Wall crossing for moduli spaces

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(Joint with K. Ascher & K. DeVleming.)

§ K-stability & K-moduli

(X, D) log Fano pair, $-K_X - D$ ample.

Ex $(\mathbb{P}^n, cD_d)$, $D_d \subseteq \mathbb{P}^n$ hypersurface of deg d,

$$0 \leq c < \max\{1, \tfrac{n+1}{d}\}.$$

$$-K_{\mathbb{P}^n} - cD_d = \mathcal{O}(n+1-cd).$$

K-stability algebraic theory to characterize existence of canonical Kähler-Einstein metric on (X, D).

$$\mathrm{Ric}(\omega) = \omega + [D],$$

where ω is a KE metric on $X \setminus \mathrm{supp}(D)$.

If $\mathrm{coeff}_{D_i}(D) = c_i \in [0,1]$, then ω has cone angle $2\pi(1-c_i)$ along D_i.

assume this

Thm K-polystability of (X, D) $\iff$ existence of canonical KE metric on (X, D).

K-moduli thm

Fix numerical invariants $n = \dim X$, $V = (-K_X - D)^n$,

c = coeff of D.

Then $\exists$ a finite type Artin stack $\mathcal{M}^K_{n,V,c}$ parametrizing K-semistable log Fano pairs (X, D).

Moreover, $\mathcal{M}^K_{n,V,c}$ admits a projective good moduli space

$M^K_{n,v,c}$ parametrizing K-polystable log Fano pairs.

<u>Wall crossing for K-moduli spaces</u>

<u>Thm</u> (Ascher - De Vleming - Liu, Zhou)

Assume X klt Fano variety, $D \geq 0$ $\mathbb{Q}$-Cartier Weil divisor s.t. $D \sim_{\mathbb{Q}} -rK_X$ for some $r \in \mathbb{Q}_{>0}$.

Consider (X, cD) as a log Fano pair $(0 \leq c < \min\{1, r^{-1}\})$.

For each c, we have a K-moduli space M^K_c.

<u>Then</u> (1) $\exists$ finitely many walls $0 < c_1 < \cdots < c_k < \min\{1, r^{-1}\}$ s.t. for any $c \in (c_i, c_{i+1})$, M^K_c is indep of choice of c.

(2) $\exists$ wall crossing (proj) morphisms

$$M^K_{c_i - \varepsilon} \longrightarrow M^K_{c_i} \longleftarrow M^K_{c_i + \varepsilon}$$

(3) $\exists$ étale local VGIT presentation for each wall crossing diagram

We will focus on 3 examples:

$(\mathbb{P}^n, cD_4)$ for $n = 2, 3, 4$.

$\uparrow$ quadratic hypersurface

<u>Thm</u> (ADL, GMGS, Zhou)

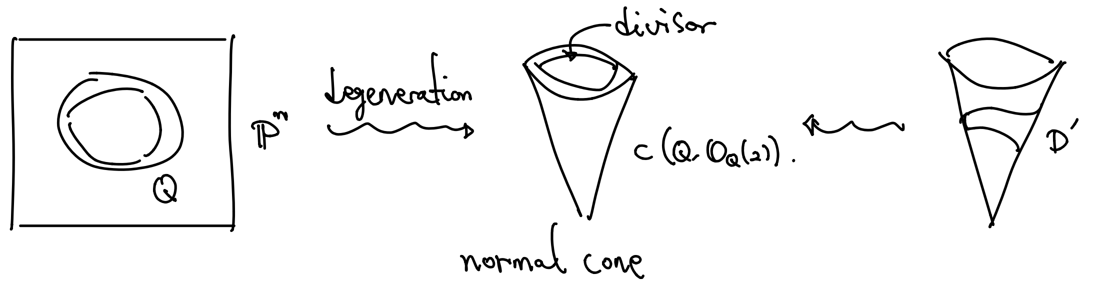
$M^K_\varepsilon \cong M^{GIT}$: GIT of hypersurfaces in $\mathbb{P}^n$.

$(0 < \varepsilon \ll 1)$

<u>First wall</u> (Conj in higher dim) $c_1 = \frac{n+1}{4n}$.

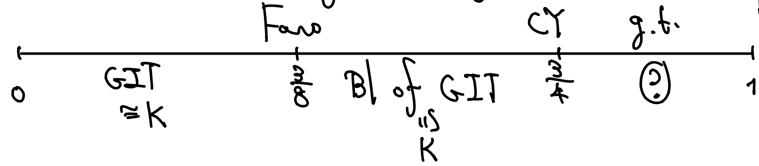
$(\mathbb{P}^n, 2Q) \rightsquigarrow (C(Q, \mathcal{O}_Q(2)), 2Q_\infty) \leftsquigarrow (C(Q), D')$

Q: smooth hyperquadric.



If $n=2$: canonical curves ($g=3$) $\rightsquigarrow$ $2Q$ double conic $\leftsquigarrow$ D' ($g=3$) hyperelliptic curves

Thm (ADL) Assume $n=2$, $(\mathbb{P}^2, cD_4)$, $0<c<\frac{3}{4}$, log Fano domain.
D_4: quartic curve
Then $c_1 = \frac{3}{8}$ is the only wall for K-moduli spaces.



Rmk If we go further to $c \in (\frac{3}{4}, 1)$, then we can recover Hassett-Keel program.

If $n=3$: $(\mathbb{P}^3, cD_4)$, D_4: quartic surface, K3. $(0<c<1)$

- $M^K_\varepsilon \cong M^{GIT}$ birat'l
- $M^{CY}_1 \cong M^{BB}$ Baily-Borel compactification.

Q (Laza-O'Grady) Understand the birat'l map between M^{GIT} & M^{BB}.
$\rightsquigarrow$ they introduced H-K-Looijenga program.

<u>Thm</u> (ADL) Assume $n=3$.

M_c^K interpolates between M^{GIT} & M^{BB}.

Moreover, we have explicit description of wall crossing

$c_1 = \frac{1}{3}$. replacing $(\mathbb{P}^3, 2Q)$ by $(C(Q), D')$ ← hyperelliptic K3.

only 2 blow-ups ⋮ flips

$c_8 = \frac{9}{13}$, replacing $(\mathbb{P}^3, T)$ by (X_u, D'')

flips ⋮ T: tangent developable of twisted cubic; X_u: 3-fold; D'': elliptic K3 w/ a section (unigonal K3).

In total 9 walls.

<u>If $n=4$</u>: $(\mathbb{P}^4, cD_4)$, D_4: quartic 3-fold.

Indeed, $c_{max} = \frac{n+1-r^{-1}}{n} = \frac{15}{16}$.

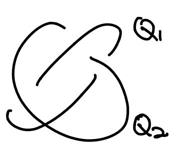
$M^K_{\frac{15}{16}} \cong$ K-moduli of D_4.

<u>First wall</u> (conj) $c_1 = \frac{5}{16}$.

$(\mathbb{P}^4, 2Q) \rightsquigarrow (C(Q), 2Q_\infty) \leftsquigarrow (C(Q), D')$ ← hyperelliptic 3-fold.

Joint work with Abban, Cheltsov, Kasprzyk, Petracci:

$(\mathbb{P}^4, Q_1 + Q_2) \rightsquigarrow (C(Q_1), Q_{1,\infty} + C(S))$



$\rightsquigarrow (C(C(S)), C_1(S) + C_2(S)) \leftsquigarrow (C(C(S)), D')$

↑ K-moduli wall crossing morphism at $c = \frac{5}{6}$

$S = Q_1 \cap Q_2$ dP surface of deg 4.

- D'' is a $(2,2,4)$-complete intersection in $\mathbb{P}(1,1,1,1,2,2)$. ← $C(C(S))$.

<u>Thm</u> (ACKLP) K-moduli compactification of D'' is isom to a K-moduli compactification of $(S, \frac{1}{16}C)$ with $C \in |-4K_S|$.