# On F-S LLC for some supercuspidal representations of $Sp_6$

Yoichi Mieda

(Joint with Masao Oi)

## §1 Introduction

$G$ conn red gp / $\mathbb{Q}_p$, $q$-split for simplicity.

$\hat{G}$ dual gp / $\mathbb{C}$ of $G$.

$\mathrm{Irr}(G) := \{\text{irr sm rep of } G(\mathbb{Q}_p)\}/\sim$.

$\Phi(\hat{G}) := \{W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}\}/\hat{G}\text{-conj}$

$\cup\!\!\mathsf{I}$

$\Phi_{ss}(\hat{G}) := \{\phi \in \Phi(\hat{G}) : \phi|_{SL_2(\mathbb{C})} = 1\}$.

<u>Conjectural LLC</u> $\exists$ surj $\mathrm{Irr}(G) \twoheadrightarrow \Phi(\hat{G})$ with finite fiber
$\pi \longmapsto \phi_\pi$ (with many properties.).

<u>Known results</u>
- $GL_n$ : Harris-Taylor
- $GSp_4$, $Sp_4$ : Gan-Takeda
- $Sp_{2n}$, $SO_n$ : Arthur
- $GSp_{2n}$, $GSO_n$ : Xu
- $G_2$ : Gan-Savin

By automorphic methods.

Fargues-Scholze constructed

$$\mathrm{Irr}(G) \longrightarrow \Phi_{ss}(\hat{G}) \text{ for any } G$$
$$\pi \longmapsto \phi_\pi^{FS}$$

via $p$-adic geometry.

- only semisimple param
- no control of the fibers
- no known compatibility with Langlands functoriality.

<u>Problem</u> $\phi_\pi^{ss} \overset{?}{=} \phi_\pi^{FS}$ when LLC for $G$ is known

Here $\phi_\pi^{ss} : W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \xrightarrow{\phi_\pi} \hat{G}$.

$$w \longmapsto \left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right)$$

(okay for $G = GL_n$, $\underbrace{GSp_4, Sp_4}_{\text{need } p \neq 2}$. by F-S & Hamann.)

<u>Main thm</u> Assume $p \neq 2, 3$.

$\pi$: <u>Simple supercuspidal</u> rep of $Sp_6(\mathbb{Q}_p)$ with trivial unit char.

Then $\phi_\pi^{SS} = \phi_\pi^{FS}$ & the similar holds for $GSp_6(\mathbb{Q}_p)$.

$\quad \overset{\shortparallel}{\phi_\pi}$

SSC = irred sc rep'n with minimal positive depth $(=\frac{1}{6})$

$= \pi$ s.t. $\phi_\pi : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \longrightarrow SO_7(\mathbb{C})$

satisfying $\phi_\pi|_{I_{\mathbb{Q}_p}^{1/6}} \neq 1$, $\phi_\pi|_{I_{\mathbb{Q}_p}^{1/6,+}} = 1$.

Application to the cohom of RZ space:

$Sh_\infty$ := Siegel mod var of $\infty$-level of deg 3

$\overline{Sh}_\infty$ : mod $p$ reduction

$\overline{Sh}_\infty^{ss} \subset \overline{Sh}_\infty$ : supersingular locus.

$\pi$: autom rep of $GSp_6(\mathbb{A}_\mathbb{Q})$ with triv central char s.t. $\pi_p$ is SSC.

$\Rightarrow H_c^i(Sh_\infty, \bar{\mathbb{Q}}, \mathcal{L}_\xi)[\pi^\infty] \cong H_c^i(\overline{Sh}_\infty, \bar{\mathbb{F}}_p, R\psi\mathcal{L}_\xi)[\pi^\infty]$.

described by GLC $\qquad$ Lan-Stroh

suitable $\ell$-adic coeff ($\ell \neq p$).

$$\cong \underbrace{H_c^i(\overline{Sh}_\infty^{ss}, \bar{\mathbb{F}}_p, R\psi\mathcal{L}_\xi)}[\pi^\infty].$$

Cor of the main thm. $\qquad$ related to RZ sp for $GSp_6$.

$\rightsquigarrow$ useful to determine

$$H^i_c(\text{RZ for } GSp_6)[\text{ssc}] \quad (\text{in progress}).$$

$\rightsquigarrow$ similar result for inner form of $Sp_6$ or $GSp_6$.

<u>Rem</u> For this application, we only need to know $\phi^{FS}_\pi$ is descrete

$\rightsquigarrow$ can drop the assumption on the cent char of $\pi$.

## <u>§2 Basic strategy</u>

Fix $\sigma$: ssc of $Sp_6(\mathbb{Q}_p)$ with triv. cent char.

$\pi$: ssc of $GSp_6(\mathbb{Q}_p)$ with triv. cent char s.t. $\sigma \subset \pi|_{Sp_6(\mathbb{Q}_p)}$.

$\rightsquigarrow \exists\, \Pi$ autom rep'n of $GSp_6(\mathbb{A}_\mathbb{Q})$ with triv cent char

s.t. $\cdot\ \Pi_p \cong \pi$,

$\cdot\ \exists\, v_0$ fin place s.t. $\Pi_{v_0} \cong St$

$\cdot\ \Pi_\infty$ discrete series

<u>Thm</u> (Kret-Shin)

(1) $\exists\, \rho_\pi : \Gamma_\mathbb{Q} \longrightarrow Spin_7(\overline{\mathbb{Q}}_\ell)$ "corr to $\pi$" s.t. $\mathrm{pr}(\rho_\pi)_p = \phi_\sigma$.

(2) $H^6_c(Sh_{\infty,\overline{\mathbb{Q}}}, \mathcal{L}_\xi)(3)[\pi^\infty] \cong (\mathrm{std}\circ \mathrm{spin}\circ \rho_\pi)^\vee$

Here

$$\begin{array}{ccccc} Spin_7 & \xrightarrow{\mathrm{Spin}} & SO_8 & \xrightarrow{\mathrm{Std}} & GL_8 \\ {\scriptstyle 2:1}\big\downarrow {\scriptstyle \mathrm{pr}} & & & & \\ SO_7 & & & & \end{array}$$

Define $\phi_\pi := (\rho_\pi)_p$

By $\mathrm{pr}\circ\phi_\pi = \phi_\sigma$ & $\mathrm{pr}\circ\phi^{FS}_\pi \underset{\uparrow}{=} \phi^{FS}_\sigma$, it suffices to show $\phi_\pi = \phi^{FS}_\pi$.

(by F-S)

<u>Thm</u> (Koshikawa)

$\tau$: irr sm rep of $W_{\mathbb{Q}_p}$,

$\tau$ is a subquotient of $\underbrace{H^i_c(Sh_{\infty,\overline{\mathbb{Q}}}, \mathcal{L}_\xi)(3)[\pi^\infty]}_{\text{rep'n of } \Gamma_{\mathbb{Q}}}$.

$\Rightarrow$ $\tau^\vee$ is an irred comp of $(\mathrm{std} \circ \mathrm{Spin} \circ \phi_\pi^{FS})^\vee$

<u>Cor</u> $\{$irred comp of $\mathrm{std} \circ \mathrm{spin} \circ \phi_\pi\} \subseteq \{$irred comp of $\mathrm{std} \circ \mathrm{spin} \circ \phi_\pi^{FS}\}$.

So we want to show that

(1) $\mathrm{Std} \circ \mathrm{spin} \circ \phi_\pi$ is multi-free

$\Rightarrow$ $\mathrm{std} \circ \mathrm{spin} \circ \phi_\pi = \mathrm{Std} \circ \mathrm{spin} \circ \phi_\pi^{FS}$. $\quad (*)$.

(2) When $(*)$ implies $\phi_\pi = \phi_\pi^{FS}$ ?

<u>§3 L-parameter of $G_2$-type</u>

<u>Dual side</u>

$$\begin{tikzcd}
G_2 = SO_7 \cap \mathrm{Spin}_7 \arrow[r, hook, "j"] \arrow[d, "i"] & SO_7 \arrow[d] \\
\mathrm{Spin}_7 \arrow[r, "\mathrm{Spin}"] \arrow[d, "pr"] & SO_8 \\
SO_7. &
\end{tikzcd}$$

$\Phi_{sc}(G) := \{\phi \in \Phi_{ss}(G) : \phi \text{ is discrete}\}$.

$i_* : \Phi(G_2) \hookrightarrow \Phi(\mathrm{Spin}_7)$

$\Phi_{sc, G_2}(\mathrm{Spin}_7) \subseteq \Phi_{G_2}(\mathrm{Spin}_7) := \mathrm{Im}(i_*)$

<u>Prop A</u> $\phi \in \Phi_{G_2}(\mathrm{Spin}_7)$

(1) $\mathrm{Std} \circ \mathrm{Spin} \circ \phi = (\mathrm{Std} \circ \mathrm{pr} \circ \phi) \oplus \mathbb{1}$.

(2) If $\phi$ is sc, $\mathrm{std} \circ \mathrm{spin} \circ \phi$ is multi-free

(3) Assume $\phi$ sc, $\phi' \in \Phi(\mathrm{Spin}_7)$ satisfies

$$\mathrm{Std} \circ \mathrm{Spin} \circ \phi = \mathrm{std} \circ \mathrm{Spin} \circ \phi'.$$

Then $\phi' \in \Phi_{sc,G_2}(\mathrm{Spin}_7)$ and $\mathrm{pr}\circ\phi = \mathrm{pr}\circ\phi'$.

<u>Prop B</u> $\eta : W_{\mathbb{Q}_p} \longrightarrow \{\pm 1\}$ unram quad char.

$\pi$ = SSC of $GSp_6(\mathbb{Q}_p)$ w/ triv cent char

$\Rightarrow$ exactly one of $\phi_\pi$ or $\phi_\pi\eta$ belongs to $\Phi_{sc,G_2}(\mathrm{Spin}_7)$.