Monodromy of subrep'ns and irreducibility of low-degree automorphic Galois rep'ns.

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<u>Goals</u> (1) Present big image results of Galois (sub)rep'ns.

(2) Prove new cases of the irred conj.

<u>Notations</u>
- K number field, $Gal_K := Gal(\bar{K}/K)$.
- Σ_K set of finite places in K.
- E_λ : λ-adic completion, char $E_\lambda = \ell$.
- $\mathbb{F}_\lambda$: res field of E_λ.
- $\rho_\lambda : Gal_K \to GL_n(E_\lambda)$ semisimple.

  $\Gamma_\lambda = \mathrm{Im}\,\rho_\lambda$ monodromy grp of ρ_λ.
- G_λ = Zariski closure of Γ_λ in GL_{n,E_λ} (red subgrp of GL_{n,E_λ}).
- $G_\lambda^{der} = [G_\lambda^\circ, G_\lambda^\circ]$ semisimple grp.

Say ρ_λ is residually irred. if $\bar{\rho}_\lambda^{ss} : Gal_K \xrightarrow{\text{semisimple}} GL_n(\mathbb{F}_\lambda)$ irred.

ρ_λ is of type A if $\mathrm{Lie}(G_\lambda^{der})$ has only type-A factors.

<u>Thm</u> (Serre) Let $\{\rho_\ell : Gal_K \to GL_2(\mathbb{Q}_\ell)\}_\ell$ be the compatible system of ℓ-adic rep'ns attached to elliptic curve E/K with no CM pt over K (i.e. $End_K(E) = \mathbb{Z}$).

Then (1) ρ_ℓ is irred $\forall \ell$,

and $\forall \ell \gg 0$, ρ_ℓ is residually irred.

(2) If $\mathrm{End}(E_{\bar{K}}) = \mathbb{Z}$, then $\forall \ell$, $G_\ell = GL_2$,

and $\Gamma_\ell \cong GL_2(\mathbb{Z}_\ell)$, $\ell \gg 0$.

Serre compatible system of K over E:

Def'n A family of n-dim'l λ-adic rep'ns

$$\{\rho_\lambda : \mathrm{Gal}_K \longrightarrow GL_n(E_\lambda)\}_{\lambda \in \Sigma_E}$$

is called an <u>E-rational</u> Serre compatible system of K

unram outside a finite subset $S \subseteq \Sigma_K$ if

$\forall v \in \Sigma_K \setminus S$, $\exists P_v(t) \in E[t]$ s.t.

$\forall \lambda \in \Sigma_E$, ρ_λ is unram at all $v \in (\Sigma_K \setminus S) \cup S_\ell$

$$S_\ell = \{v \in \Sigma_K : v \mid \ell\}.$$

and $\det(1 - t\rho_\lambda(\mathrm{Frob}_v)) = P_v(t) \in E[t]$.

E.g. $V_\ell = H^w_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}_K$ $\rightsquigarrow$ $\left\{ \rho_\ell : \mathrm{Gal}_K \longrightarrow GL(V_\ell) \supseteq GL_n(\mathbb{Q}_\ell) \right\}_\ell$

(Deligne 1974)

Thm (Hui) Let $\{\rho_\lambda : \mathrm{Gal}_K \to GL_n(E_\lambda)\}_\lambda$ be a ss E-valued Serre compatible system.

<u>Suppose</u> $\exists N_1, N_2 \in \mathbb{Z}_{\geq 0}$ and a fin ext'n K'/K s.t.

(a) (Bounded tame inertia weights)

For almost all λ and $\forall v \in S_\ell$ (places above ℓ)

$(\bar{\rho}_\lambda^{ss} \otimes \bar{\varepsilon}_\ell^{N_1})|_{\mathrm{Gal}_{K_v}}$ has tame inertia wts $\in [0, N_2]$.

$\bar{\varepsilon}_\ell : \mathrm{Gal}_K \to \mathbb{F}_\ell^\times$ (mod ℓ cycl char).

(b) (Potential semistability)

For almost all λ and $\forall\, w \in \Sigma_K$, $w \nmid \ell$,

$$\bar\rho_\lambda^{ss}\big|_{\mathrm{Gal}_{K_w}} \text{ is unram.}$$

Then for almost all λ,

if σ_λ is the type A irred. subrep of $\rho_\lambda \otimes \bar{\mathbb{Q}}_\ell \supseteq E_\lambda$

then σ_λ is residually (absolutely) irred.

Cor $\dim \sigma_\lambda \le 3$, irred. $\Rightarrow$ res irred. ($\ell \gg 0$).

Idea Method of algebraic envelops G_ℓ.

- Restriction of scalar:

Assume $E = \mathbb{Q}$, $\lambda = \ell$.

$$\Big(\prod_{\lambda | \ell} \rho_\ell\Big) : \mathrm{Gal}_K \to \prod \mathrm{GL}_n(E_\lambda) \subseteq \mathrm{GL}_n(\mathbb{Q}_\ell).$$

When $\ell \gg 0$,

$$\begin{array}{ccc} \Gamma_\ell = \rho_\ell(\mathrm{Gal}_K) & \hookrightarrow & G_\ell \subseteq \mathrm{GL}_{n,\mathbb{Q}_\ell} \\ & & \rightsquigarrow \sigma_\lambda \subseteq \bar{\mathbb{Q}}_\ell^n \\ \downarrow & & \\ \bar\rho_\ell^{ss}(\mathrm{Gal}_K) & \hookrightarrow & G_\ell(\mathbb{F}_\ell) \subseteq \mathrm{GL}_{n,\mathbb{F}_\ell}. \\ & & \rightsquigarrow \sigma_\lambda \subseteq \mathbb{F}_\ell^n. \end{array}$$

Can form a relation indep'd of $\ell \gg 0$.

| G conn red grp
| $\cup\!\!\mid$ H of type A
| $\rightsquigarrow \mathrm{rk}^{ss} H = \mathrm{rk}^{ss} G$
| $\Rightarrow H = G$.

Autom Gal rep'ns

K totally real in a CM field.

π regular alg polarized cusp autom rep of $\mathrm{GL}_n(\mathbb{A}_K)$.

$\overset{(\ldots)}{\Longrightarrow}$ $\exists$ a CM field E & $\{\rho_{\lambda,\pi} : \mathrm{Gal}_K \to \mathrm{GL}_n(E_\lambda)\}_\lambda$ semisimple E-val'd strictly compatible system.

Irreducibility conj.

(1) $\forall \lambda$, $\rho_{\lambda,\pi}$ abs irred

(2) For almost all λ, $\bar{\rho}^{ss}_{\lambda,\pi}$ is abs irred.

Known cases

- Ribet 1977: classical MFs
- Taylor 1995: Hilb MFs
- $n=3$, K tot real: Blasus-Rogawsk 1992
- $n \le 5$, K tot real: Calegari-Gee 2013.

Thm (Hui) $n \le 6$. Then

(1) $\rho_{\lambda,\pi}$ abs irred, $\forall$ almost all λ

(2) Assume $K = \mathbb{Q}$. Then $\bar{\rho}_{\lambda,\pi}$ abs irred. (almost all λ).