The quadratic Artin conductor of motivic spectrum

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(Joint with Enlin Yang).

Grothendieck-Ogg-Shafarevich (GOS) formula:

$k = \bar{k}$. C/k sm proj curve, $\Lambda = \mathbb{F}_\ell$.

$\mathcal{F} \in Sh(C_{\acute{e}t}, \Lambda)$ locally const outside $\{x_1, \dots, x_n\}$.

$$\rightsquigarrow \chi(\mathcal{F}) = rk(\mathcal{F}) \cdot \underbrace{\chi(C)}_{= \chi(\Lambda_C)} - \sum Art_{x_i}(\mathcal{F}) \in \mathbb{Z}. \qquad (*)$$

Goal $GW(k) = K_0(\text{nondeg sym bilin forms } / k)$.

$$\begin{array}{c} GW(k) \\ \downarrow \deg \\ \mathbb{Z} \end{array} \quad \rightsquigarrow \text{lift } (*) \text{ to } GW(k).$$

§1 Motivic homotopy theory

Stable homotopy $\mathcal{H}_\bullet = \{\text{pointed top spaces}\}/\text{homotopy}$

$$\begin{array}{c} \mathcal{H}_\bullet \\ \downarrow \\ \underline{SH}_{top} \underset{N}{\overset{K}{\leftrightarrows}} D(Ab) \end{array}$$

stable Dold-Kan corr.

SH_{top}: objects $= S^1$-spectra

$$E = \begin{cases} (E_n)_{n \in \mathbb{N}}, \ E_n \in \mathcal{H}_\bullet. \\ \sigma_n : S^1 \wedge E_n \to E_{n+1} \ \text{suspection maps.} \end{cases}$$

Morphisms $= f : E \to F \iff (E_n \xrightarrow{f_n} F_n)_{n \in \mathbb{N}}$ commutes with σ_n.

$\pi_n(E) = \operatorname{Colim}_i \pi_{n+i}(E_i)$ stable homotopy grps.

$f: E \to F$ is called a $\underbrace{\text{stable weak equiv.}}$

if $\forall n$, $\pi_n(f)$ isom.

$$SH_{top} = (S^1\text{-Spectra})[\text{s.w.e}]^{-1}$$

- This is a triangulated cat: $X[1] = X \wedge S^1$.

- $\forall E \in SH_{top}$, $E^n(X) = [X, E \wedge S^n]_{SH_{top}}$.

  coh theory rep'd by E.

E.g. $X \in \mathcal{H}$. $\Sigma^\infty X \in SH$ infinite suspection, spectra.

$$(\Sigma^\infty X)_i = X \wedge S^i.$$

In particular, sphere spectrum $\mathbb{S} = \Sigma^\infty(pt)$.

- A ring, $HA \in SH_{top}$. Eilenberg-Maclane spectrum.

<u>Motivic homotopy</u> $S \in Sch$, qcqs.

"Spaces = cat of spaces" (e.g. CW complexes), sSets.

motivic spaces $/S$ = presheaf of spaces over $\underbrace{Sm/S}_{\text{cat of sm schs}}$.

$$\mathcal{H}.(S) = \{\text{motivic spaces}/S\}[Nis, \mathbb{A}^1]^{-1}.$$

<u>Homotopy sheaves</u> $X \in \mathcal{H}.(S)$

$\pi^{\mathbb{A}^1}_{a,b}(X)$ = Nis sheaf on Sm/S assoc to

$$U \longmapsto [U \wedge (S^1)^{a-b} \wedge \mathbb{G}_m^b, X]_{\mathcal{H}(S)}.$$

<u>$\mathbb{P}^1$-spectra</u> $\begin{cases} \mathbb{E} = (\mathbb{E}_n), & \mathbb{E}_n \in \mathcal{H}(S) \\ \sigma_n : \mathbb{P}^1 \wedge \mathbb{E}_n \longrightarrow \mathbb{E}_{n+1} \end{cases}$.

- $f: \mathbb{E} \to \mathbb{F}$ is stable motivic w.e. if $\pi^{\mathbb{A}^1}_{a,b}(f)$ isom.

$$SH(S) = (\mathbb{P}^1\text{-Spectra})[\text{s.m.w.e}]^{-1}$$

Stable motivic homotopy cat.

- 2 spheres: $\mathbb{P}^1 \overset{\mathbb{A}^1}{\sim} S^1 \wedge \mathbb{G}_m$

$$\underset{\mathbb{1}(1)[2]}{\mathbb{P}^1} \quad \underset{\mathbb{1}(1)}{\mathbb{G}_m}\ \ \underset{\mathbb{1}(1)[1]}{}$$

$\mathbb{P}^1 = \mathbb{1}(1)[2]$, $S^1 = \mathbb{1}(1)$, $\mathbb{G}_m = \mathbb{1}(1)[1]$.

- $\mathbb{E} \in SH(S)$ represents a bounded coh theory.

$X \in Sm/S$, $\mathbb{E}^{p,q}(X) = [\Sigma^\infty X, (S^1)^{n(p-q)} \wedge (\mathbb{G}_m)^{nq} \wedge \mathbb{E}]_{SH(S)}$.

E.g. · $H\mathbb{Z}$ motivic E-M spectra $\rightsquigarrow$ motivic coh.

- H_ℓ étale coh.

$SH_c(X) \to D^b_{eff}(X_{\acute{e}t}, \Lambda)$ étale regularization.

- KGL: homotopy K-theory
- MGL: alg cobordism.

- 6 functors formalism: $f_*, f^*, f_!, f^!, \otimes, \underline{Hom}$.

SH = univ cat with 6 functors.

(Drew – Gallauer).

<u>Thom space</u> V/S vector bundle.

$$Th_S(V) = V/V-0 \in SH(S)$$

(<u>Rmk</u>

$$\underset{PSh(Sm/S,\, Spectra)}{SH(S)} \rightleftarrows \underset{PSh(Sm/S,\, C(Ab))}{D^{\mathbb{A}^1}(S)} = \mathbb{A}^1\text{-derived cat}.$$

↑ stable motivic Dold-Kan.)

II. Milnor–Witt K-theory

Def (Morel) F field.

$K^{MW}_*(F)$ = graded assoc ring with

- generators : $-[u]$ for $a \in F\setminus\{0\}$, $\deg[u]=1$.
  - η : $\deg \eta = -1$.
- relations :
  1. (Steinberg relation) $\forall a \in F\setminus\{0,1\}$, $[a]\cdot[1-a]=0$
  2. $\forall a,b \in F\setminus\{0\}$, $[ab]=[a]+[b]+\eta\cdot[a]\cdot[b]$.
  3. $\forall a \in F\setminus\{0\}$, $[a]\cdot\eta = \eta\cdot[a]$.
  4. $\eta\cdot(\eta\cdot[-1]+2)=0$.

Rmk (1) $K^{MW}_*(F)/\eta = K^M_*(F)$ Milnor K-theory.

(2) char $F \neq 2$.

$K^{MW}_0(F) \simeq GW(F)$ $\qquad \left(GW(F)/h = W(F)\right)$.

$1+\eta\cdot[a] \longleftarrow\!\shortmid \langle a\rangle$

$\underset{\substack{+\\2}}{\eta\cdot[-1]} \longmapsto 1+\langle -1\rangle = h$ (hyperbolic form $\leftrightarrow \begin{pmatrix}1&0\\0&-1\end{pmatrix}$).

(3) η = motivic Hopf map

$$\begin{array}{c} \mathbb{G}_m \wedge \mathbb{P}^1 \underset{\mathbb{A}^1}{\simeq} \mathbb{A}^2-\{0\} \longrightarrow \mathbb{P}^1 \\ \Big\downarrow \mathbb{P}^1\text{-deloop} \\ \mathbb{G}_m \longrightarrow 1. \end{array}$$

Def'n $\mathbf{K}^{MW}_n$ = unram M-W sheaf

= Nis sheaf on Sm/S assoc to norm-residue

$$X \longmapsto \ker\Big(\bigoplus_{x\in X^{(0)}} K^{MW}_n(k(x)) \longrightarrow \bigoplus_{x\in X^{(1)}} K^{MW}_{n-1}(k(x))\Big).$$

Thm (Morel) k = field.

$$\pi^{\mathbb{A}^1, st}_{n,n} = \underline{K}^{MW}_n \quad \left(\text{In particular, } [\mathbb{1}_k, \mathbb{1}_k]_{SH(k)} = GW(k).\right).$$

Milnor-Witt spectrum

k = infinite field (Hopefully, can be dropped.)

$\widetilde{DM}(k) = Psh^{MW}_{tr}(Sm/k, Ab)$. Milnor-Witt motives

$\downarrow$ $\uparrow$ presheaves with M-W transfers.

$DM(k)$ = presheaves with transfers

$$SH(k) \underset{K}{\overset{N}{\rightleftarrows}} D^{\mathbb{A}^1}(k) \underset{\gamma_*}{\overset{\gamma^*}{\rightleftarrows}} \widetilde{DM}(k).$$

- M-W spectrum: $H\widehat{\mathbb{Z}}_{MW} = K\gamma_*\mathbb{1}_k \in SH(k)$.

  $\downarrow$

  $H\mathbb{Z}$

- H_{MW} represents MW motivic cohom

$$H^{n,0}_{MW}(X) = H^n(X, \underline{K}^{MW}_0)$$

- Rationally, $\mathbb{1}_{k,\mathbb{Q}} \simeq H_{MW,\mathbb{Q}}$ (Deligne-Fasel-Ji-Khan).

III. Quadratic GOS formula

char $k = p$.

- $f^\Delta := \mathrm{cofib}(f^*(-) \otimes f^!\mathbb{1} \longrightarrow f^!(-))$
- $\mathbb{E} = H_{MW} \in SH(k)[\frac{1}{p}]$.

$X \xrightarrow{f} k$ sm. $\rightsquigarrow$ $\mathbb{E}^!_{X/k} = f^!\mathbb{E} \in SH(X)[\frac{1}{p}]$

$\cup$

$f^*\mathbb{E} = \mathbb{E}_X$.

$\rightsquigarrow \mathbb{E}(X/k) = [\mathbb{1}_X, \mathbb{E}^!_{X/k}]_{SH(X)}$

Borel-Moore $\mathbb{E}$-homology.

- $\delta : X \to X \times_E X$ Euler cofib sequence.

$$\mathbb{E}_X \xrightarrow[\uparrow]{e(T_{X/k})} \mathbb{E}^!_{X/k} \longrightarrow \delta^\Delta \delta_* \mathbb{E}^!_{X/k}$$

Euler class.

- $Z \overset{i}{\hookrightarrow} X \hookleftarrow U$, $K \in SH_c(X)$ s.t. $K|_U$ dualizable.

Def'n

$$\begin{tikzcd}
i_* \mathbb{1}_Z \arrow[r] \arrow[drrr, bend right] & i_* i^* \delta^\Delta \delta_* \mathbb{1}_X \arrow[r] & i_* i^* \delta^\Delta (D(K) \boxtimes K) \simeq & \delta^\Delta (D(K) \boxtimes K) \arrow[d, "i_U"] \\
 & & & \delta^\Delta \delta_* \mathbb{E}^!_{X/k}.
\end{tikzcd}$$

$$B^Z_X(K, \mathbb{E}) \in [i_* \mathbb{1}_Z, \delta^\Delta \delta_* \mathbb{E}^!_{X/k}].$$

Theorem If (1) $\mathbb{Z}[\frac{1}{2p}]$-coeff, (2) $\mathrm{codim}_Z X \geq 2$, (3) $\dim X$ odd,

Then $\exists !\, C^Z_X(K, \mathbb{E}) \in \mathbb{E}(Z/k)[\frac{1}{p}]$ s.t.

$$\begin{tikzcd}
 & C^Z_X(K,\mathbb{E}) \arrow[r, dashed] & \mathbb{E}^!_{X/k} \arrow[d] \\
i_* \mathbb{1}_Z \arrow[ur, dashed] \arrow[rr, "B^Z_X(X{,}\mathbb{E})"'] & & \delta^\Delta \delta_* \mathbb{E}^!_{X/k}.
\end{tikzcd}$$

Def'n X/k sm proper. $f : X \to k$.

$$\mathrm{Art}(K)[\tfrac{1}{p}] := \int_Z C^Z_X(K, \mathbb{E}) \in \mathbb{E}(k/k)[\tfrac{1}{p}] = GW(k)[\tfrac{1}{p}],$$

$\rightsquigarrow$ char $k \neq 2$:

$$\begin{tikzcd}
\mathrm{Art}(K) \arrow[rr, mapsto] \arrow[d, bend right] & & \mathrm{Art}(K_{et}) \\
 & GW(k) \arrow[r, "rk"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z} \arrow[d] \\
\mathrm{Art}(K)[\frac{1}{p}] & GW(k)[\frac{1}{p}] \arrow[r] & \mathbb{Z}[\frac{1}{p}].
\end{tikzcd}$$

Thm $f : X \to k$.

$$\chi(f_* K) = rk(K_{et}) \cdot \underbrace{\chi(f_* \mathbb{1}_X)}_{= f_* e(T_f)} - \mathrm{Art}(K) \in GW(k).$$

note $\dim X$ odd $\Rightarrow e(T_X, HW) = 0$.

Ingredient.

$$\begin{tikzcd} H_{MW}\Lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H\Lambda \arrow[d] \\ H_W\Lambda \arrow[r] & \Lambda/2\Lambda \end{tikzcd}$$

Additivity : $K \to L \to M$ dist triang $\Rightarrow C^Z_X(L) = C^Z_X(K) + C^Z_X(M)$.

(This is invalid in arbitrary Δ-cat)

Higher-cat cases : Moy, Jin-Yang.