_t^* W\widetilde{\omega}^\bullet_{\mathsf{X},\mathbb{Q}_p} \arrow[r, "(B.6)"] \arrow[d] & \widetilde{\omega}_{(\mathsf{U},\mathsf{X})} \arrow[d, "(B.8)"] \\
\mathsf{F}_{t!}\mathsf{F}_t^*\nu^{d-1}_{\mathsf{X},\mathbb{Q}_p}[-d] \arrow[r] & \mathsf{F}_{t!}\mathsf{F}_t^* W\Xi^\bullet_{\mathsf{X},\mathbb{Q}_p}[-1] \arrow[r, "(B.9)"] & \omega^+_{(\mathsf{U},\mathsf{X})}[-1]
\end{tikzcd}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}},\mathbb{Q}_p)$, the left square commutes as shown in the proof of [Sat13, Proposition 9.10] (followed by taking limit and applying $\mathsf{F}_{t!}\mathsf{F}_t^*$), and the right square is (B.10). It follows that

$$\rho_{2d}(\sigma_c)\in\ker\left(\frac{\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}}}{N\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}_t,\mathsf{X})/W^\circ)_{\mathbb{L}}}\to\mathbf{H}^{2d-1}(\mathsf{X}_{\text{ét}},\omega^+_{(\mathsf{U}_t,\mathsf{X})})_{\mathbb{L}}\right).$$

By Lemma B.10 and the diagram (B.19), we have $\rho_{2d}(t^*\sigma_c)=t^*\rho_{2d}(\sigma_c)=0$. Finally, by Proposition B.9, we have $\mathrm{res}_{K_U}(\alpha_{2d}(t^*\sigma_c))=0$. Since $\mathrm{res}_{K_U}(\alpha_{2d}(t^*\sigma_c))$ coincides with the image of $\mathrm{res}_{K_U}(t^*\beta_c)$ under the map

$$\mathrm{H}^1(K_U,\mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U},\mathbb{L}(d)))\to\mathrm{H}^1(K_U,\mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{cris}})_{\mathbb{L}},$$

$\mathrm{res}_{K_U}(t^*\beta_c)$ belongs to $\mathrm{H}^1_f(K_U,\mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U},\mathbb{L}(d)))$. The proposition is proved. □

Proof of Theorem A.8. We consider the localized cohomology $\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'}$, which is a direct summand of $\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))$ in the category $\mathbf{M}_K(\mathbb{L})$. By the C_{st}-comparison theorem for $\mathcal{X}$, Mokrane's weight spectral sequence [Mok93] and (A.3), we know that $\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'}$ is either zero or a semistable representation of G_K pure of weight -1 (Definition A.11). In particular, [Nek93, Proposition 1.25] implies the following

(∗) For every short exact sequence $0\to\mathbb{L}(1)\to\mathrm{E}\to\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d))\to0$ in $\mathbf{M}_K(\mathbb{L})$ such that $[\mathrm{E}^\vee(1)]$ belongs to $\mathrm{H}^1_f(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d')))\cap\mathrm{H}^1(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'})$, we have a short exact sequence

$$0\to\mathrm{H}^1_f(K^\dagger,\mathbb{L}(1))\to\mathrm{H}^1_f(K^\dagger,\mathrm{E})\to\mathrm{H}^1_f(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d)))\to0$$

for every finite extension $K^\dagger/K$ contained in $\overline{K}$.

Let $\mathbb{J}'\subseteq\mathbb{T}$ be the annihilator of

$$\bigoplus_{d'=1}^{n-1}\mathrm{Ker}\left(\mathrm{H}^1(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d')))\to\mathrm{H}^1(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'})\right).$$

In particular, we have $[\mathrm{E}^{t_1'^*c'}]\in\mathrm{H}^1(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'})$ for every $t_1'\in\mathbb{J}'$. We need to apply Proposition B.11 three times.

First, we apply Proposition B.11 to the case $c=t_1'^*c'$, $d=d'$, and $\mathcal{U}=\mathcal{X}$. Then for every $t_2'\in\mathbb{I}^{4n-5}$, the class $\beta_{t_2'^*t_1'^*c'}=t_2'^*\beta_{t_1'^*c'}$ belongs to $\mathrm{H}^1_f(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d')))$. In other words, we have

$$[\mathrm{E}^\vee_{t'^*c'}(1)]\in\mathrm{H}^1_f(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d')))\cap\mathrm{H}^1(K,\mathrm{H}^{2d'-1}(\overline{X},\mathbb{L}(d'))_{\mathfrak{m}'}),$$

where $t':=t_1't_2'$. By (∗), we have a short exact sequence

$$0\to\mathrm{H}^1_f(K^\dagger,\mathbb{L}(1))\to\mathrm{H}^1_f(K^\dagger,\mathrm{E}_{t'^*c'})\to\mathrm{H}^1_f(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d)))\to0$$

for every finite extension $K^\dagger/K$ contained in $\overline{K}$. Now we denote by $\mathrm{H}^1_\sharp(K^\dagger,\mathrm{E}_{t'^*c'})$ the inverse image of the subspace $\mathrm{H}^1_f(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d)))$ under the map $\mathrm{H}^1(K^\dagger,\mathrm{E}_{t'^*c'})\to\mathrm{H}^1(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d)))$.[34] Then the diagram

(B.21)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1_f(K^\dagger,\mathbb{L}(1)) \arrow[r] \arrow[d] & \mathrm{H}^1_f(K^\dagger,\mathrm{E}_{t'^*c'}) \arrow[r] \arrow[d] & \mathrm{H}^1_f(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d))) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{H}^1(K^\dagger,\mathbb{L}(1)) \arrow[r] & \mathrm{H}^1_\sharp(K^\dagger,\mathrm{E}_{t'^*c'}) \arrow[r] & \mathrm{H}^1_f(K^\dagger,\mathrm{H}^{2d-1}(\overline{X},\mathbb{L}(d))) \arrow[r] & 0
\end{tikzcd}$$

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[34] In fact, $\mathrm{H}^1_\sharp(K^\dagger,\mathrm{E}_{t'^*c'})=\mathrm{H}^1_{\mathrm{st}}(K^\dagger,\mathrm{E}_{t'^*c'})$.

is a pushout of extensions.

Second, we apply Proposition B.11 to the case $c = c$, $d = d$, and $\mathcal{U} = \mathcal{X}$. Then for every $t_1 \in \mathbb{I}^{4n-5}$, the class $\beta_{t_1^*c} = t_1^*\beta_c$ belongs to $\mathrm{H}^1_f(K, \mathrm{H}^{2d-1}(\overline{X}, \mathbb{L}(d)))$. In other words, $[\mathrm{E}^{t_1^*c}] \in \mathrm{H}^1_f(K, \mathrm{H}^{2d-1}(\overline{X}, \mathbb{L}(d)))$ and hence $[\mathrm{E}^{t_1^*c}_{t'^*c'}] \in \mathrm{H}^1_\sharp(K, \mathrm{E}_{t'^*c'})$.

Third, we apply Proposition B.11 to the case $c = t_1^*c$, $d = d$, and $\mathcal{U} = \mathcal{X} \setminus (\operatorname{supp} C')^{t'}$. Then for every $t_2 \in \mathbb{I}^{4n-5}$, the class $\mathrm{res}_{K_U}(\beta_{t_2^*t_1^*c}) = \mathrm{res}_{K_U}(t_2^*\beta_{t_1^*c})$ belongs to $\mathrm{H}^1_f(K_U, \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U_t}, \mathbb{L}(d)))$, where $t \coloneqq t_1t_2$. Note that the fact that $\mathcal{U}$ is d-dense follows from condition (2), and that $\operatorname{supp} C \subseteq \mathcal{U}_t$ follows from condition (1). Since $[\mathrm{E}^{t^*c}_{t'^*c'}]$ is the image of β_{t^*c} under the pushout map

$$\mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U_t}, \mathbb{L}(d))) \to \mathrm{H}^1(K, \mathrm{E}_{t'^*c'}),$$

we have $\mathrm{res}_{K_U}([\mathrm{E}^{t^*c}_{t'^*c'}]) \in \mathrm{H}^1_f(K_U, \mathrm{E}_{t'^*c'})$. Since the inverse image of $\mathrm{H}^1_f(K_U, \mathbb{L}(1))$ under res_{K_U} coincides with $\mathrm{H}^1_f(K, \mathbb{L}(1))$, we conclude that $[\mathrm{E}^{t^*c}_{t'^*c'}] \in \mathrm{H}^1_f(K, \mathrm{E}_{t'^*c'})$ by the diagram (B.21) (for $K^\dagger = K, K_U$).

From the above discussion, the conclusion of the theorem holds for every pair of elements $t \in (\mathbb{I}^{4n-5} \setminus \mathfrak{m})^2$ and $t' \in (\mathbb{J}' \setminus \mathfrak{m}') \cdot (\mathbb{I}^{4n-5} \setminus \mathfrak{m}')$. It is clear that $\mathbb{J}' \setminus \mathfrak{m}' \neq \emptyset$. By (A.3), we also have $\mathbb{I} \setminus \mathfrak{m} \neq \emptyset$ and $\mathbb{I} \setminus \mathfrak{m}' \neq \emptyset$.

The theorem is proved. □

B.5. **Further preparation.** Let **A** be a W-linear additive category. Following Fontaine, we say that a (φ, N)-*module* in **A** is an object C in **A** with a W-semi-linear endomorphism $\varphi_C \colon C \to C$ (called the Frobenius operator) and a W-linear endomorphism $N_C \colon C \to C$ (called the monodromy operator) satisfying that $N_C \circ \varphi_C = p \cdot \varphi_C \circ N_C$. A map between (φ, N)-modules is a map that commutes with both Frobenius operators and monodromy operators.

Suppose that **A** admits W-linear tensors. For two (φ, N)-modules C and D in **A**, we equip $C \otimes_W D$ with a (φ, N)-module structure with the obvious W-action, together with

$$\varphi_{C\otimes_W D} \coloneqq \varphi_C \otimes \varphi_D, \quad N_{C\otimes_W D} \coloneqq N_C \otimes 1 + 1 \otimes N_D.$$

When **A** is an abelian category, we say that a (φ, N)-module is *nilpotent* if it has finite length and the monodromy operator is nilpotent.

Definition B.12. Let **D** be a triangulated category. For a finite complex

$$\mathscr{C}_0 \to \mathscr{C}_1 \to \cdots \to \mathscr{C}_m$$

in **D**, we have a canonical diagram

$$\begin{tikzcd}
\mathscr{C}_{-1} \arrow[r] \arrow[d] & 0 \arrow[d] & & & & \\
\mathscr{C}_0 \arrow[r] & \bullet \arrow[r] \arrow[d] & 0 \arrow[d] & & & \\
& \mathscr{C}_1 \arrow[r] & \bullet \arrow[r] \arrow[d] & \ddots \arrow[d] & & \\
& & \ddots \arrow[r] & \bullet \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& & & \mathscr{C}_{m-2} \arrow[r] & \bullet \arrow[r] \arrow[d] & 0 \arrow[d] \\
& & & & \mathscr{C}_{m-1} \arrow[r] & \mathscr{C}_m
\end{tikzcd}$$

in **D**, in which every square is a homotopy fiber. We call $\mathscr{C}_{-1}$ the *successive homotopy fiber* of the complex $\mathscr{C}_0 \to \mathscr{C}_1 \to \cdots \to \mathscr{C}_m$.

If $\mathscr{C}_0 \to \mathscr{C}_1 \to \cdots \to \mathscr{C}_m$ is a complex of (φ, N)-modules in a W-linear triangulated category, then $\mathscr{C}_{-1}$ is canonically a (φ, N)-module.

Next, we review some period rings. For every $l \geqslant 1$, let R_l be the PD envelop of O_K/p^l in $W[t]/p^l$, and let P_l be the R_l-ring wit a discrete action by G_K defined in [Kat94, (3.2)] or [Tsu99, §1.6]. Then $R_\bullet$ and $P_\bullet$ are (φ, N)-modules in $\mathbf{M}(W_\bullet)$. Put

$$\mathbb{K} := \left(\varprojlim_l R_l\right) \otimes_W K_0 \in \mathbf{M}(K_0), \quad \widehat{\mathbb{B}_{\mathrm{st}}^+} := \left(\varprojlim_l P_l\right) \otimes_W K_0 \in \mathbf{M}_K(K_0),$$

which are again (φ, N)-modules in relevant categories. See [Bre97, §2 & §4] for an explicit description.[35]

Lemma B.13. *The following holds.*

1. *There is a canonical isomorphism $\mathbb{B}_{\mathrm{st}}^+ \simeq (\widehat{\mathbb{B}_{\mathrm{st}}^+})^{N-\mathrm{nilp}}$.*
2. *The ring $\widehat{\mathbb{B}_{\mathrm{st}}^+}$ is flat over $\mathbb{K}$.*
3. *The monodromy operator $N\colon \widehat{\mathbb{B}_{\mathrm{st}}^+} \to \widehat{\mathbb{B}_{\mathrm{st}}^+}$ is surjective.*

Proof. For (1), this is [Tsu99, Proposition 4.1.3].

For (2), this is [Tsu99, Proposition 4.1.5].

For (3), we have $N = \left(\varprojlim_l N_l\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where N_l is the monodromy operator on P_l [Kat94, Definition 3.4]. Then (3) follows from the fact that N_l is surjective and $\mathrm{R}^1 \varprojlim_l \mathrm{Ker}\, N_l = 0$ [Kat94, Corollary 3.6]. □

For two (φ, N)-modules C and D in $\mathbf{M}(K_0)$, the (φ, N)-module structure on $C \otimes_W D$ clearly descends to $C \otimes_{K_0} D$. Now we consider a nilpotent (φ, N)-module D in $\mathbf{M}(K_0)$. By Lemma B.13(1,3), we have a Frobenius equivariant diagram

$$\begin{tikzcd}
0 \arrow[r] & \left(D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0} \arrow[r] \arrow[d, equal] & D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+ \arrow[r, "N"] \arrow[d, hook] & D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+ \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \left(D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0} \arrow[r] & D \otimes_{K_0} \widehat{\mathbb{B}_{\mathrm{st}}^+} \arrow[r, "N"] & D \otimes_{K_0} \widehat{\mathbb{B}_{\mathrm{st}}^+} \arrow[r] & 0
\end{tikzcd}$$

in $\mathbf{M}_K(K_0)$, in which the two rows are short exact sequences. It induces a diagram

$$\begin{tikzcd}
\dfrac{D}{ND} \arrow[r, hook] \arrow[d] & \mathrm{H}^1\left(K, \left(D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0}\right) \arrow[d, equal] \\
\dfrac{D \otimes_{K_0} \mathbb{K}}{N(D \otimes_{K_0} \mathbb{K})} \arrow[r, hook] & \mathrm{H}^1\left(K, \left(D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0}\right)
\end{tikzcd}$$

of edge maps as $(\mathbb{B}_{\mathrm{st}}^+)^{\mathrm{G}_K} = K_0$ and $(\widehat{\mathbb{B}_{\mathrm{st}}^+})^{\mathrm{G}_K} = \mathbb{K}$ [Bre97, Corollaire 4.1.3].

Lemma B.14. *For every integer r, the restricted edge map*

$$\left(\frac{D \otimes_{K_0} \mathbb{K}}{N(D \otimes_{K_0} \mathbb{K})}\right)^{\varphi = p^r} \to \mathrm{H}^1\left(K, (D \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+)^{N=0}\right)$$

factors through the map

$$\frac{D \otimes_{K_0} \mathbb{K}}{N(D \otimes_{K_0} \mathbb{K})} \to \frac{D}{ND}$$

induced by the specialization map $\mathbb{K} \to K_0$ at $t = 0$.

Proof. This follows from the same proof of [Lan99, Lemma 2.6].[36] □

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[35] Our $\mathbb{K}$ and $\widehat{\mathbb{B}_{\mathrm{st}}^+}$ are denoted as S_{min} and $\widehat{B_{\mathrm{st}}^+}$ in [Bre97], respectively.

[36] In [Lan99], the author works with $\widehat{K\langle t\rangle}$, which is different from our $\mathbb{K}$ when $K/\mathbb{Q}_p$ is ramified. However, such difference will not affect the proof in view of the explicit description of $\mathbb{K}$ in [Bre97, §4].

Remark B.15. Lemma B.14 is certainly wrong without the restriction to the part $\varphi = p^r$ since the specialization map $\mathbb{K}/N\mathbb{K} \to K_0$ has a large kernel. However, we do not know whether the edge map $D \otimes_{K_0} \mathbb{K} \to \mathrm{H}^1\left(K, (D \otimes_{K_0} \mathbb{B}_{\mathrm{st}})^{N=0}\right)$ factors though the quotient D/ND, which is equivalent to the inclusion $\mathrm{Ker}(\mathbb{K} \to K_0) \subseteq N(\widehat{\mathbb{B}_{\mathrm{st}}})^{\mathrm{G}_K}$ where $\widehat{\mathbb{B}_{\mathrm{st}}} \coloneqq \widehat{\mathbb{B}^+_{\mathrm{st}}} \otimes_{\mathbb{B}^+_{\mathrm{cris}}} \mathbb{B}_{\mathrm{cris}}$. See [Bre97, §5] for the mystery of $(\widehat{\mathbb{B}_{\mathrm{st}}})^{\mathrm{G}_K}$.

B.6. **Proof of Proposition B.9.** This subsection is devoted to the proof of Proposition B.9, for which we use de Jong's alterations. We may assume $1 \leqslant d \leqslant n-1$.

By [dJ96, Theorem 6.5], we may find a finite extension K' of K (depending on U) contained in $\overline{K}$, a projective strictly semistable scheme $\mathcal{X}'$ over $O_{K'}$ and a generically finite morphisms $\mathscr{A}\colon \mathcal{X}' \to \mathcal{X}$ over O_K,[37] such that $(\mathcal{X}', A^{-1}U)$ is a strict semistable pair [dJ96, §6.3]. Let $\mathcal{F}'\colon \mathcal{U}' \to \mathcal{X}'$ be the open subscheme with $U' = A^{-1}U$ and such that $\mathcal{X}' \setminus \mathcal{U}'$ is flat over $O_{K'}$. Note that $\mathcal{U}'$ may strictly contain $\mathscr{A}^{-1}\mathcal{U}$. We have objects with respect to $(\mathcal{X}', \mathcal{U}')$ and we put a *prime* for their notations.

Lemma B.16. *Suppose that $n < p$. Then for every $q \geqslant 0$, the composite map*

$$\mathbf{H}^q(\mathsf{X}'_{\text{ét}}, \mathsf{F}'_!\mathsf{F}'^*\mathscr{S}(d)_{\mathcal{X}',\mathbb{Q}_p}) \to \mathbf{H}^q(\mathsf{X}'_{\text{ét}}, \widetilde{\omega}_{(\mathsf{U}',\mathsf{X}')}) \to \mathbf{H}^q(\mathsf{X}'_{\text{ét}}, \omega_{(\mathsf{U}',\mathsf{X}')}) = \mathrm{H}^q_{\mathrm{rig}}((\mathsf{U}',\mathsf{X}')/W'^\circ)$$

(Definition B.4) vanishes on $\mathbf{H}^q(\mathsf{X}'_{\text{ét}}, \mathsf{F}'_!\mathsf{F}'^*\mathscr{S}(d)_{\mathcal{X}',\mathbb{Q}_p})^\heartsuit$.

Note that this lemma does not follow from Lemma B.8 even for $q = 2d$, since $\mathcal{U}'$ is not necessarily d-dense anymore. By Lemma B.16, we have the map

$$\rho'_q\colon \mathbf{H}^q(\mathsf{X}'_{\text{ét}}, \mathsf{F}'_!\mathsf{F}'^*\mathscr{S}(d)_{\mathcal{X}',\mathbb{Q}_p})^\heartsuit \to \frac{\mathrm{H}^{q-1}_{\mathrm{rig}}((\mathsf{U}',\mathsf{X}')/W'^\circ)}{N\mathrm{H}^{q-1}_{\mathrm{rig}}((\mathsf{U}',\mathsf{X}')/W'^\circ)} \tag{B.22}$$

similar to (B.18) for every $q \geqslant 0$.

Lemma B.17. *Suppose that $n < p$. Then*

$$\mathrm{Ker}(\rho'_q) \subseteq \mathrm{Ker}(\alpha'_q)$$

holds for every $q \geqslant 0$.

We now prove Proposition B.9 assuming the above two lemmas, whose proofs are postponed later.

Proof of Proposition B.9. Take $K_U = K'$. It is also clear that we may take K_X to be K. We have the commutative diagram

$$\begin{tikzcd}
\dfrac{\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}',\mathsf{X}')/W'^\circ)}{N\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U}',\mathsf{X}')/W'^\circ)} & \mathbf{H}^{2d}(\mathsf{X}'_{\text{ét}}, \mathsf{F}'_!\mathsf{F}'^*\mathscr{S}(d)_{\mathcal{X}',\mathbb{Q}_p})^\heartsuit \arrow[l] \arrow[r] & \mathrm{H}^1(K', \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U'}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}) \\
\dfrac{\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)}{N\mathrm{H}^{2d-1}_{\mathrm{rig}}((\mathsf{U},\mathsf{X})/W^\circ)} \arrow[u, "\mathsf{A}^*"] & \mathbf{H}^{2d}(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^\heartsuit \arrow[l] \arrow[r] \arrow[u, "\mathsf{A}^*"] & \mathrm{H}^1(K, \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}) \arrow[u, "A^*"']
\end{tikzcd}$$

of $\mathbb{Q}_p$-vector spaces.[38] By Lemma B.17, to prove the proposition, it suffices to show that the map

$$\mathrm{H}^1(K', \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}) \to \mathrm{H}^1(K', \mathrm{H}^{2d-1}_{\mathrm{c}}(\overline{U'}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}})$$

is injective. However, this follows from the fact that the map $\mathrm{H}^q_{\mathrm{c}}(\overline{U}, \mathbb{Q}_p) \to \mathrm{H}^q_{\mathrm{c}}(\overline{U'}, \mathbb{Q}_p)$ in the category $\mathbf{M}_{K'}(\mathbb{Q}_p)$ admits a section, which is a consequence of the usual Poincaré duality for étale cohomology of $\overline{U}$ and $\overline{U'}$. The proposition is proved. □

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[37] The letter $\mathscr{A}$ stands for *alteration*.

[38] Note that U' may properly contain $\mathsf{A}^{-1}\mathsf{U}$. By A^*, we mean the restriction map with a possible composition of the pushforward map along the inclusion $\mathsf{A}^{-1}\mathsf{U} \subseteq \mathsf{U}'(\subseteq \mathsf{X}')$ of log rigid cohomology (Definition B.4) or étale cohomology with proper support.

It remains to show Lemma B.16 and Lemma B.17. Since we will only study $(\mathcal{X}', \mathcal{U}')$ from now on, we will suppress the *prime* from all notation to release some burden. Put $\mathcal{V} := \mathcal{X} \setminus \mathcal{U}$, and for every $h \geqslant 1$, let $\mathcal{V}^{(h)}$ be the disjoint union of intersections of h different irreducible components of $\mathcal{V}$, which is either empty or a strictly semistable scheme over O_K of pure (absolute) dimension $n-h$. For notational convenience, we also put $\mathcal{V}^{(0)} := \mathcal{X}$ and $\mathcal{V}^{(-1)} := (\mathcal{U}, \mathcal{X})$. Denote by $\mathcal{G}^{(h)} \colon \mathcal{V}^{(h)} \to \mathcal{X}$ the obvious morphism for $h \geqslant 0$.

Lemma B.18. *For every $h \geqslant 0$, the pullback of the log structure $L_{\mathcal{X}}^{\mathsf{X}}$ for $\mathcal{X}$ to $\mathcal{V}^{(h)}$ coincides with $L_{\mathcal{V}^{(h)}}^{\mathsf{V}^{(h)}}$.*

Proof. The question is local in the Zariski topology. By [dJ96, §6.4], Zariski locally $\mathcal{X}$ is smooth over $O_K[t_1, \dots, t_i, s_1, \dots, s_j]/(t_1 \cdots t_i - \varpi)$. We may assume $j \geqslant h$ since otherwise $\mathcal{V}^{(h)}$ is empty in this chart. It suffices to consider the open and closed subscheme $\mathcal{T}$ of $\mathcal{V}^{(h)}$ defined by $s_{j-h+1} = \cdots = s_j = 0$. Now locally $L_{\mathcal{X}}^{\mathsf{X}}$ and $L_{\mathcal{T}}^{\mathsf{T}}$ are the log structures associated with the pre-log structures $\mathbb{N}^i \to \mathscr{O}_{\mathcal{X}}$ and $\mathbb{N}^i \to \mathscr{O}_{\mathcal{T}}$ sending 1 in the i'-th factor to the pullback of $t_{i'}$ for $1 \leqslant i' \leqslant i$, respectively. The lemma follows immediately. □

Our first step is to construct, for every $r \geqslant 0$, a syntomic complex $\mathscr{S}(r)_{\overline{\mathcal{V}^{(-1)}}} \in \mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$, together with a period map

$$\mathscr{S}(r)_{\overline{\mathcal{V}^{(-1)}}} \to \bar{i}^* \mathrm{R}\bar{j}_* \overline{F}_!(\mu_{p^\bullet}^{\otimes r})_{\overline{U}} \tag{B.23}$$

when $0 \leqslant r < p-1$, that becomes an equivalence when $n-1 \leqslant r < p-1$. The construction is inspired by the observation in the following remark.

Remark B.19. After choosing an order on the (finite) set of irreducible components of $\mathcal{V}$, we have an exact sequence

$$0 \to \overline{F}_!(\mu_{p^\bullet}^{\otimes r})_{\overline{U}} \to \overline{G^{(0)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(0)}}} \to \overline{G^{(1)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(1)}}} \to \overline{G^{(2)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(2)}}} \to \cdots \to \overline{G^{(n-1)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(n-1)}}} \to 0$$

in $\mathbf{M}_{\mathrm{G}_K}(\overline{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$. In particular, $\overline{F}_!(\mu_{p^\bullet}^{\otimes r})_{\overline{U}}$ is canonically equivalent to the successive homotopy fiber of the complex

$$\overline{G^{(0)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(0)}}} \to \overline{G^{(1)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(1)}}} \to \overline{G^{(2)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(2)}}} \to \cdots \to \overline{G^{(n-1)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(n-1)}}}.$$

In order to unify the notation, we put

$$\mathscr{N}(r)_{\overline{\mathcal{V}^{(-1)}}} := \bar{i}^* \mathrm{R}\bar{j}_* \overline{F}_!(\mu_{p^\bullet}^{\otimes r})_{\overline{U}} \in \mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet}), \quad \mathscr{N}(r)_{\mathcal{V}^{(-1)}} := i^* \mathrm{R} j_* F_!(\mu_{p^\bullet}^{\otimes r})_U \in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet}),$$

and

$$\mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}}} := \bar{i}^* \mathrm{R}\bar{j}_* \overline{G^{(h)}}_*(\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(h)}}} \in \mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet}), \quad \mathscr{N}(r)_{\mathcal{V}^{(h)}} := i^* \mathrm{R} j_* G^{(h)}_*(\mu_{p^\bullet}^{\otimes r})_{V^{(h)}} \in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet}),$$

for $h \geqslant 0$.[39]

To define $\mathscr{S}(r)_{\overline{\mathcal{V}^{(-1)}}}$, we need to consider all extensions of K in $\overline{K}$. A *Galois embedding system* for $O_{\overline{K}}^{\mathrm{can}}/W[t]^\circ$ consists of

- an increasing tower $K = K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of finite Galois extensions of K with $\bigcup_m K_m = \overline{K}$ (and we regard $O_{K_m}^{\mathrm{can}}$ as a log-scheme over O_K^{can}),

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[39] The letter $\mathscr{N}$ stands for *nearby*.

- for every $m \geqslant 1$, an embedding system $\{O_{K_m}^{\mathrm{can}} \hookrightarrow (\mathcal{Z}_m^{\flat}, N_m^{\flat})\}$ for $O_{K_m}^{\mathrm{can}}/W[t]^{\circ}$ with a compatible action of $\mathrm{Gal}(K_m/K)$ that fits into a commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
O_{K_{m+1}}^{\mathrm{can}} \arrow[r, hook] \arrow[d] & (\mathcal{Z}_{m+1}^{\flat}, N_{m+1}^{\flat}) \arrow[d] \\
O_{K_m}^{\mathrm{can}} \arrow[r, hook] \arrow[d] & (\mathcal{Z}_m^{\flat}, N_m^{\flat}) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

of log-schemes over $W[t]^{\circ}$ that is G_K-equivariant.

It is clear that Galois embedding system for $O_{\overline{K}}^{\mathrm{can}}/W[t]^{\circ}$ exists.

We now choose a Galois embedding system for $O_{\overline{K}}^{\mathrm{can}}/W[t]^{\circ}$ as above, and write $\kappa = \kappa_1 \subseteq \kappa_2 \subseteq \kappa_3 \subseteq \cdots$ for the induced tower of residue fields. We also choose an embedding system $\{(\mathcal{X}^{\star}, L^{\star}) \hookrightarrow (\mathcal{Z}^{\star}, N^{\star})\}$ for $(\mathcal{X}, L_{\mathcal{X}}^{\mathsf{X}})/W[t]^{\circ}$. For $m \geqslant 1$, put

$$(\mathcal{Z}_m^{\star}, N_m^{\star}) \coloneqq (\mathcal{Z}^{\star}, N^{\star}) \times_{W[t]^{\circ}} (\mathcal{Z}_m^{\flat}, N_m^{\flat}).$$

Let $\mathcal{T}$ be an irreducible component of $\mathcal{V}^{(h)}$ for some $h \geqslant 0$. Lemma B.18 implies that

$$\left\{(\mathcal{T}, L_{\mathcal{T}}^{\mathsf{T}}) \times_{(\mathcal{X}, L_{\mathcal{X}}^{\mathsf{X}})} (\mathcal{X}^{\star}, L^{\star}) \hookrightarrow (\mathcal{Z}^{\star}, N^{\star})\right\}$$

is an embedding system for $(\mathcal{T}, L_{\mathcal{T}}^{\mathsf{T}})/W[t]^{\circ}$. For every $m \geqslant 1$, let $\mathcal{E}_{m,l}^{\star}$ be the PD envelop of $\mathcal{T}^{\star} \otimes_{O_K} O_{K_m}/p^l$ in $\mathcal{Z}_m^{\star} \otimes \mathbb{Z}/p^l$. For $i \geqslant 1$, let $\mathscr{J}_{m,l}^{[i]} \subseteq \mathscr{O}_{\mathcal{E}_{m,l}^{\star}}$ be the i-th divided power of the ideal $\mathscr{J}_{m,l} \coloneqq \mathrm{Ker}\left(\mathscr{O}_{\mathcal{E}_{m,l}^{\star}} \to \mathscr{O}_{\mathcal{T}^{\star} \otimes_{O_K} O_{K_m}/p^l}\right)$. For $i \leqslant 0$, we put $\mathscr{J}_{m,l}^{[i]} \coloneqq \mathscr{O}_{\mathcal{E}_{m,l}^{\star}}$. We have the complex

$$B(r)_{\mathcal{T},m} \colon \mathscr{J}_{m,\bullet}^{[r]} \to \mathscr{J}_{m,\bullet}^{[r-1]} \otimes_{\mathscr{O}_{\mathcal{Z}_m^{\star}}} \Omega^1_{(\mathcal{Z}_m^{\star}, N_m^{\star})/W^{\mathrm{triv}}} \to \mathscr{J}_{m,\bullet}^{[r-2]} \otimes_{\mathscr{O}_{\mathcal{Z}_m^{\star}}} \Omega^2_{(\mathcal{Z}_m^{\star}, N_m^{\star})/W^{\mathrm{triv}}} \to \cdots \tag{B.24}$$

regarded as an object in $\mathbf{C}^+_{\mathrm{G}_K}((\mathsf{T} \times_{\mathsf{X}} \mathsf{X}^{\star} \otimes_{\kappa} \kappa_m)_{\text{ét}}, W_{\bullet})$, where $\mathscr{J}_{m,\bullet}^{[r]}$ is placed in degree 0. Put

$$B(r)_{\overline{\mathcal{T}}} \coloneqq \varinjlim_m B(r)_{\mathcal{T},m}|_{\overline{\mathsf{T}} \times_{\mathsf{X}} \mathsf{X}^{\star}}, \tag{B.25}$$

where the colimit is taken in the abelian category $\mathbf{C}^+_{\mathrm{G}_K}((\overline{\mathsf{T}} \times_{\mathsf{X}} \mathsf{X}^{\star})_{\text{ét}}, W_{\bullet})$. For every $h \geqslant 0$, we put

$$B(r)_{\overline{\mathcal{V}^{(h)}}} \coloneqq \bigoplus_{\mathcal{T}} B(r)_{\overline{\mathcal{T}}},$$

where the direct sum is taken over all irreducible components of $\mathcal{V}^{(h)}$, regarded as an element in $\mathbf{C}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^{\star}}_{\text{ét}}, W_{\bullet})$ via pushforward along closed immersions $\mathsf{T} \to \mathsf{X}$. Then parallel to Remark B.19, we have a complex

$$B(r)_{\overline{\mathcal{V}^{(0)}}} \to B(r)_{\overline{\mathcal{V}^{(1)}}} \to B(r)_{\overline{\mathcal{V}^{(2)}}} \to \cdots \to B(r)_{\overline{\mathcal{V}^{(n-1)}}}$$

in $\mathbf{C}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^{\star}}_{\text{ét}}, W_{\bullet})$.

Take $h \geqslant 0$. Put $C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}} \coloneqq B(0)_{\overline{\mathcal{V}^{(h)}}}$. Then we have the canonical map $B(r)_{\overline{\mathcal{V}^{(h)}}} \to C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$ given by the inclusions $\mathscr{J}_{m,l}^{[i]} \to \mathscr{J}_{m,l}^{[0]}$. We also have the crystalline complex $C_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}}$, which is obtained in the same way as $C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$ except that in the definition of $B(0)_{\mathcal{T},m}$ (B.24), we replace $\Omega^q_{(\mathcal{Z}_m^{\star}, N_m^{\star})/W^{\mathrm{triv}}}$ by $\Omega^q_{(\mathcal{Z}_m^{\star}, N_m^{\star})/W[t]^{\circ}}$. We have natural Frobenius operators on both $C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$ and $C_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}}$, and a distinguished triangle

$$C^{\triangle}_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}} \colon \qquad C_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}}[-1] \to C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}} \to C_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}} \xrightarrow{N} C_{\overline{\mathcal{V}^{(h)}}/W[t]^{\circ}} \tag{B.26}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^\star}_{\text{ét}}, W_\bullet)$, where the first arrow is given by $\wedge\,\mathrm{d}\log t$, the second arrow is the canonical one (which is Frobenius equivariant), and the third arrow is the connecting map, such that $C_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$ becomes a (φ, N)-module in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^\star}_{\text{ét}}, W_\bullet)$. We define $S(r)_{\overline{\mathcal{V}^{(h)}}}$ to be the homotopy fiber of the map

$$1 - p^{-r}\varphi_{r+\bullet}\colon B(r)_{\overline{\mathcal{V}^{(h)}}} \to C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$$

(see [Tsu00, Page 540] for more details) in the category $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^\star}_{\text{ét}}, \mathbb{Z}_{p\bullet})$.

Let $B(r)_{\overline{\mathcal{V}^{(-1)}}}$ be the successive homotopy fiber (Definition B.12) of the complex

$$B(r)_{\overline{\mathcal{V}^{(0)}}} \to B(r)_{\overline{\mathcal{V}^{(1)}}} \to B(r)_{\overline{\mathcal{V}^{(2)}}} \to \cdots \to B(r)_{\overline{\mathcal{V}^{(n-1)}}}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}^\star}_{\text{ét}}, W_\bullet)$, and similarly for $C_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}}}$, $C_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ}$, and $S(r)_{\overline{\mathcal{V}^{(-1)}}}$. Then $C_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}}}$ is a (φ, N)-module and we have a similar distinguished triangle (B.26) for $h = -1$.

Finally, for $h \geqslant -1$, put

$$\mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}} \coloneqq \mathrm{R}\bar{\mathsf{u}}_* S(r)_{\overline{\mathcal{V}^{(h)}}}, \qquad \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}} \coloneqq \mathrm{R}\bar{\mathsf{u}}_* C_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}, \qquad \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ} \coloneqq \mathrm{R}\bar{\mathsf{u}}_* C_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$, $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$, $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$, respectively. In particular, we have a canonical map

$$\bar{\xi}_r\colon \mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}} \to \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}} \tag{B.27}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$, and a distinguished triangle

$$\mathscr{C}^{\triangle}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}\colon \qquad \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}[-1] \to \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}} \to \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ} \xrightarrow{N} \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ} \tag{B.28}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$.

When $0 \leqslant r < p - 1$, the usual period maps for $\mathcal{V}^{(h)}$ with $h \geqslant 0$ give a commutative diagram

$$\begin{tikzcd}
\mathscr{S}(r)_{\overline{\mathcal{V}^{(0)}}} \arrow[r] \arrow[d] & \mathscr{S}(r)_{\overline{\mathcal{V}^{(1)}}} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{S}(r)_{\overline{\mathcal{V}^{(n-1)}}} \arrow[d] \\
\overline{\mathsf{G}^{(0)}}_* \overline{i^{(0)}}^* \mathrm{R}\overline{j^{(0)}}_* (\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(0)}}} \arrow[r] & \overline{\mathsf{G}^{(1)}}_* \overline{i^{(1)}}^* \mathrm{R}\overline{j^{(1)}}_* (\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(1)}}} \arrow[r] & \cdots \arrow[r] & \overline{\mathsf{G}^{(n-1)}}_* \overline{i^{(n-1)}}^* \mathrm{R}\overline{j^{(n-1)}}_* (\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(n-1)}}}
\end{tikzcd}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$, where $i^{(h)}\colon \mathsf{V}^{(h)} \to \mathcal{V}^{(h)}$ and $j^{(h)}\colon V^{(h)} \to \mathcal{V}^{(h)}$ denote the special and generic fibers, respectively, for $h \geqslant 0$. However, since the natural map

$$\mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}}} = \bar{i}^* \mathrm{R}\bar{j}_* \overline{G^{(h)}}_* (\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(h)}}} \to \overline{\mathsf{G}^{(h)}}_* \overline{i^{(h)}}^* \mathrm{R}\overline{j^{(h)}}_* (\mu_{p^\bullet}^{\otimes r})_{\overline{V^{(h)}}}$$

is an equivalence for every $h \geqslant 0$, we obtain the period map

$$\bar{\pi}_r\colon \mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}} \to \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}}} \tag{B.29}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ for every $h \geqslant -1$ by Remark B.19 and the process of taking successive homotopy fibers. If $n - 1 \leqslant r < p - 1$, then (B.29) is an equivalence. The desired map (B.23) is simply (B.29) for $h = -1$.

To proceed, we need versions of syntomic and crystalline complexes for $\mathcal{V}^{(h)}$ rather than $\overline{\mathcal{V}^{(h)}}$. The construction is similar to $\mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}}$ and $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$ but only taking $m = 1$ without the colimit (B.25). More precisely, for $h \geqslant -1$, we have

- $\mathscr{S}(r)_{\mathcal{V}^{(h)}}$ in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$, which is obtained in the same way as $\mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}}$ but only taking $m = 1$,[40]
- $\mathscr{C}_{\mathcal{V}^{(h)}/W^{\mathrm{triv}}}$ in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$, which is obtained in the same way as $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}}}$ but only taking $m = 1$,[41]
- $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$, which is obtained in the same way as $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$ but only taking $m = 1$;[42] it is a (φ, N)-module,

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[40] In particular, $\mathscr{S}(d)_{\mathcal{V}^{(0)}}$ coincides with $\mathscr{S}(d)_{\mathcal{X}}$ from §B.4.

[41] In particular, $\mathscr{C}_{\mathcal{V}^{(0)}/W^{\mathrm{triv}}}$ coincides with $\mathscr{C}_{l,\mathcal{X}/W^{\mathrm{triv}}}$ from §B.4.

[42] In particular, $\mathscr{C}_{\mathcal{V}^{(0)}/W[t]^\circ}$ coincides with $\mathscr{C}_{l,\mathcal{X}/W[t]^\circ}$ from §B.4.

- $\widetilde{\mathscr{C}}_{\mathsf{V}^{(h)}/W^\circ}$ and $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}$ in $\in \mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$, which are obtained after we replace $B(0)_{\mathcal{T},1}$ (B.24) by the following complexes

  $$C_{\mathsf{T}/W^\circ}\colon \mathscr{O}_{\mathcal{D}^\star_\bullet} \to \mathscr{O}_{\mathcal{D}^\star_\bullet} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \Omega^1_{(\mathcal{Z}^\star,N^\star)/W^\circ} \to \mathscr{O}_{\mathcal{D}^\star_\bullet} \otimes_{\mathscr{O}_{\mathcal{Z}^\star}} \Omega^2_{(\mathcal{Z}^\star,N^\star)/W^\circ} \to \cdots$$

  and

  $$C_{\mathsf{T}/W^\circ}\colon \mathscr{O}_{\mathcal{D}^\star_\bullet} \to \mathscr{O}_{\mathcal{D}^\star_\bullet} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \Omega^1_{(\mathcal{Y}^\star,M^\star)/W^\circ} \to \mathscr{O}_{\mathcal{D}^\star_\bullet} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \Omega^2_{(\mathcal{Y}^\star,M^\star)/W^\circ} \to \cdots$$

  respectively, where $(\mathcal{Y}^\star, M^\star) \coloneqq (\mathcal{Z}^\star, N^\star) \times_{W[t]^\circ} W^\circ$ as in §B.2 and $\mathcal{D}^\star_l$ denotes the PD envelop of T in $\mathcal{Y}^\star_l$ for $l \geqslant 1$.[43]

By construction, we have maps

$$\xi_r\colon \mathscr{S}(r)_{\mathcal{V}^{(h)}} \to \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}}}, \quad \pi_r\colon \mathscr{S}(r)_{\mathcal{V}^{(h)}} \to \mathscr{N}(r)_{\mathcal{V}^{(h)}}$$

$\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ similar to (B.27) and (B.29), and a distinguished triangle

$$\mathscr{C}^{\triangle}_{\mathcal{V}^{(h)}/W[t]^\circ}\colon \qquad \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}[-1] \to \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}}} \to \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ} \xrightarrow{N} \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ} \tag{B.30}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ similar to (B.28), without *bar*.

In order to prove Lemma B.16 and Lemma B.17, we need to connect the syntomic cohomology to the log rigid cohomology via crystalline complexes we have just constructed. By construction, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{S}(r)_{\mathcal{V}^{(h)},\mathbb{Q}_p} \arrow[r, "\xi_r"] & \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}},K_0} \arrow[r] \arrow[d] & \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0} \arrow[d] \\
& \widetilde{\mathscr{C}}_{\mathsf{V}^{(h)}/W^\circ,K_0} \arrow[r] & \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}
\end{tikzcd} \tag{B.31}$$

in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, \mathbb{Q}_p)$ for every $r \geqslant 0$ and every $h \geqslant -1$, similar to (B.16). Now we study the cohomology of various crystalline complexes.

We start from $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}$. Note that, as in §B.2, for $h \geqslant 0$, the object $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}$ is equivalent to the modified de Rham–Witt complex $W\omega^\bullet_{\mathsf{V}^{(h)}}$. By the construction and (B.5), we have

- a distinguished triangle

  $$\mathscr{C}^{\triangle}_{\mathsf{V}^{(h)}/W^\circ}\colon \qquad \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}[-1] \to \widetilde{\mathscr{C}}_{\mathsf{V}^{(h)}/W^\circ} \to \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ} \xrightarrow{N} \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}$$

  in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ for every $h \geqslant 0$ hence also $h = -1$, similar to (B.30), so that $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}$ is a (φ, N)-module,
- a commutative diagram

  $$\begin{tikzcd}
  \boldsymbol{\omega}^{\triangle}_{\mathsf{V}^{(-1)}} \arrow[r] \arrow[d] & \mathsf{G}^{(0)}_{*}\boldsymbol{\omega}^{\triangle}_{\mathsf{V}^{(0)}} \arrow[r] \arrow[d, "\simeq"] & \mathsf{G}^{(1)}_{*}\boldsymbol{\omega}^{\triangle}_{\mathsf{V}^{(1)}} \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & \mathsf{G}^{(n-1)}_{*}\boldsymbol{\omega}^{\triangle}_{\mathsf{V}^{(n-1)}} \arrow[d, "\simeq"] \\
  \mathscr{C}^{\triangle}_{\mathsf{V}^{(-1)}/W^\circ,K_0} \arrow[r] & \mathscr{C}^{\triangle}_{\mathsf{V}^{(0)}/W^\circ,K_0} \arrow[r] & \mathscr{C}^{\triangle}_{\mathsf{V}^{(1)}/W^\circ,K_0} \arrow[r] & \cdots \arrow[r] & \mathscr{C}^{\triangle}_{\mathsf{V}^{(n-1)}/W^\circ,K_0}
  \end{tikzcd} \tag{B.32}$$

  of distinguished triangles in $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, K_0)$, in which terms in the top row are from (B.1); note that all vertical arrows starting from the second are equivalences.

Lemma B.20. *The first vertical arrow in* (B.32) *is also an equivalence. In particular, for every $h \geqslant -1$ and $q \geqslant 0$, we have a canonical isomorphism*

$$\mathrm{H}^q_{\text{rig}}(\mathsf{V}^{(h)}, W^\circ) \simeq \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0})$$

of K_0-vector spaces which commutes with monodromy operators.

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[43] In particular, $\widetilde{\mathscr{C}}_{\mathsf{V}^{(0)}/W^\circ}$ and $\mathscr{C}_{\mathsf{V}^{(0)}/W^\circ}$ coincide with $\widetilde{\mathscr{C}}_{(\mathsf{X},L^{\mathsf{X}}_{\mathcal{X}}|\mathsf{X})/W^\circ}$ and $\mathscr{C}_{(\mathsf{X},L^{\mathsf{X}}_{\mathcal{X}}|\mathsf{X})/W^\circ}$ from §B.2, respectively.

Proof. It suffices to show that the map $\omega_{\mathsf{V}^{(-1)}} = \omega_{(\mathsf{U},\mathsf{X})}$ is the successive homotopy fiber of the complex

$$\mathsf{G}_*^{(0)}\omega_{\mathsf{V}^{(0)}} \to \mathsf{G}_*^{(1)}\omega_{\mathsf{V}^{(1)}} \to \cdots \to \mathsf{G}_*^{(n-1)}\omega_{\mathsf{V}^{(n-1)}}.$$

However, this follows from the easy fact that for an embedding system $\{(\mathsf{X}^\star, L^\star) \hookrightarrow (\mathcal{Y}^\star, M^\star)\}$ for $(\mathsf{X}, L_{\mathcal{X}}^{\mathsf{X}}|_{\mathsf{X}})/W^\circ$, the complex

$$\begin{aligned}0 \to \mathsf{f}^!_{(\mathsf{U}^\star,\mathsf{X}^\star)}\left(\Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}}\right) \to \Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{X}^\star[_{\mathfrak{Y}^\star}} \\ \to \Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{V}^{(1)\star}[_{\mathfrak{Y}^\star}} \to \cdots \to \Omega^\bullet_{(\mathcal{Y}^\star,M^\star)/W^\circ} \otimes_{\mathscr{O}_{\mathcal{Y}^\star}} \mathscr{O}_{]\mathsf{V}^{(n-1)\star}[_{\mathfrak{Y}^\star}} \to 0\end{aligned}$$

in $\mathbf{C}^+(\mathfrak{Y}^\star_{\eta,\text{qét}}, K_0)$ is exact. Here, for $h \geqslant 1$,

$$\mathscr{O}_{]\mathsf{V}^{(h)\star}[_{\mathfrak{Y}^\star}} := \bigoplus_{\mathcal{T}} \mathscr{O}_{]\mathsf{T}^\star[_{\mathfrak{Y}^\star}}$$

where the direct sum is taken over all irreducible components of $\mathcal{V}^{(h)}$. The lemma is proved. □

Recall that by [Tsu99, (4.5.1)], we have the canonical identification

$$P_\bullet = \Gamma(\operatorname{Spec}\overline{\kappa}, \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W[t]^\circ}), \quad \widehat{\mathbb{B}_{\text{st}}^+} = \Gamma(\operatorname{Spec}\overline{\kappa}, \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W[t]^\circ,K_0})$$

of (φ, N)-modules in $\mathbf{M}_K(W_\bullet)$ and $\mathbf{M}_K(K_0)$, respectively.

Lemma B.21. *The following holds for every $h \geqslant -1$.*

(1) The object $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ of $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ is admissible (Definition A.3).

(2) There is a canonical isomorphism

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0}) \simeq \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{K}$$

of (φ, N)-modules in $\mathbf{M}(K_0)$ for every $q \geqslant 0$.

(3) The object $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$ of $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$ is admissible.

(4) The natural map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0}) \otimes_{K_0} \widehat{\mathbb{B}_{\text{st}}^+} \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,K_0})$$

of (φ, N)-modules induced by functoriality and cup product descends to an isomorphism

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0}) \otimes_{\mathbb{K}} \widehat{\mathbb{B}_{\text{st}}^+} \xrightarrow{\sim} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,K_0})$$

of (φ, N)-modules in $\mathbf{M}_K(K_0)$ for $q \geqslant 0$. In particular, the natural map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,K_0})$$

is injective.

(5) The object $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}$ of $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$ is admissible.[44]

(6) The distinguished triangle $\mathscr{C}^\triangle_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$ (B.28) induces a Frobenius equivariant isomorphism

$$\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}},K_0}) \simeq \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\text{st}}^+\right)^{N=0}$$

in $\mathbf{M}_K(K_0)$ for every $q \geqslant 0$.

Moreover, the isomorphisms in (2,4,6) are compatible with h in the obvious sense.

Proof. It is clear that condition (1) of Definition A.3 holds for $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ trivially as G is the trivial group and holds for $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}$ and $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}$ since they are defined as injective limits over terms fixed by open subgroups of G_K. Thus, it remains to check condition (2) of Definition A.3 for the three objects. Below in the proof, we say that two objects $\mathscr{C}$ and $\mathscr{C}'$ in either $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ or $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_\bullet)$ are almost equivalent if there exists a map $\mathscr{C} \to \mathscr{C}'$ whose fiber is annihilated by some power of p. It is clear that condition (2) of Definition A.3 is preserved under almost equivalence.

First, we prove (1) and (2). Note that [HK94, Lemma 5.2] is applicable to $\mathcal{V}^{(h)}$ with $h \geqslant 0$, and hence also to the case for $h = -1$ by long exact sequences induced by the successive homotopy fiber. In other words, for every $h \geqslant -1$, $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ and $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ} \otimes^{\mathrm{L}}_{W_\bullet} R_\bullet$ are almost equivalent. It is already known by the proof of [Sat13,

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[44] However, the object $\mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}}}$ of $\mathbf{D}^+(\mathsf{X}_{\text{ét}}, W_\bullet)$ is in general *not* admissible.

Proposition A.3.1(1)] that, when $h \geqslant 0$, $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ} \otimes^{\mathrm{L}}_{W_\bullet} R_\bullet$ (hence $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$) satisfies condition (2) of Definition A.3 and we have the canonical isomorphism in (2). To pass the same argument for $h = -1$, it suffices to show that $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,l})$ is finite for every $q \geqslant 0$ and $l \geqslant 1$, which follows from long exact sequences induced by the successive homotopy fiber.

Second, we prove (3) and (4). It is known by [Tsu99, Proposition 4.5.4] that for $h \geqslant 0$, the natural map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}) \otimes_{W_\bullet} \Gamma(\operatorname{Spec} \overline{\kappa}, \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W[t]^\circ}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ})$$

of (φ, N)-modules descends to an isomorphism

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}) \otimes_{R_\bullet} \Gamma(\operatorname{Spec} \overline{\kappa}, \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W[t]^\circ}) \xrightarrow{\sim} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ}) \tag{B.33}$$

of (φ_N)-modules in $\mathbf{M}_{\mathrm{G}_K}(W_\bullet)$ for every $q \geqslant 0$. The same holds for $h = -1$ by the long exact sequences induced by successive homotopy fiber and the fact that $\Gamma(\operatorname{Spec} \overline{\kappa}, \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W[t]^\circ}) = P_\bullet$ is flat over $R_\bullet$ [Tsu99, Proposition 4.1.5].

To show (3), since $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ and $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ} \otimes^{\mathrm{L}}_{W_\bullet} R_\bullet$ are almost equivalent for $h \geqslant -1$, it suffices to show that $\mathrm{R}^1 \varprojlim_l \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,l}) \otimes_{W/p^l} P_l = 0$ which has been argued in the proof of [Sat13, Proposition A.3.1(2)].

To show (4), by (B.33) (for $h \geqslant -1$), we have

$$\varprojlim_l \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,l}) \otimes_{R_l} P_l \xrightarrow{\sim} \varprojlim_l \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,l}).$$

Since both $R_\bullet$ and $P_\bullet$ have surjective transition maps, we have

$$\left(\varprojlim_l \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,l}) \otimes_{R_l} P_l\right) \otimes_W K_0 = \left(\varprojlim_l \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,l}) \otimes_W K_0\right) \otimes_{\mathbb{K}} \widehat{\mathbb{B}^+_{\mathrm{st}}}.$$

By (1) and (3), we have

$$\left(\varprojlim_l \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,l})\right) \otimes_W K_0 = \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ,K_0}),$$

$$\left(\varprojlim_l \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,l})\right) \otimes_W K_0 = \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W[t]^\circ,K_0}),$$

respectively. Together we obtain (4); and the injectivity follows from Lemma B.13(2).

Finally, we prove (5) and (6). It is well-known that when $h \geqslant 0$, the monodromy operator on $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ})$ is nilpotent for every $q \geqslant 0$. The same holds for $h = -1$ by long exact sequences induced by the successive homotopy fiber. Since the monodromy map on $P_\bullet$ is surjective [Kat94, Corollary 3.6], the monodromy map on $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}) \otimes_{W_\bullet} P_\bullet$ is surjective for every $h \geqslant -1$.

To show (5), since $\mathscr{C}_{\mathcal{V}^{(h)}/W[t]^\circ}$ and $\mathscr{C}_{\mathsf{V}^{(h)}/W^\circ} \otimes^{\mathrm{L}}_{W_\bullet} R_\bullet$ are almost equivalent and by (B.33) for every $h \geqslant -1$, it suffices to show that

$$\mathrm{R}^1 \varprojlim_l \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ}) \otimes_{W_\bullet} P_\bullet\right)^{N=0} = 0,$$

which follows from the argument in the proof of [Sat13, Proposition A.3.1(3)].

To show (6), we first note that

$$\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}^+_{\mathrm{st}}\right)^{N=0} = \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \widehat{\mathbb{B}^+_{\mathrm{st}}}\right)^{N=0}$$

by Lemma B.13(1). Then by (2) and (4), it suffices to show that the monodromy map

$$N\colon \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \widehat{\mathbb{B}^+_{\mathrm{st}}} \to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \widehat{\mathbb{B}^+_{\mathrm{st}}}$$

is surjective for every $q \geqslant 0$. However, this follows from Lemma B.13(3) and the fact that the monodromy operator on $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0})$ is nilpotent. □

Lemma B.22. *Consider integers r satisfying $n-1 \leqslant r < p-1$. For every $h \geqslant -1$ and every $q \geqslant 0$, the $\mathbb{B}_{\mathrm{st}}$-linear extension of the composite map*

$$\begin{aligned}\text{(B.34)}\qquad \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) &\xrightarrow{\sim} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-r)\\ &\xrightarrow{\sim} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-r)\\ &\to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-r)\\ &\xrightarrow{\sim} \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-r)\\ &\to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}\end{aligned}$$

is independent of r and induces an isomorphism

$$\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{st}} \simeq \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}$$

of (φ, N)-modules in $\mathbf{M}_K(K_0)$. Here in (B.34), the second arrow is induced by the inverse of the period map $\overline{\pi}_r$ (B.29), the third arrow is induced by the map $\overline{\xi}_r$ (B.27), the fourth arrow is the isomorphism from Lemma B.21(6), and the last arrow is induced by the canonical map $\mathbb{Q}_p(-r) \hookrightarrow \mathbb{B}_{\mathrm{st}}$.

In particular, the above isomorphism induces a Frobenius equivariant isomorphism

$$\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}} \simeq \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}\right)^{N=0}$$

in $\mathbf{M}_K(K_0)$.

Proof. For $h \geqslant 0$, the statement follows from [Tsu99, Theorem 4.10.2] (the usual C_{st}-comparison theorem for proper strictly semistable schemes) together with the compatibility properties [Tsu99, Corollaries 4.8.8 & 4.9.2] for the independence of r (which is at least $\dim V^{(h)}$) of the map. The case for $h = -1$ follows from the series of long exact sequences induced by the successive homotopy fiber.[45] □

Lemma B.23. *Suppose that $n < p$. For every $h \geqslant -1$, every $q \geqslant 0$, and every $0 \leqslant r < p-1$, the following diagram*

$$\begin{tikzcd}
\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \arrow[r, "\overline{\xi}_r", "\text{(B.27)}"'] \arrow[d, "\overline{\pi}_r"', "\text{(B.29)}"] & \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0}) \arrow[r, "\sim"] & \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+\right)^{N=0} \arrow[d, hook] \\
\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \arrow[r, "\mathbb{Q}_p(r)\hookrightarrow\mathbb{B}_{\mathrm{cris}}"] & \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}} \arrow[r, "\sim"] & \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^\circ,K_0}) \otimes_{K_0} \mathbb{B}_{\mathrm{st}}\right)^{N=0}
\end{tikzcd}$$

in $\mathbf{M}_K(\mathbb{Q}_p)$ commutes, in which the equivalence in the first row is from Lemma B.21(6), and the equivalence in the second row is from Lemma B.22.

Proof. When $h \geqslant 0$, the commutativity follows from the compatibility properties [Tsu99, Corollaries 4.8.8 & 4.9.2]. The case for $h = -1$ follows from the series of long exact sequences induced by the successive homotopy fiber. □

Lemma B.24. *Suppose that $n < p$. The map $\mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \to \mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} = \mathscr{S}(d)_{\mathcal{V}^{(0)},\mathbb{Q}_p}$ factors though $\mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}$. In particular,*

(1) the map (B.14) factors as

$$\mathrm{R}\Gamma(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathrm{R}\Gamma(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}) \to \mathrm{R}\Gamma(\mathsf{X}_{\text{ét}}, \mathscr{N}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}) = \mathrm{R}\Gamma_{\mathrm{c}}(U, \mathbb{Q}_p(d)).$$

when $d < p-1$;

(2) the map (B.17) factors as

$$\mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p} \to \mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p} \to \widetilde{\mathscr{C}}_{\mathsf{V}^{(-1)}/W^\circ,K_0} \simeq \widetilde{\omega}_{(\mathsf{U},\mathsf{X})}$$

in which the last equivalence comes from Lemma B.20.

---

[45] Such an isomorphism for $h = -1$ has already been obtained in [Yam11]. The results there are much stronger than ours and in particular they contain a C_{dR}-comparison isomorphism. Thus, the log structure on $\mathcal{X}$ used there is $L_{\mathcal{X}}^{\mathsf{X}\cup\mathcal{V}}$, which makes things more complicated.

Proof. Since the complex $\mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p}$ is supported on V for every $h \geqslant 1$, the factorization follows from the construction. □

For every $h \geqslant -1$, every $q \geqslant 0$, and every $0 \leqslant r < p-1$, put

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(r)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} := \operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(r)_{\mathcal{V}^{(h)},\mathbb{Q}_p}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right). \tag{B.35}$$

Now we can give a proof of Lemma B.16.

Proof of Lemma B.16. By Lemma B.24(1), $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^{\heartsuit}$ coincides with the kernel of the composite map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(-1)}},\mathbb{Q}_p}) \xrightarrow{\pi_r} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(-1)}},\mathbb{Q}_p}).$$

By Lemma B.23, it is also the kernel of the composite map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p}) \to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(-1)}},\mathbb{Q}_p}) \xrightarrow{\bar{\xi}_r} \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\text{triv}},K_0}).$$

By Lemma B.24(2) and (B.31), we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(-1)}},\mathbb{Q}_p}) \arrow[r, "\bar{\xi}_r"] & \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\text{triv}},K_0}) \arrow[r, "\nu"] & \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^{\circ},K_0}) \\
\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p}) \arrow[u] \arrow[r, "\xi_r"] & \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(-1)}/W^{\text{triv}},K_0}) \arrow[u] \arrow[r] \arrow[d] & \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^{\circ},K_0}) \arrow[u, "\mu"'] \arrow[d] \\
 & \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \widetilde{\mathscr{C}}_{\mathcal{V}^{(-1)}/W^{\circ},K_0}) \arrow[r] & \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(-1)}/W^{\circ},K_0})
\end{tikzcd}$$

in which μ and ν are injective by Lemma B.21(4) and (6), respectively. Thus, $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{\mathcal{X},\mathbb{Q}_p})^{\heartsuit}$ maps to zero all the way to the lower-right corner $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(-1)}/W^{\circ},K_0}) = \mathrm{H}^q_{\text{rig}}((\mathsf{U},\mathsf{X})/W^{\circ})$. The lemma is proved. □

In order to prove Lemma B.17, we need to compare edge maps in both étale and crystalline settings. For every $h \geqslant -1$, every $q \geqslant 0$, and every $r \geqslant 0$, put

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{N}(r)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^0 := \operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{N}(r)_{\mathcal{V}^{(h)},\mathbb{Q}_p}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right),$$
$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}},K_0})^0 := \operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}},K_0}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}},K_0})\right).$$

Suppose that $n < p$. Then by definition, π_d induces a map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \to \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{N}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^0. \tag{B.36}$$

By Lemma B.23 and the definition of the syntomic complex, ξ_d induces a map

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \to \left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathcal{V}^{(h)}/W^{\text{triv}},K_0})^0\right)^{\varphi=p^d}. \tag{B.37}$$

As $\mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}}}$ is clearly an admissible object of $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, \mathbb{Z}_{p\bullet})$ and $\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}$ is an admissible object of $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}}, W_{\bullet})$ by Lemma B.21(5), we have the spectral sequences and hence the corresponding edges maps for them from Lemma A.4. Composing (B.36) and (B.37) with the corresponding edge maps, we obtain maps

$$\beta_q\colon \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \to \mathrm{H}^1\left(K, \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right),$$
$$\gamma_q\colon \mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \to \mathrm{H}^1\left(K, \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}},K_0})\right),$$

respectively.

The following lemma is an “Abel–Jacobi” version of Lemma B.23.

Lemma B.25. *Suppose that $n < p$. For every $h \geqslant -1$ and every $q \geqslant 0$, the following diagram*

$$\begin{tikzcd}
\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \arrow[r, "\gamma_q"] \arrow[d, "\beta_q"'] \arrow[dr, dashed] & \mathrm{H}^1\left(K, \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}},K_0})\right) \arrow[r, "\sim"] & \mathrm{H}^1\left(K, \left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0}) \otimes_{K_0} \mathbb{B}_{\text{st}}^{+}\right)^{N=0}\right) \arrow[d] \\
\mathrm{H}^1\left(K, \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right) \arrow[r, "\mathbb{Q}_p(d)\hookrightarrow\mathbb{B}_{\text{cris}}"] & \mathrm{H}^1\left(K, \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{cris}}\right) \arrow[r, "\sim"] & \mathrm{H}^1\left(K, \left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}}, \mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0}) \otimes_{K_0} \mathbb{B}_{\text{st}}\right)^{N=0}\right)
\end{tikzcd}$$

of $\mathbb{Q}_p$-vector spaces commutes, in which the equivalence in the first row is from Lemma B.21(6), and the equivalence in the second row is from Lemma B.22.

This lemma does not follow immediately from Lemma B.23 since $\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit}$, by our definition (B.35), is in general larger than

$$\operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}}, \mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p}) \to \mathbf{H}^q(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right).$$

Proof. Take an integer r satisfying $(1 \leqslant d \leqslant) n - 1 \leqslant r < p - 1$ (which exists as $n < p$). We have

- the map $\overline{\pi}_{r-d} \colon \mathscr{S}(r-d)_{\overline{\operatorname{Spec} O_K}} \xrightarrow{\sim} \mathscr{N}(r-d)_{\overline{\operatorname{Spec} O_K}}$ in $\mathbf{D}^+_{\mathrm{G}_K}(\mathbb{Z}_{p\bullet})$, both equivalent to $\mathbb{Z}_{p\bullet}(r-d)$,
- the object $\mathscr{C}_{\overline{\operatorname{Spec} O_K}/W^{\text{triv}}}$ in $\mathbf{D}^+_{\mathrm{G}_K}(W_\bullet)$, which is equivalent to $P_\bullet^{N=0}$,
- the map $\overline{\xi}_{r-d} \colon \mathscr{S}(r-d)_{\overline{\operatorname{Spec} O_K}} \to \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W^{\text{triv}}}$ in $\mathbf{D}^+_{\mathrm{G}_K}(\mathbb{Z}_{p\bullet})$, which is equivalent to the natural map $\mathbb{Z}_{p\bullet}(r-d) \hookrightarrow P_\bullet^{N=0}$.

For every $h \geqslant -1$, consider the following diagram

(B.38)
$$\begin{tikzcd}
\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathscr{C}_{\overline{\operatorname{Spec} O_K}/W^{\text{triv}}} \arrow[r] & \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}) \\
\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(h)}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathscr{S}(r-d)_{\overline{\operatorname{Spec} O_K}} \arrow[r] \arrow[u, "\overline{\xi}_d\otimes\overline{\xi}_{r-d}"] \arrow[d, "\overline{\pi}_d\otimes\overline{\pi}_{r-d}"'] & \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(r)_{\overline{\mathcal{V}^{(h)}}}) \arrow[u, "\overline{\xi}_r"'] \arrow[d, "\overline{\pi}_r"] \\
\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathscr{N}(r-d)_{\overline{\operatorname{Spec} O_K}} \arrow[r] & \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(r)_{\overline{\mathcal{V}^{(h)}}})
\end{tikzcd}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\mathbb{Z}_{p\bullet})$, in which all horizontal maps are induced by cup products. We claim that (B.38) commutes. For $h \geqslant 0$, the upper square commutes by definition, and the lower square commutes by the compatibility of period maps with cup products (see [Tsu99, §3.1] in a more general context). The case for $h = -1$ follows from the process of taking successive homotopy fiber.

Using the equivalences

$$\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(h)}}}) \simeq \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(h)}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathscr{S}(r-d)_{\overline{\operatorname{Spec} O_K}} \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathbb{Z}_{p\bullet}(d-r),$$
$$\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}}}) \simeq \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathscr{N}(r-d)_{\overline{\operatorname{Spec} O_K}} \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathbb{Z}_{p\bullet}(d-r),$$

and the fact that $\overline{\pi}_r$ (for $\overline{\mathsf{X}}_{\text{ét}}$) is an equivalence, we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{S}(d)_{\overline{\mathcal{V}^{(h)}}}) \arrow[d] \arrow[dr] & \\
\mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}}}) \arrow[r, "(\overline{\xi}_r\circ\overline{\pi}_r^{-1})\otimes\mathbb{Z}_{p\bullet}(d-r)"] & \mathrm{R}\Gamma(\overline{\mathsf{X}}_{\text{ét}}, \mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\text{triv}}}) \otimes^{\mathrm{L}}_{\mathbb{Z}_{p\bullet}} \mathbb{Z}_{p\bullet}(d-r)
\end{tikzcd}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\mathbb{Z}_{p\bullet})$ from (B.38).[46] Composing with $\mathrm{R}(\Gamma_{\mathrm{G}_K}\circ\underleftarrow{\lim})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and taking edge maps, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{S}(d)_{\mathcal{V}^{(h)},\mathbb{Q}_p})^{\heartsuit} \arrow[d] \arrow[dr] & \\
\mathrm{H}^1\left(K,\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\right) \arrow[r, "(\overline{\xi}_r\circ\overline{\pi}_r^{-1})\otimes\mathbb{Q}_p(d-r)"] & \mathrm{H}^1\left(K,\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0})\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(d-r)\right).
\end{tikzcd}$$

The lemma follows since we have the commutative diagrams

$$\begin{tikzcd}
\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{N}(d)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p}) \arrow[r, "(\overline{\xi}_r\circ\overline{\pi}_r^{-1})\otimes\mathbb{Q}_p(d-r)"] \arrow[dd, "\mathbb{Q}_p(d)\hookrightarrow\mathbb{B}_{\mathrm{cris}}"] & \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0})\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(d-r) \arrow[d, "\simeq"', "\text{Lemma B.21(6)}"] \\
 & \left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}^+_{\mathrm{st}}\right)^{N=0}\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(d-r) \arrow[d, "\mathbb{Q}_p(d-r)\hookrightarrow\mathbb{B}_{\mathrm{st}}"] \\
\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{N}(0)_{\overline{\mathcal{V}^{(h)}},\mathbb{Q}_p})\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{cris}} \arrow[r, "\sim", "\text{Lemma B.22}"'] & \left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}\right)^{N=0}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0}) \arrow[r] \arrow[d, "\simeq"', "\text{Lemma B.21(6)}"] & \mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(h)}}/W^{\mathrm{triv}},K_0})\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(d-r) \arrow[d] \\
\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}^+_{\mathrm{st}}\right)^{N=0} \arrow[r] & \left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(h)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}\right)^{N=0}
\end{tikzcd}$$

in which the upper horizontal arrow is induced by the canonical map $\mathbb{Q}_p(r-d)\hookrightarrow(\mathbb{B}^+_{\mathrm{st}})^{N=0}\simeq\mathscr{C}_{\overline{\operatorname{Spec}O_K}/W^{\mathrm{triv}},K_0}$ and the cup product, and the right vertical arrow is the composition of two right vertical arrows in the previous diagram. ☐

Proof of Lemma B.17. By Lemma B.24, we may replace the source of both α_q and ρ_q, which is originally $\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathsf{F}_!\mathsf{F}^*\mathscr{S}(d)_{X,\mathbb{Q}_p})^{\heartsuit}$, by $\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p})^{\heartsuit}$. Then α_q (B.15) coincides with the dashed arrow in Lemma B.25 (with $h=-1$). By Lemma B.25, we have

$$\operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p})^{\heartsuit}\to\mathrm{H}^1\left(K,\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}^+_{\mathrm{st}}\right)^{N=0}\right)\right)\subseteq\operatorname{Ker}(\alpha_q).$$

Thus, the lemma will follow if we can show

$$\operatorname{Ker}(\rho_q)=\operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p})^{\heartsuit}\to\mathrm{H}^1\left(K,\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^{\circ},K_0})\otimes_{K_0}\mathbb{B}^+_{\mathrm{st}}\right)^{N=0}\right)\right). \tag{B.39}$$

Lemma B.21(4,6) implies that

$$\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0=\operatorname{Ker}\left(\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})\to\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^{\circ},K_0})\right),$$

which induces an isomorphism

$$\frac{\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^{\circ},K_0})}{N\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^{\circ},K_0})}\xrightarrow{\sim}\mathbf{H}^q(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0$$

by the distinguished triangle $\mathscr{C}^{\triangle}_{\mathcal{V}^{(-1)}/W[t]^{\circ},K_0}$ (B.30), under which the Frobenius operator on the right side corresponds to p times the one on the left side.

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[46] The above commutative diagram is parallel to [Sat13, (A.6.4)]. However, somehow unfortunately, the roles of d and r here are switched from those there.

Consider the following diagram

$$\begin{tikzcd}
\left(\dfrac{\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})}{N\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})}\right)^{\varphi=p^{d-1}} \arrow[r, "\sim"] \arrow[d] & \left(\mathbf{H}^{q}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0\right)^{\varphi=p^{d}} \arrow[d] \\
\dfrac{\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})}{N\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})} \arrow[r, hook, "-\delta"] & \mathrm{H}^1\left(K,\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}^+\right)^{N=0}\right)
\end{tikzcd}\tag{B.40}$$

in which

- the map δ is the edge map induced from the short exact sequence

$$0\longrightarrow\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}^+\right)^{N=0}\longrightarrow\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}^+\xrightarrow{N}\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}^+\longrightarrow 0$$

  in $\mathbf{M}_K(K_0)$,
- the left vertical arrow is the specialization map at $t=0$, and
- the right vertical arrow is the composite map

$$\mathbf{H}^{q}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0\to\mathrm{H}^1\left(K,\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}},K_0})\right)\xrightarrow{\sim}\mathrm{H}^1\left(K,\left(\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})\otimes_{K_0}\mathbb{B}_{\mathrm{st}}^+\right)^{N=0}\right)$$

  in which the isomorphism is from Lemma B.21(6).

We show that (B.40) commutes. Applying Lemma A.5 to $S=\overline{\mathsf{X}}_{\text{ét}}$, the distinguished triangle

$$\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ}[-1]\xrightarrow{-N[-1]}\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ}[-1]\to\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}}}\xrightarrow{+1}\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ}$$

in $\mathbf{D}^+_{\mathrm{G}_K}(\overline{\mathsf{X}}_{\text{ét}},W_\bullet)$ (which is a shift of the distinguished triangle $\mathscr{C}^{\triangle}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ}$ (B.28)) in which all objects are admissible by Lemma B.21(3,5), we know that the image of an element $c_1\in\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})$ under the composite map

$$\begin{aligned}\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})&\to\mathbf{H}^{q}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0\\&\to\mathbf{H}^{q}_K(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}},K_0})^0\to\mathrm{H}^1\left(K,\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W^{\mathrm{triv}},K_0})\right)\end{aligned}$$

can be represented by the (continuous) 1-cocycle $g\mapsto g\overline{c}_0-\overline{c}_0$ for $g\in\mathrm{G}_K$, where $\overline{c}_0$ is an arbitrary element in $\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ,K_0})$ satisfying that $-N(\overline{c}_0)$ coincides with

$$c_1\in\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})\subseteq\mathbf{H}^{q-1}(\overline{\mathsf{X}}_{\text{ét}},\mathscr{C}_{\overline{\mathcal{V}^{(-1)}}/W[t]^\circ,K_0}).$$

Then the commutativity of (B.40) follows from Lemma B.14 (with $D=\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})$) and Lemma B.21(2).

Now (B.39) follows since the composite map

$$\begin{aligned}\mathbf{H}^{q}(\mathsf{X}_{\text{ét}},\mathscr{S}(d)_{\mathcal{V}^{(-1)},\mathbb{Q}_p})^{\heartsuit}&\xrightarrow{\text{(B.37)}}\left(\mathbf{H}^{q}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W^{\mathrm{triv}},K_0})^0\right)^{\varphi=p^d}\\&\xrightarrow{\sim}\left(\frac{\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})}{N\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathcal{V}^{(-1)}/W[t]^\circ,K_0})}\right)^{\varphi=p^{d-1}}\to\frac{\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})}{N\mathbf{H}^{q-1}(\mathsf{X}_{\text{ét}},\mathscr{C}_{\mathsf{V}^{(-1)}/W^\circ,K_0})}\end{aligned}$$

is nothing but ρ_q (B.22) composed with the isomorphism from Lemma B.20.

Lemma B.17 is proved. □

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