Comparing local Langlands correspondences

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F nonarch local field, res char p. G/F conn red gp.

<u>LLC (vague form)</u> Expect a natural map

$$\begin{array}{ccc} \Pi(G) & \longrightarrow & \Phi(G) \\ \pi & \longmapsto & \phi_\pi \end{array}$$

$\Pi(G)$ = sm rep'ns of $G(F)$ $\qquad$ $\Phi(G)$ = L-parameters $W_F \times SL_2 \to {}^L G$

with finite fibers, explicit image, and many other good properties.

<u>BIG PICTURE of approaches</u>

Global / autom methods:

Harris-Taylor, Henniart, Laumon-Rapoport-Stuhler, Arthur, Moeglin, Gan's collaborators, KMSW.

- GL_n's inner forms / any F
- Sp_{2n}, SO_n, GSp_4, U_n, G_2, ..., & char $F = 0$ (fair)

Local geometric F-S: $\pi \longmapsto \phi_\pi$ semisimple (any F, any G).

Genestier-Lafforgue: char $F = p$, any G. (goal)

Local explicit: Howe, Bushnell-Henniart, Yu, Kim, Reeder, Kaletha, Fintzen, ...

Cleanest when "$p \nmid |W|$" & G is tame. (terrible.)

How to reconcile all these?

<u>Thm</u> (Li-Huerta 23)

When char $F = p$, the GL constr & FS constr always agree.

<u>Idea</u> GL uniquely char'd by compatibility with V. Lafforgue's global parametrization over fin fields.

Essential step: to prove the FS constr satisfies the same compatibility.

Cartoon: $\underset{(FS)}{\mathrm{Sht}^{loc}} \xrightarrow{\text{uniformization}} \underset{(V.\ Lafforgue)}{\mathrm{Sht}^{glob}}$

$\rightsquigarrow H^*(\mathrm{Sht}^{glob}) \longrightarrow H^*(\mathrm{Sht}^{loc}) \circlearrowleft W_F$

$\rightsquigarrow$ local and global excursion actions are compatible.

When char $F = 0$, global shtukas don't exist (yet local sht does)

Shimura vars, much less flexible than Shtuka spaces

$\rightsquigarrow$ only partial results comparing FS with global autom pictures.

Inner forms for GL_n?

<u>Thm</u> (1) [FS 21] For GL_n / any F, the FS parameter φ_π is the semisimplification of the "true parameter".

(2) [HKW 21] Same result for inner forms when char $F = 0$.

(3) [Li-Huerta] Same result for inner forms when char $F = p$.

<u>Ideas</u> (1) The "true" LLC appears in the cohom of the LT / Drinfeld tower $\subseteq$ a Hecke corr on Bun_{GL_n}.

(2) Progress on Kottwitz Conj

$\Rightarrow$ Can realize the local JL corr
using Hecke ops on Bun_{GL_n}
and "Hecke operators commute w/ excursion operators".
(3) globalization + global JL Corresp + Chebotarev.

<u>Rmk</u> (2): used by Koshikawa in his (re)proof of CS vanishing
for cohom of global shimura varieties.

Other grps? (char $F = 0$)

<u>Thm</u> The FS parameter is the ss'n of the "known" global/autom parameter
in the following cases:

(1) <u>Hamann 21</u>: GSp_4 or $GU_2(D)$, $F/\mathbb{Q}_p$ unram & $p>2$.
(Known: Gan-(Takeda, Tantono, Chen).)

(2) <u>Bertolini-Meli-Hamann-Nguyen 22</u>: unram $U_{2n+1}/\mathbb{Q}_p$, $p>2$.
(Known: KMSW, Mok.)

(3) <u>Hansen 23</u> SO_{2n+1} (unique inner form, $F/\mathbb{Q}_p$ unram), $p>2$.
(Known: Arthur, Moeglin, Ishimoto.)

<u>Ingredients</u> p-adic uniformization of basic loci in ab type Shimura vars
compatibility of FS & known with parabolic induction (HKW).
KSZ stable trace formula $\Rightarrow$ globalization.
(Luck & miracles?)

Given G. Requires (at least) the existence of a Shimura datum (G, X)
of ab type s.t. $\cdot$ $G_{\mathbb{Q}_p} = \mathrm{Res}_{F/\mathbb{Q}_p} G$
$\cdot$ $B(G_{\mathbb{Q}_p}, \mu_X)$ is totally HN reducible.

· $\Phi(G_{op}) \xrightarrow{r_x} \Phi(GL_N)$ is (close to) injective.

<u>Problem</u> Drop the annoying conditions on F & p.

<u>Problem</u> Treat more general unitary gps / more general F.

<u>Problem</u>* Prove general results for $(G)Sp_{2n}$ or SO_{2n}.

<u>Local explicit</u> GL_n: many papers by Breuil-Henniart,

Complete results for GL_n w/ $p \nmid n$.

For G tame, $p \nmid |W|$, work by many people (in particular Kaletha)

regular supercuspidals $\subseteq$ non-singular supercuspidals

$\updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\updownarrow$

(S, θ) : $S \subseteq G$ tame elliptic max torus $\subseteq$ (S, θ, ρ), S same

$\theta : S(F) \longrightarrow \mathbb{C}^\times$ $\qquad\qquad$ θ "nonsingular", $\rho \in \mathrm{Irr}_\theta N_G(S)(F)$

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

regular supercuspidal $\qquad\qquad$ <u>all</u> supercuspidal parameters

<u>Problem</u> Compute this constr with

FS constr or the global autom constr (when available)

explicit v.s. global autom: Some very explicit results.

e.g. for π a ssc of $SO_{2n+1}(F)$, $\varphi_\pi^{\mathrm{expli}} = \varphi_\pi^{\mathrm{Art}}$