Drinfeld's lemma for F-isocrystals

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(with K. Kedlaya).

Motivation Langlands corr / function fields

X sm curve $/\mathbb{F}_p$, $F = \mathbb{F}_p(X)$.

$\ell \neq p$. $\mathbb{A}_F = \bigotimes'_{x \in |X|} F_x$.

V Lafforgue autom to Galois

- G split red grp $/ F$.
- $\check{G}$ Langlands dual grp $/ \bar{\mathbb{Q}}_\ell$.

$(G = GL_n, SO_{2n}, SO_{2n+1}, \check{G} = GL_n, SO_{2n}, Sp_{2n})$.

Consider $C_c^{cusp}(G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)\Xi, \bar{\mathbb{Q}}_\ell) \simeq \bigoplus_{\substack{\sigma: \pi_1(X) \to \check{G}(\bar{\mathbb{Q}}_\ell) \\ \text{semisimple}}} \mathfrak{h}_\sigma$. $(*)$

where $\Xi \leq \mathrm{Bun}_Z(\mathbb{F}_p)$ finite index lattice.

$Z = Z_G$, σ Langlands parameters.

Conj This decomp $(*)$ is indep of the choice of ℓ.

(replace $\bar{\mathbb{Q}}_\ell$ by $\bar{\mathbb{Q}}$
$\rightsquigarrow$ should have a decomp w.r.t. "motivic Langlands parameters".)

Ingredients (1) Geom Satake equiv.

$\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\check{G}) \simeq \mathrm{Perv}_{L^+G}(Gr_G)$ $\quad Gr_G = LG/L^+G$.

where LG^+ (resp. LR) $: R \mapsto G(R[[t]])$ (resp. $G(R((t)))$.)

(1)' Motivic version (Zhu, Richarz - Scholbach).

(2) Cohomology of shtukas.

(ℓ-adic, p-adic, v)

(3) Drinfeld's lemma:

X_1, X_2 connected flat sch $/\mathbb{F}_p$. $X = X_1 \times_{\mathbb{F}_p} X_2$.

F_{X_i} abs Frob of X_i.

$F_1 = F_{X_1} \times id_{X_2}$, $F_2 = id_{X_1} \times F_{X_2} : X \to X$ partial Frob.

Def'n $\mathcal{C}(X/\Phi)$: category of pairs

$$\left(\begin{array}{l} T \to X \\ \text{finite étale}, \end{array} \quad \begin{array}{l} F_{\{i\}} : T \times_{X, F_i} X \longrightarrow T \\ \text{s.t. } F_T = F_{\{1\}} \circ F_{\{2\}} = F_{\{2\}} \circ F_{\{1\}} \end{array} \right).$$

⇝ this is a Galois category.

⇝ $\pi_1(X/\Phi)$ Galois grp.

Have $\text{Féf}(X_1) \longrightarrow \mathcal{C}(X/\Phi) ⇝ \pi_1(X/\Phi) \longrightarrow \pi_1^{\text{ét}}(X_1)$.

$T_1 \longmapsto T_1 \times X_2$

Thm (Drinfeld's lemma).

$$\pi_1(X/\Phi) \xrightarrow{\sim} \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2).$$

Application ℓ-adic rep'ns of $\pi_1(X/\Phi)$

⟺ ℓ-adic local systems on X with partial Frob.

⟺ ℓ-adic rep'ns of $\pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$.

(applied to cohom of shtukas).

$\ell = p$: p-adic Weil coh for $X/\mathbb{F}_p$.

H^*_{rig} rigid cohom: it unifies

(1) Crystalline cohom for sm proper varieties.

$X/\mathbb{F}_p$ sm proper, if $\exists$ sm lift $\mathfrak{X}/\mathbb{Z}_p$ (proper formal sch)

then $H^*_{cris}(X/W)[\frac{1}{p}] = \mathbb{H}^*(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}/\mathbb{Z}_p})[\frac{1}{p}]$.

(2) Monsky-Washnitzer coh for affine vars:

e.g. $X = \mathbb{A}^1_{\mathbb{F}_p}$, $A^\dagger = \bigcup_{r>1} \left(K\langle \frac{t}{r} \rangle = \sum_{i\geq 0} a_i \left(\frac{t}{r}\right)^i \,\middle|\, |a_i| \to 0 \right)$.

ring of p-adic function on a closed disc of radius $r>1$.

$d: A^\dagger \to A^\dagger$ by $t \mapsto dt$,

$d(t^p) = p \cdot t^{p-1} dt$.

$H^*_{rig}(\mathbb{A}^1_{\mathbb{F}_p}) \simeq H^*(A^\dagger \xrightarrow{d} A^\dagger dt)$.

Assume $\exists\, \mathfrak{X}/\mathbb{Z}_p$ sm formal lift

and $F_{\mathfrak{X}}: \mathfrak{X} \to \mathfrak{X}$ a lift of Frob.

(1) A convergent F-crystal is (M, ∇, φ)

- $M \in Coh(\mathfrak{X}^{rig})$
- $\nabla: M \to M \otimes \Omega^1_{\mathfrak{X}^{rig}}$ integrable connection
- $\varphi: F^*_{\mathfrak{X}}(M, \nabla) \xrightarrow{\sim} (M, \nabla)$ (Frob structure).

(2) Suppose $\mathfrak{X}$ admits smooth compactification $\bar{\mathfrak{X}}$

overconvergent isocrystal (M, ∇)

Denote $F = Iso(X)$

- $M \in Coh(\mathcal{O}_V)$, V strict nbhd of $]X[_{\bar{\mathfrak{X}}}$ in $\bar{\mathfrak{X}}^{rig}$.
- $\nabla: M \to M \otimes \Omega^1_V$ satisfying overconvergent condition.

Denote $F = Iso(X)$.

$\rightsquigarrow Iso^\dagger(X)$ is functorial on X.

Frob str on $(M, \nabla) \in Iso^\dagger(X)$ is

$$\varphi : F_X^*(M,\nabla) \xrightarrow{\sim} (M,\nabla).$$

$\left(\begin{array}{l} F\text{-}\mathrm{Iso}^+(X) : \text{triples } (M,\nabla,\varphi) \\ \quad \text{cat of overconv } F\text{-isocrystals.} \end{array} \right)$

<u>Thm</u> (Kedlaya) $F\text{-}\mathrm{Iso}^+(X) \longrightarrow F\text{-}\mathrm{Iso}(X)$ is fully faithful.

e.g. $f : E \to S$ family of ordinary elliptic curves $/\mathbb{F}_p$

$R^1 f_{rig,*}(\mathcal{O}_E) \in F\text{-}\mathrm{Iso}^+(X)$ is irred. of rk 2.

But : $\exists$ slope filtration $\in F\text{-}\mathrm{Iso}(X)$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_0 \arrow[r] & R^1 f_{rig,*}\mathcal{O}_E \arrow[r] & \mathcal{E}_1 \arrow[r] & 0 \\ & \begin{array}{c}\text{slope}=0\\ rk=1\end{array} \arrow[u] & & \begin{array}{c}\text{slope}=1\\ rk=1.\end{array} \arrow[u] & \end{tikzcd}$$

<u>Drinfeld lemma for F-isocrystals</u>:

(*) : $\pi_1^{et}(X)$ is not enough.

<u>Thm</u> (Crew) $\mathrm{Rep}_{\mathbb{Q}_p}(\pi_1^{et}(X)) \simeq \left\{ \begin{array}{c} \text{Unit-root convergent} \\ F\text{-isocrystals} \end{array} \right\}$.

Unit-root : Frob-slope $= 0$.

<u>Tannakian grp</u>

$F\text{-}\mathrm{Iso}^+(X)$, $F\text{-}\mathrm{Iso}(X)$. Tannakian cats $/\mathbb{Q}$.

$\omega : F\text{-}\mathrm{Iso}^+(X) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$.

$\rightsquigarrow \pi_1^{F\text{-}\mathrm{Iso}}(X)$, $\pi_1^{F\text{-}\mathrm{Iso}^+}(X)$ Tannakian grps.

Tannakian $\pi_1(X/\Phi)$:

<u>Def</u>: $\mathcal{E} \in \mathrm{Iso}^+(X)$ (resp. $\mathrm{Iso}(X)$).

a partial Frob on $\mathcal{E}$ is (ϕ_1, ϕ_2). $\phi_i : F_i^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

s.t. $\phi_1 \circ F_1^*(\phi_2) = \phi_2 \circ F_2^*(\phi_1)$.

$\rightsquigarrow$ a Frob str on $\mathcal{E}$.

It turns out:

$\Phi\text{-Iso}^+(X)$, $\Phi\text{-Iso}(X)$

cats of overconv / conv isocrystals w/ partial Frobs

are Tannakian cats / $\mathbb{Q}$.

<u>Thm</u> (Kedlaya-Xu)

X_i smooth geom connected / $\mathbb{F}_p$.

(1) $\pi_1^{\Phi\text{-Iso}}(X) \xrightarrow{p_1^\circ \times p_2^\circ} \pi_1^{F\text{-Iso}^\circ}(X_1) \times \pi_1^{F\text{-Iso}^\circ}(X_2)$

(2) Apply $\pi_0(-)$

$\rightsquigarrow \pi_1(X/\Phi) \xrightarrow{\sim} \pi_1(X_1) \times \pi_2(X_2)$.

<u>Drinfeld motivic L-parameters</u>

F: char 0, alg closed.

$\text{Pro-red}(F)$: groupoid of pro-reductive grps / F.

map $G_1 \xrightarrow{\sim} G_2$

up to conjugate by $x \mapsto gxg^{-1}$, $g \in G_2^\circ$.

$\text{Pro-ss}(F)$: $F_1 \to F_2$ alg closed fields.

$\rightsquigarrow \text{Prored}(F_1) \xrightarrow{\sim} \text{Prored}(F_2)$.

<u>Thm</u> (Drinfeld) X sm var / $\mathbb{F}_p$.

$\exists\, \widehat{\pi}_X \in \text{Pross}(\bar{\mathbb{Q}})$ $(\pi_0(\widehat{\pi}_X) \simeq \pi_1^{et}(X))$.

$\xrightarrow{\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p} (\pi_1^{F\text{-Iso}^+}(X) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)^{ss}$ (isom is unique)

$\xrightarrow{\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell}$ (Tannakian grp def'd by lisse ℓ-adic sheaves)^{ss}.

$\rightsquigarrow$ motivic parameters $\hat{\pi}_X^{(\cdot)} \longrightarrow \check{G}$.

upgrade $(*)$ to an isom b/w $\hat{\pi}_X$ in $\mathrm{Pross}(\bar{\mathbb{Q}})$.