I-cohomology of Grassmannians

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Motivation: To find "singular cohom" in alg geom.

Let X sm $/\mathbb{C}$,

Comparison thm: Chow grps, étale coh, motivic coh involved.

When X sm $/\mathbb{R}$:

Nice candidate: I-cohom (can define / any field char $\neq 2$).

$$\begin{array}{c} H^i(X, I^j) \\ \uparrow \ \text{"}\cong\text{"} \\ H^i_{sing}(X(\mathbb{R}), \mathbb{Z}) \end{array} \qquad \left(\begin{array}{l} \text{Jacobson 2017} \\ \text{Hornbostel - Wendt - Xie - Zibrowius} \end{array}\right)$$

depending on j. ($\cong$ when $j \gg 0$).

Another reason to study I-cohom:

Thm (Morel 2012) A = smooth affine k-alg, $\dim A = d$.

P = proj A-mod of rank d s.t. $\det P = A$.

Then $P \cong Q \oplus A \iff$ Euler class $e(P) = 0 \in \widetilde{CH}^d(X)$.

(Rmk: $\widetilde{CH}^i(X)$ "=" $H^i(X, I^i) \times_{CH^i(X)/2CH^i(X)} CH^i(X)$)

F field of char $\neq 2$.

$Q(F)$ set of non-degenerate quadratic form $[V, q]/F$

(monoid w.r.t. orthogonal sum $\perp$).

$L(F)$ submonoid generated by hyperbolic forms $[(V, q) \perp (V, -q)]$.

Def'n Witt grp $W(F) := Q(F)/L(F)$.

inverse: $-[V, q] = [V, -q]$.

Grothendieck-Witt grp $\quad GW(F) = K_0(Q(F))$.

<u>Rmk</u> $\cdot$ $W(F) \cong GW(F)/\langle L(F)\rangle$.

$\cdot$ $W(F)$ is a ring w.r.t. $\otimes$.

<u>Example</u> $W(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$, $W(\mathbb{R}) = \mathbb{Z}$,

$GW(\mathbb{C}) = \mathbb{Z}$, $GW(\mathbb{R}) = \mathbb{Z}\times\mathbb{Z}$.

Construct $\mathrm{rank}: W(F) \longrightarrow \mathbb{Z}/2\mathbb{Z}$.

$$[V,q] \longmapsto \mathrm{rk}(V).$$

<u>Def'n</u> $I(F) := \ker(W(F) \xrightarrow{\mathrm{rk}} \mathbb{Z}/2\mathbb{Z})$ fundamental ideal.

$I^n(F)$ = nth power of $I(F)$.

(<u>Thm</u> (Milnor Conj, Voevodsky 1996)

$$I^n(F)/I^{n+1}(F) \cong K_n^M(F)/2K_n^M(F).$$).

<u>Residue</u> [MH73]

R DVR, $F = \mathrm{Frac}\, R$, k res field.

$\mathfrak{m}$ max'l ideal, π uniformizer.

<u>Thm</u> [MH73] (a) $W(F)$ is generated by $\langle a\rangle$, for some $a \in F^\times$.

(b) $\delta_\pi : W(F) \longrightarrow W(k)$ is a grp hom.

$$\langle \pi^i u\rangle \longmapsto \begin{cases} \langle \bar{u}\rangle, & 2\mid i \\ 0. & 2\nmid i \end{cases}$$

reciprocity.

$$\rightsquigarrow \; 0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \longrightarrow \bigoplus_{p\neq\infty} W(\mathbb{F}_p) \to 0$$

<u>Rost-Schmid 1998</u>

$$\delta : W(F) \longrightarrow W(k, (\mathfrak{m}/\mathfrak{m}^2)^*) := W(k)\otimes_{k^\times}((\mathfrak{m}/\mathfrak{m}^2)^* - \{0\}).$$

$$\langle a\rangle \longmapsto (\delta_\pi(\langle a\rangle)\otimes\pi^*).$$

Transfer $k \hookrightarrow F \hookrightarrow L$ finite ext'ns, $tr: L \to F$.

Thm (Scharlau 1985)

$$t: I^j(L, \omega_{L/k}) \longrightarrow I^j(F, \omega_{F/k}) \text{ is a grp hom.}$$
$$[q] \longmapsto [tr \circ q].$$

X sch of fin type/k. $\dim X = n$. L line bun / X.

$X^{(i)}$ = set of codim i pts in X.

- $x \in X^{(i)}$, $y \in X^{(i+1)}$.

$$I^j(k(x), \omega_x^L) \xrightarrow{\oplus \delta} \bigoplus_{\tilde{y}} I^{j-1}(k(\tilde{y}), \omega_{\tilde{y}}^L) \longrightarrow I^{j-1}(k(y), \omega_y^L)$$

∂ (define this)

where $\tilde{y} \in \widetilde{\{x\}}$ normalization of $\{\bar{x}\}$ in $k(x)$ and $\tilde{y}$ dominates y.

Thm [RS98]

$$0 \to \bigoplus_{x \in X^{(0)}} I^j(k(x), \omega_x^L) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} I^{j-1}(k(x), \omega_x^L) \to \cdots$$
$$\to \bigoplus_{x \in X^{(n)}} I^{j-n}(k(x), \omega_x^L) \to 0.$$

Its cohom is def'd to be

$$H^i(X, I^j(L)).$$

Thm (Ji?, HWXZ 21)

$$\mathrm{Sing}: H^i(X, I^j(L)) \xrightarrow{\cong} H^i_{sing}(X(\mathbb{R}), \mathbb{Z}^L), \quad j \geq \dim X + 1.$$

local coeff system.

sending Euler classes to Euler classes.

Computation Fasel 2013: proj space.

HXZ 2020: quadratic

W 2020: Grassmannian / $\mathbb{R}$.

Thm X/k. Assume

$$\begin{tikzcd}
Z \arrow[r, hook, "i"'] & X & U \arrow[l, hook', "v"'] \arrow[dl, "\tilde{v}"] \arrow[d, "\alpha"] \\
E \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{i}"'] & Bl \arrow[u, "\pi"] \arrow[r, dashed, "\tilde{\alpha}"'] & Y
\end{tikzcd}$$

α: $\mathbb{A}^*$-bundle

and $\mathrm{Pic}(Bl) = \mathbb{Z} \oplus \mathrm{Pic}(X)$.

$$\tilde{\alpha}^*(\alpha^*)^{-1} v^*(L) = (c(L), \pi^* L).$$

(a) If $2 \mid c(L)$, then

$$H^i(X, I^j(L)) \cong H^{i-d}(Z, I^{j-d}(\omega \otimes L)) \oplus H^i(Y, I^j(L)).$$

(b) If $2 \nmid c(L)$, then

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-d}(Z, I^{j-d}(\omega\otimes L)) \arrow[r, "i_*"] & H^i(X, I^j(L)) \arrow[r, "v^*"] & H^i(U, I^j(L)) \arrow[dll, bend left, "\partial"'] \\
& H^{i-d+1}(Z, I^{j-d}(\omega\otimes L)) & \circlearrowleft & H^i(Y, I(L)). \arrow[u, "\alpha^*"'] \arrow[dll, bend right] \\
& H^i(E, I^j(L)) \arrow[u] & &
\end{tikzcd}$$

__Application__ $G_d(E)$ Grassman bundle. E/S vector bundle, $\mathrm{rk}\, E = n+d$.

$G_d(n) := G_d(\mathcal{O}_S^{n+d})$.

$$\begin{tikzcd}
G_d(n-1) \arrow[r, hook] & G_d(n) & U_d(n) \arrow[l, hook'] \arrow[d] \\
\mathbb{P}(N) = E \arrow[u] \arrow[r] & Bl \arrow[u] \arrow[r, dashed, "\text{proj. bundle}"'] & G_{d-1}(E)
\end{tikzcd}$$

$$H^{tot}(X) := \bigoplus_{L \in \mathrm{Pic}(X)/2\mathrm{Pic}(X)} \bigoplus_{i,j} H^i(X, I^j(L)).$$

$$H^{tot}(S) := \bigoplus_{i,j} H^i(S, I^j/I^{j+1}).$$

__Thm__ $$H^{tot}(G_d(n)) = \Big(\bigoplus_{a<b} H^{tot}(S)\Big) \oplus \Big(\bigoplus_{a\geq b} H^{tot}(S)\Big).$$

$$a = \binom{n+d}{d}, \quad b = \binom{\lfloor \frac{d}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}.$$