Homology of intertwining operator

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<u>Outline</u> Questions / Known cases / Applications / Method.

<u>Setup</u> G conn red quotient split grp / F.

$\int$ F p-adic local field.

$P = MN$ max'l standard parabolic subgrp with M Levi.

$\bar{P} = M\bar{N}$ the unique opposite para subgrp of P.

$\rightsquigarrow$ <u>Induced rep'n</u> · σ irred rep'n of M

· $\gamma \in X^*(M)$ F-rational char of M.

$\leftrightarrow s\tilde{\gamma}$ $(s \in \mathbb{C})$,

for a specific F-rat char of M.

$$\rightsquigarrow I_P^G(\sigma_s) := \left\{ f : G \xrightarrow{\text{Smooth}} V_\sigma \;\middle|\; \begin{array}{l} f(mng) = \delta_P(m)^{\frac{1}{2}} |\tilde{\gamma}(m)|^s \sigma(m) f(g) \\ \forall m \in M, n \in N, g \in G \end{array} \right\}.$$

<u>Intertwining operator</u> $w_0 = w_0^G w_0^M$ with

w_0^G (resp. w_0^M) the longest Weyl element in G (resp. M).

$$\rightsquigarrow A(s,\sigma,w_0) : I_P^G(\sigma_s) \longrightarrow I_{w_0.\bar{P}}^G(w_0(\sigma_s)).$$

$$f \longmapsto \int_{N_{w_0}} f(w_0^{-1} n g)\, dn.$$

Here N_{w_0} = the unipotent part of the std parabolic subgrp of $w_0 \cdot \bar{P}$.

<u>Properties</u> (1) $A(s,\sigma,w_0)$ converges for $\mathrm{Re}(s) \gg 0$.

and has mero conti to $\mathbb{C}$.

(2) $A(s,\sigma,w_0) \neq 0$ as an operator.

<u>Question</u>: Singularity of $A(s, \sigma, w_0)$?

<u>Known cases</u> (classical results)

(3) For σ supercuspidal, $s \in \mathbb{R}$,

if $w_0\sigma \not\cong \sigma$, $A(s, w_0, \sigma)$ is always holo.

Otherwise, $A(s, w_0, \sigma)$ has a unique simple pole ?

$\Updownarrow$ proof has gaps / issues.

$I_P^G(\sigma)$ is irred. (Silberger, Ann. Math, 1980).

(4) (Reduction property)

$w_0 = w_1 \cdots w_t$ reduced decomp with w_i elementary reflection

i.e. $\ell(w_0) = t$.

$\sigma \hookrightarrow I_{M \cap P_0}^M(\rho_\nu)$ with ρ s.c. $/M_0$, $\nu \in X^*(M_0)$

$$\begin{tikzcd}
I_P^G(\sigma_s) \arrow[r, hook] \arrow[d] & I_{P_0}^G(\rho_{\nu,s}) \arrow[d, dotted] \\
I_{w_0 P}^G(w_0(\sigma_s)) \arrow[r, hook] & I_{w_0, \bar{P_0}}^G(w_0(\rho_{\nu,s}))
\end{tikzcd}$$

$$\Rightarrow A(s, w_0, \sigma) = \prod_{i=1}^{t} A(s, w_i, \sigma_i)$$

<u>Issues</u> (1) subgrps, not whole spaces

(2) $I_P^G(\tau_s)$ with τ tempered.

(a) $I_P^G(\chi_s)$: $\sigma = \chi$ char of $M \hookrightarrow G$

(Siegel parabolic in a classical grp).

(Pre-homogeneous v.s. method).

By Piatetski-Shapiro-Rallis (1987).

(b) σ generic supercuspidal : Langlands - Shahidi theory (answered (2)).

L-factors : $M^\vee$ Langlands dual grp

$M^\vee \overset{\mathrm{adj}}{\curvearrowright} \mathfrak{n}^\vee$ Lie alg of $N^\vee = \bigoplus_{i=1}^{\ell} V_i$

with V_i irred. ordered in a specific way.

Langlands - Shahidi L-functions

$\rightsquigarrow L(s, \sigma, V_i)$ Artin L-func.

Thm (Shahidi) (local coefficient theory)

For σ generic s.c. $\prod_{i=1}^{\ell} L(s, \sigma, V_i)^{-1} A(s, \sigma, w_0)$

is holo, and (optimally) nonzero.

Casselman - Shahidi conj :

$\prod_{i=1}^{\ell} L(s, \sigma, V_i)^{-1} A(s, \sigma, w_0)$ is holo for σ generic tempered.

Applications

Goal To control the poles of autom L-functions & Eisenstein series.

(a) Theta correspondence

$P \hookrightarrow G$ Siegel of classical grp.

$\begin{cases} JH(I_P^G(\chi_s)) & \text{degenerate P.S. } (\chi \text{ char}). \\ L(s, \pi, \mathrm{Std}) & \pi/G \text{ gcd definition} \end{cases}$

under doubling method (Yamana, Inv. Math., 2013).

① $I_P^G(\chi_s)$:

(b) Rankin - Selberg L-func for $\pi \otimes \tau / G \times GL_n$ both generic.

$L(s, \pi \times \tau)$ g.c.d. def'n (Kaplan, Compos. Math. 2013) (But Incomplete)

② $I_P^G(\tau_s)$ with τ tempered:

(c) generated doubling method ([CFGK] Inv. Math, 2019)

$L(s, \pi\times\tau)$ no restriction on π.

$L(s, \pi\times\tau)$ g.c.d. def'n (In progress).

③ $I_P^G(\rho_c(\tau)_s)$ with $\rho_c(\tau)$ Speh rep'n ass to τ discrete.

$\left(\rho_c(\pi) \hookrightarrow \tau|\cdot|^{\frac{n-1}{2}} \times \cdots \times \tau|\cdot|^{-\frac{n-1}{2}}.\right)$

Methods

- σ tempered $A(s,\sigma,w_0)$ has no pole for $\mathrm{Re}\, s>0$.
- Claim $A^*(s,\sigma,w_0)$ holo at $\mathrm{Re}(s)=0$.

Observation (Tempered L-func theorem).

(Heiermann-Opdam, 2013)

$L(s,\sigma,V_i)$ has no pole for $\mathrm{Re}\, s>0$ (*).

(Casselman-Shahidi, 1998) (*) true for grps of classical type.

Rmk C-S conj $\Leftrightarrow$ $A^*(s,\sigma,w_0)$ holo for $\mathrm{Re}\, s<0$.

Rough idea: $w_0 = w_1 w_2$, w_1 & $w_2 \neq$ elementary reflection

P_1-side $\leftarrow$ w_1 = "w_0" of lower dim'l subgrps

P_2-side $\leftarrow$ $= w_3\cdot w_4\cdot w_5 \cdots$

If "$(P_1,P_2)=1$", it follows from the inductive arguement.

*

+ multiplicity property + Heiermann-Opdam $\rightsquigarrow$ reduces to σ discrete (tempered case).

Example (GL_n)

$G = GL_{na} \supseteq M = GL_{m_1 a} \times GL_{m_2 a}$

$\sigma = \sigma_1 \otimes \sigma_2$ discrete series

with $\sigma_i \hookrightarrow \rho_i |\cdot|^{\frac{m_i-1}{2}} \times \cdots \times \rho|\cdot|^{-\frac{m_i-1}{2}}$ $(\rho_1 = \rho_2$ self-dual$)$

$A_{GL}(s, m_1, m_2) : \sigma_1|\cdot|^s \times \sigma_2|\cdot|^s \longrightarrow \sigma_2|\cdot|^s \times \sigma_1|\cdot|^s$

$$\alpha_{GL}(s, m_1, m_2) = L(2s, \sigma_1 \otimes \sigma_2) = \prod_{j=\left|\frac{m_1-m_2}{2}\right|}^{\frac{m_1+m_2}{2}-1} L(2s+j, \rho \otimes \check{\rho}) = \text{"} \prod_j \frac{1}{2s+j} \text{"}.$$

Assume $m_1 \geq m_2$. $\sigma_i = \rho_{m_i}$.

$$\begin{tikzcd}
\sigma_1|\cdot|^s \times \sigma_2|\cdot|^s \arrow[r, hook] \arrow[dd] \arrow[ddr, phantom, "\circlearrowright"] & \sigma_1|\cdot|^s \times \rho|\cdot|^{\frac{m_2-1}{2}-s} \times \rho_{m_2-1}|\cdot|^{-s-\frac{1}{2}} \arrow[d] \\
 & \rho|\cdot|^{-s+\frac{m_2-1}{2}} \times \rho_{m_1}|\cdot|^s \times \rho_{m_2-1}|\cdot|^{-s-\frac{1}{2}} \arrow[d] \\
\sigma_2|\cdot|^s \times \sigma_1|\cdot|^s \arrow[r, hook] & \rho|\cdot|^{-s+\frac{m_2-1}{2}} \times \rho_{m_2-1}|\cdot|^{s-t} \times \rho_{m_1}|\cdot|^s
\end{tikzcd}$$

$\Rightarrow$ $P_2(s) = L\left(2s - \frac{m_2-1}{2}, \rho \times \check{\rho}\right)$ only has pole at $\mathrm{Re}(s) > 0$.