ℓ-adic cohom & ultraproduct

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<u>Begin</u> Weil cohom: k field, K field, char $K=0$.

$$\rightsquigarrow \quad (\mathrm{SmProj\,Sch}/k)^{op} \longrightarrow \mathrm{GrVec}(K).$$
$$X \longmapsto H^*(X)$$

<u>Properties</u> (1) Künneth formula: $H^*(X\times Y) \simeq H^*(X)\otimes H^*(Y)$.

(2) Poincaré duality: $H^i(X)\times H^{2d-i}(X)(d) \longrightarrow K$

($d=\dim X$) perfect pairing

(3) $\exists$ cycle class map.

<u>Classical Weil coh</u>

- char $k=0$: de Rham: $H^*_{dR}(X/k) = \mathbb{H}^*(X, \Omega^\bullet_{X/k})$, $K=k$

  ($k\subseteq\mathbb{C}$) Betti: $H^*_{Betti}(X) = H^*(X(\mathbb{C}),\mathbb{Q})$, $K=\mathbb{Q}$.

  $\rightsquigarrow H^*_B(X)\otimes_{\mathbb{Q}}\mathbb{C} \simeq H^*_{dR}(X/k)\otimes_k \mathbb{C}$.

- char $k=p>0$: ℓ-adic: $H^*_\ell(X) = H^*_{et}(X\otimes_k \bar{k}, \mathbb{Q}_\ell)$, $K=\mathbb{Q}_\ell$.

  p-adic/crys: $H^*_{cris}(X) = H^*(X/W(k))\otimes K$, $K=W(k)[\frac{1}{p}]$.

<u>Thm</u> (Deligne) $\dim H^i_\ell(X)$ indep of ℓ.

$\forall i$, $\exists$ $\mathbb{Q}$-v.s. V s.t. $H^i_\ell(X) \underset{\text{non-canonical}}{\simeq} V\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$.

When $K=\mathbb{Q}_\ell\times\mathbb{Q}_{\ell'}$: $H^i_{\ell\ell'}(X) = H^i_\ell(X)\times H^i_{\ell'}(X)$

$H^i_{\ell\ell'}(X)\otimes_K \mathbb{Q}_\ell \simeq H^i_\ell(X)$. ($\Rightarrow$ $\forall \ell,\ell'$, $H^i_{\ell\ell'}(X)$ free K-mod.

Take $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}^{(p,\infty)} = \prod'_{\substack{\ell \neq p \\ \ell \neq \infty}} \mathbb{Q}_\ell = \mathbb{Q} \otimes \prod_{\ell \neq p} \mathbb{Z}_\ell$.

<u>Thm</u> $H^i_{\mathbb{A}}(X)$ free $\mathbb{A}$-mod of finite rk,

$$H^i_\ell(X) = H^i_{\mathbb{A}}(X) \otimes_{\mathbb{A}} \mathbb{Q}_\ell.$$

<u>Rmk</u> Take $\mathbb{A} \twoheadrightarrow \mathbb{Q}_\ell$,

Thm $\Leftrightarrow$ $\begin{cases} \text{(Deligne)} \ \dim H^i_\ell(X) \text{ indep of } \ell \\ \text{(Gabber)} \ H^i(X_{\bar{k}}, \mathbb{Z}_\ell) \text{ torsion-free for } \ell \gg 0. \end{cases}$

$\Leftrightarrow \forall i, \exists \mathbb{Z}\text{-mod } M$ s.t. $H^i(X_{\bar{k}}, \mathbb{Z}_\ell) \underset{\substack{\uparrow \\ \text{non-can.}}}{\cong} M \otimes \mathbb{Z}_\ell$

<u>More generally</u>

Define the datum $M = (X, e, r)$, $e^2 = e$, $r \in \mathbb{Z}$,

$$\rightsquigarrow CH^d(X \times X), \ d = \dim_k X.$$

$\exists \otimes$-functor: $CHM(k)_{\mathbb{Q}} \longrightarrow GrVec_K$.

<u>Thm</u> $H^i_{\mathbb{A}}(M)$ free $\mathbb{A}$-mod of finite rk.

$\Rightarrow \dim H^i_\ell(M)$ indep of ℓ (André - Kahn).

A ring R is <u>semiprimary</u> if $\mathrm{rad}(R)$ is nilp

& $R/\mathrm{rad}(R)$ semi-simple.

<u>Fact</u> $\mathrm{Mor}_{H^*}(M, M)$ is semiprimary.

$$cl_{H^*}: Z^i(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X)(i).$$

Define $\sim$: $\underset{\substack{\cup \\ \alpha}}{} \alpha \sim \beta \Leftrightarrow cl_{H^*}(\alpha) = cl_{H^*}(\beta)$.

<u>Standard conj.</u> (Grothendieck) $\sim_{H^*} = \sim_{num}$.

$\left(\alpha \sim_{num} \beta \iff \forall \gamma,\ (\alpha,\gamma) = (\beta,\gamma).\right)$.

In particular, $\sim_{H^*_\ell}$ is indep of ℓ.

<u>Rmk</u> $k = \overline{\mathbb{F}}_p$. $\sim_{H^*_\ell}$ indep of $\ell \iff \forall X$ sep of fin type $/k$.

$\uparrow$ by van Dobben de Bruyn. $\forall i$, $\dim H^i_c(X, \mathbb{Q}_\ell)$ indep of ℓ.

<u>Thm</u> X proper. $IH^i(X, \mathbb{A})$ free $\mathbb{A}$-mod of finite rk.

(Gabber) $\dim IH^i(X, \mathbb{Q}_\ell)$ indep of ℓ.

(Cadoret - Zheng) $IH^i(X, \mathbb{Z}_\ell)$ torsion-free, for $\ell \gg 0$.

<u>Ultraproduct</u>

$$\begin{array}{ccl} \mathbb{A} & \longrightarrow & \mathbb{Q}_\ell, \quad \dim \mathbb{A} = \infty. \\ \downarrow & & \\ R = \mathbb{Q} \otimes \prod_{\ell \neq p} \mathbb{F}_\ell, & & \dim R = 0. \end{array}$$

$\rightsquigarrow \prod_{p \neq \ell} \mathbb{F}_\ell \longrightarrow k$, $\mathcal{L} = \{\ell : \ell \neq p\}$.

$\rightsquigarrow \{0,1\}^{\mathcal{L}} \xrightarrow{\psi} \{0,1\}$ Boolean alg hom.

$\quad\ \ \| $
$\quad P(\mathcal{L})$ power set

$\rightsquigarrow$ ultrafilter (looks like $\ker \psi$):

$u \subset P(\mathcal{L})$ s.t. (1) $\mathcal{L} \in u$,

(2) $A, B \in u \Rightarrow A \cap B \in u$

(3) exactly one of A and $\mathcal{L} \setminus A \in u$.

$\rightsquigarrow m_u = \left\{ (a_\ell) \in \prod_{\ell \in \mathcal{L}} \mathbb{F}_\ell \,\middle|\, \{\ell \mid a_\ell = 0\} \in u \right\}$.

Fact $\beta\mathcal{L} = \operatorname{Spec}\left(\prod_{\ell\in\mathcal{L}} \mathbb{F}_\ell\right) \xleftrightarrow{1-1} \{\text{ultrafilters on } \mathcal{L}\}$

$$m_u \longleftarrow\!\shortmid \; u$$

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{L} \arrow[rr] \arrow[dr] & & X \\ & \beta\mathcal{L} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \quad \text{compact Hausdorff.}$$

$\mathcal{U}$ = {non-principal ultrafilters}.

Define $\forall u \in \mathcal{U}$, $\mathbb{Q}_u = \prod_{\ell\in\mathcal{L}} \mathbb{F}_\ell / m_u := \prod_u \mathbb{F}_\ell$. $\quad \operatorname{char} \mathbb{Q}_u = 0$.

$$V_\ell \rightsquigarrow \prod_u V_\ell := \prod_{\ell\in\mathcal{L}} V_\ell / \sim_u .$$

$$(a_\ell) \underset{u}{\sim} (b_\ell) \iff \{\ell \mid a_\ell = b_\ell\} \in u .$$

Thm $\prod_u |V_\ell| = \phi \iff \{\ell \mid V_\ell \models \phi\} \in u$.

$\Rightarrow$ Can define $H^i(X, \mathbb{Q}_u) := \prod_u H^i(X, \mathbb{F}_\ell)$.