Recent development of the Mordell conjecture

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§1 Mordell conj

K number field, X/K curve (sm, proj, geom integral) with genus $g > 1$.

<u>Mordell conj</u> (Faltings's thm) $X(K)$ finite.

<u>3 Proofs</u> (1) Faltings 1983:

Faltings ht, Shafarevich conj,

moduli space of AVs.

(2) Vojta 1990s: (inspired by the pf of Roth's thm)

Diophantine approximation.

(3) Lawrence-Venkatesh 2018:

Shafarevich conj, p-adic Hodge.

§2 Uniform Mordell

X/K, $g>1$ as before.

<u>Thm</u> (Uniform 1, Conj of Mazur).

$\exists$ const $c(g)>0$ depending <u>only</u> on $g>1$,

s.t. $|X(K)| < c(g)^{1+\mathrm{rk}\, J(K)}$, $\forall X/K$.

where $J = \mathrm{Jac}(X) \rightsquigarrow J(K) =$ Mordell-Weil gp.

Due to: Vojta (1990s), DGH (Dimitrov-Gao-Habegger) (2018), Kühne (2021).

A "stronger" conj:

Conj (Uniform 2)

$\exists$ const $c(g,[K:\mathbb{Q}])$ dep only on g & $[K:\mathbb{Q}]$

s.t. $|X(K)| < c(g,[K:\mathbb{Q}])$, $\forall X/K$.

Work progresses:

- Caporaso-Harris-Mazur:

  Bombieri-Lang conj $\Rightarrow$ Uniform 2.

  (B-L conj: Y/K (sm) proj var of general type, then $Y(K)$ not Zariski dense in Y.)

§3 Uniform Bogomolov conj

Pf of Uniform 1 (in 2 steps):

Step 1 large pts (Vojta)

$$\#\{x\in X(K) \mid \hat{h}(x) < a_1 \cdot h^+_{Fal}(x)\} < ? \qquad ①$$

where $\hat{h}: J(K)\to\mathbb{R}$ Néron-Tate height.

$h^+_{Fal}(x) := \max\{h_{Fal}(x), 1\}$

$\left(h_{Fal}(x) := h_{Fal}(J) = \text{arith deg of the Hodge bundle of } J/\mathcal{O}_K. \right)$

a real number measuring "complexity of X" (ht of coeff of equation defining X.)

Step 2 small pts (DGH, K)

$a_2(g) > 0$,

$$\#\{x\in X(\bar{K}) \mid \hat{h}(x) < a_2(g)\cdot h^+_{Fal}(x)\} < ? \qquad ②$$

Rmk (a) Northcott thm says $\{x \in X(\bar{k}) \mid \deg(x) < A_1, \hat{h}(x) < A_2\}$ is finite.

Then Vojta's bound ① ⇒ Mordell.

(b) ② is a type of uniform Bogomolov conj.

Recall Bogomolov Conj (Ullmo's thm):

X/k, $\forall \alpha \in \mathrm{Pic}^1(X_{\bar{k}})$, $\exists \varepsilon > 0$ s.t.

$\#\{x \in X(\bar{k}) \mid \hat{h}(x-\alpha) < \varepsilon\} < \infty$.

note $x - \alpha \in \mathrm{Pic}^0(X_{\bar{k}}) = J(\bar{k})$.

Thm (Uniform Bogomolov, Yuan 2022)

$\exists c_1, c_2 > 0$ depending only on $g > 1$,

s.t. $\forall K = \mathbb{Q}$ or $K = k(t)$ for any field k, $\forall C/\bar{K}$ of genus g,

$\forall \alpha \in \mathrm{Pic}^1(C_{\bar{K}})$ s.t. $(X_{\bar{K}}, \alpha)$ non-isotrivial when $K = k(t)$,

$\#\{x \in C(\bar{K}) \mid \hat{h}(x-\alpha) < c_1(h^+_{\mathrm{Fal}}(C) + \hat{h}((2g-2)\alpha - \omega_C))\} < c_2$.

Rmk (1) [DGH, K] previously proved case K number field without extra thm.

($\underbrace{\text{o-minimality, ht ineq}}_{\text{DGH}}$, $\underbrace{\text{equidistribution}}_{\text{K}}$.)

(2) [LSW] Looper - Silverman - Wilms 2021:

independently (to Yuan's pf) proved the case where $K = k(t)$ function field.

(admissible pairing on single curve.

⇝ constants c_1, c_2 surprisingly explicit.)

(3) [Yuan] adelic line bundles of Yuan–S. Zhang 2021
bigness of canonical bundle of univ curve.

Over function field, have:

<u>Thm</u> (LSW, J. Yu) K/k func field of 1 variable.

X/K curve genus $g>1$.

J/K has trivial $\bar{K}/\bar{k}$-trace,

Then

$$|X(K)| < (16g^2 + 32g + 188)\cdot(20g)^{\mathrm{rk}\, J(K)}.$$

<u>pf</u> LSW: large pt

Yu: small pts following Vojta's conj.

<u>§4 Effective Mordell conj</u> (Holy Grail)

X/K, $g>1$.

<u>Conj</u> (Effective Mordell)

X/K, $\exists$ const $A, B > 0$ s.t.

$\forall x \in X(\bar{K})$, $h(x) < A\cdot \log|D_x| + B$.

Here $h: X(\bar{K}) \to \mathbb{R}$ Weil ht associated to an embedding $X \hookrightarrow \mathbb{P}^N$.

D_x = discriminant of res field of x.

<u>Equiv to</u> abc conj, Szpiro conj, Miyaoka-Yan ($c_1^2 \leq 3c_2$).

<u>Rmk</u> (Szpiro) Eff. Mordell known over fcn fields.