Multivariable (φ,Γ)-modules and local-global compatibility

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§1 Motivation

$K/\mathbb{Q}_p$ finite, $\mathbb{F}/\mathbb{F}_p$ fin (large).

Hope

$$\begin{tikzcd}
\{\bar\rho : G_K \to GL_2(\mathbb{F})\} \arrow[rr, leftrightarrow, "?"] \arrow[dr, equal, "\substack{\text{Fontaine}\\(K=\mathbb{Q}_p)}"'] & & \{\text{adm sm rep'ns of } GL_2(K)/\mathbb{F}\} \arrow[dl, "\text{Colmez functor}"] \\
& \{\text{étale } (\varphi,\Gamma)\text{-mods} / \mathbb{F}((T))\} &
\end{tikzcd}$$

Global candidate $F/\mathbb{Q}$ tot real, p inert,

$\bar r : G_F \longrightarrow GL_2(\mathbb{F})$ abs irred. automorphic,

$$\pi(\bar r) := \varinjlim_{U_p} \mathrm{Hom}_{G_F}(\bar r, H^1_{et}(X_{U_pU^p} \times_F \bar F, \mathbb{F})) \neq 0$$

$\circlearrowleft$ $GL_2(\mathbb{Q}_p)$ adm sm.

Q: Does $\pi(\bar r)$ essentially only depend on $\bar r|_{G_{F_p}}$?

§2 Multivariable (φ,Γ)-modules

$K/\mathbb{Q}_p$ unram, res field $k = \mathbb{F}_{p^f}$.

Fix $k \hookrightarrow \mathbb{F}$. Let $Y_i := \sum_{\lambda\in k^\times} \lambda^{-p^i} \underbrace{[\lambda]}_{\text{Teich lift}} \in \mathbb{F}[[\mathcal{O}_K]]$ $(0 \le i \le f-1)$.

So $\mathbb{F}[[\mathcal{O}_K]] = \mathbb{F}[[Y_0, \dots, Y_{f-1}]]$.

Take

$$\begin{aligned}
A &:= (\mathbb{F}[[\mathcal{O}_K]]_{Y_0\cdots Y_{f-1}})^\wedge \\
&\cong \mathbb{F}((Y_0))\left\langle \left(\frac{Y_i}{Y_0}\right)^{\pm 1} : i \neq 0 \right\rangle \\
&\cong \left\{ \sum_{n=0}^{\infty} \lambda_n Y^{\underline{i}(n)} : \underline{i}(n) \in \mathbb{Z}^f,\ \sum_{j=0}^{f-1} \underline{i}(n)_j \to \infty \text{ as } n \to \infty \right\}.
\end{aligned}$$

Rmk $\mathrm{Spa}\, A \subset \mathrm{Spa}\, \mathbb{F}[\![\mathcal{O}_K]\!]$ (open)
$\quad = \{|Y_0| = \cdots = |Y_{f-1}| \neq 0\}$.

By continuity, $\varphi, \mathcal{O}_K^\times \curvearrowright A$.

Prev result Constructed exact functor

$$D_A : \{\text{certain ab cat } \mathcal{C} \text{ of adm reps of } GL_2(K) / \mathbb{F}\} \to \{\text{étale } (\varphi, \mathcal{O}_K^\times)\text{-modules } / A\}$$

s.t. if $\bar{r}|_{G_{\mathbb{F}_p}}$ is tame & "strongly generic". $\left(K = \mathbb{Q}_p,\ c < a-b < p-c,\ \bar{r}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^a & \\ & \omega^b \end{pmatrix} \right)$

Then (1) $\pi(\bar{r}) \in \mathcal{C}$

(2) For $\mathrm{tr}: A \to \mathbb{F}(\!(T)\!)$ (induced $\mathbb{F}[\![\mathcal{O}_K]\!] \xrightarrow{\mathrm{tr}} \mathbb{F}[\![\mathbb{Z}_p]\!]$),

$$D_A(\pi(\bar{r})) \otimes_{A, \mathrm{tr}} \mathbb{F}(\!(T)\!) = (\varphi, \underset{= \mathbb{Z}_p^\times}{\Gamma})\text{-mod assoc to } \mathrm{Ind}^{\otimes G_{\mathbb{Q}_p}}_{G_{\mathbb{F}_p}} (\bar{r}|_{G_{\mathbb{F}_p}}).$$

§3 Main result

Construct a functor

$$D_A^{\otimes} : \{\bar{\rho} : G_K \to GL_2(\mathbb{F})\} \to \{\text{étale } (\varphi, \mathcal{O}_K^\times)\text{-mod over } A\}.$$

Thm (BHHMS) If $\bar{r}|_{G_{\mathbb{F}_p}}$ tame + str gen,

then $\quad D_A(\pi(\bar{r})) \cong D_A^{\otimes}(\bar{r}|_{G_{\mathbb{F}_p}})$.

Rmk (i) tame + str gen should not be necessary
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(ii) $D_A^{\otimes}(\bar{\rho}) \cong \bigotimes_{\sigma: k \hookrightarrow \mathbb{F}} D_{A,\sigma}(\bar{\rho})$, where $D_{A,\sigma}(-)$ exact & fully faithful.

(iii) If $\bar{\rho}$ tame, then $D_A^{\otimes}(\bar{\rho})$ is explicit.

§4 $D_A^{\otimes}$

$G_{LT} := \mathrm{Spf}\, \mathcal{O}_K[\![T_K]\!]$ Lubin-Tate formal $\mathcal{O}_K$-mod

$\rightsquigarrow$ LT étale $(\varphi, \mathcal{O}_K^\times)$-module $D_{LT}(\bar\rho) = \bigoplus_\sigma D_{LT,\sigma}(\bar\rho)$

over $\mathbb{F}\otimes k((T_K)) = \prod_{\sigma: k\hookrightarrow \mathbb{F}} \mathbb{F}((T_K))$.

$D_{LT,\sigma}(\bar\rho)$: étale $(\varphi^f, \mathcal{O}_K^\times)$-mod over $\mathbb{F}((T_K))$.

Universal cover: $\tilde{G}_{LT,\mathbb{F}} := \varprojlim_{[p]} \underset{\substack{\circlearrowleft \\ K}}{G_{LT,\mathbb{F}}} \cong \mathrm{Spa}\, \mathbb{F}[[T_K^{1/p^\infty}]]$.

<u>Fargues–Fontaine</u> $R\in \mathrm{Perf}_{\mathbb{F}} \rightsquigarrow$ Fréchet ring $B^+(R)$.

(completion of $W(R^\circ)[\frac{1}{p}]$).

as proétale sheaves of K-v.s. on $\mathrm{Perf}_{\mathbb{F}}$.

$$\begin{cases} \tilde{G}_{LT,\mathbb{F}}(-) \cong B^+(-)^{\varphi^f = p} \\ \tilde{G}_{\mathcal{O}_K,\mathbb{F}}(-) \cong B^+(-)^{\varphi^f = p^f} \end{cases}.$$

where $G_{\mathcal{O}_K,\mathbb{F}} := \hat{\mathbb{G}}_{m,\mathbb{F}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K$

$\Rightarrow \tilde{G}_{\mathcal{O}_K,\mathbb{F}} \cong \mathrm{Spa}\, \widehat{\mathbb{F}[[\mathcal{O}_K]]^{1/p^\infty}}$.

Let $Z_{LT} := (\tilde{G}_{LT,\mathbb{F}} \setminus \underset{\substack{\uparrow \\ \text{unique non-an pt.}}}{\{0\}})^f$, $Z_{\mathcal{O}_K} := \tilde{G}_{\mathcal{O}_K,\mathbb{F}} \setminus \{0\}$

Multiplication on $B^+(-)$ induces

$$\begin{tikzcd} & & Z_{LT} \arrow[rr, "m"] & & Z_{\mathcal{O}_K} & \\ 1 \arrow[r] & \Delta \arrow[r] & (K^\times)^f \rtimes S_f \arrow[u, phantom, "\circlearrowleft"] \arrow[rr, "\text{multi.}"] & & K^\times \arrow[u, phantom, "\circlearrowleft"] \arrow[r] & 1 \end{tikzcd}$$

<u>Fargues</u> $\Delta\backslash Z_{LT} \xrightarrow{\sim} Z_{\mathcal{O}_K}$ as pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}}$.

Let $Z^{gen}_{\mathcal{O}_K} = \{|Y_0| = \cdots = |Y_{f-1}| \neq 0\} \subseteq Z_{\mathcal{O}_K}$ (open)

$= \mathrm{Spa}\, A_\infty$, $A_\infty = \widehat{A^{1/p^\infty}}$.

<u>Prop</u> m is a pro-étale torsor $\underset{\substack{\| \\ Z^{gen}_{LT}.}}{m^{-1}(Z^{gen}_{\mathcal{O}_K})} \longrightarrow Z^{gen}_{\mathcal{O}_K}$

<u>Construction</u> $\bar{\rho} \longleftrightarrow$ étale $(\varphi, \mathcal{O}_K^\times)$-mod $\mathbb{F}((T_K^{1/p^\infty})) \otimes_{\mathbb{F}((T_K))} D_{LT,\sigma_0}$.

$\rightsquigarrow$ $K^\times$-equiv v.b. $\mathcal{V}_{\bar{\rho}}$ on $\tilde{G}_{LT,\mathbb{F}} \setminus \{0\}$.

$\rightsquigarrow$ $(K^\times)^f \rtimes S_f$-equiv v.b. $\mathcal{V}_{\bar{\rho}}^{\otimes f}$ on $(\tilde{G}_{LT,\mathbb{F}} \setminus \{0\})^f = Z_{LT}$.

Descent after $\cdot|_{Z_{LT}^{gen}}$ $\rightsquigarrow$ $K^\times$-equiv v.b. $(m_* \mathcal{V}_{\bar{\rho}}^{\otimes f})^\Delta$ on $Z_{\mathcal{O}_K}^{gen} = \mathrm{Spa}\, A_\infty$.

$\longleftrightarrow$ ét $(\varphi, \mathcal{O}_K^\times)$-mod over A_∞ (and then A) $\rightsquigarrow D_A^\otimes(\bar{\rho})$.

<u>Thm</u> $\bar{r}|_{G_{\mathbb{F}_p}}$ tame + str gen $\Rightarrow D_A(\pi(\bar{r})) \cong D_A^\otimes(\bar{r}|_{G_{\mathbb{F}_p}})$.

Compute RHS: $Z_{LT}^{gen} \xrightarrow{\sigma\text{-torsor}} Z_{\mathcal{O}_K}^{gen}$