Endoscopic Classification for Unitary Groups

Sug Woo Shin (UC Berkeley)

Morningside Center, Beijing, July 10–14, 2023

based on work in progress:

KMS (Kaletha–Minguez–S.),
AGIKMS (Atobe–Gan–Ichino–Kaletha–Minguez–S.)

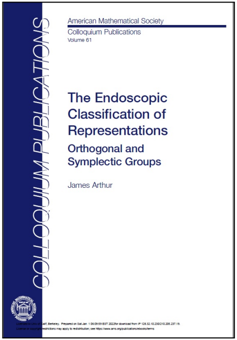
Plan

Part I. Overview

- What's been done
- Where are we now?

Part II. Local Intertwining Relation (LIR)

- What is LIR?
- LIR and Aubert–Zelevinsky duality
- Problem and Approach



I don't assume familiarity with Arthur's book.

Long-Term Goal

Endoscopic classification for connected reductive groups, including

- refined LLC (Local Langlands Correspondence),
- construction of A-packets,
- endoscopic Langlands functoriality,
- Arthur multiplicity formula,
- make things as explicit as you can.

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Too ambitious! ☹ ☹ ☹

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Endoscopic classification for classical groups, e.g., SO, Sp, U, including

- refined LLC (Local Langlands Correspondence),
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- Arthur multiplicity formula.

What has been done

qs = quasi-split, nqs = non-quasi-split

1. qs Sp, SO: Arthur's book 2013 (built on ...)
2. qs U: Mok 2015
3. qs GSp, GSO: Bin Xu 2018/21
4. nqs U: Kaletha–Minguez–S.–White 2014–now
5. nqs $\mathrm{SO}_{\mathrm{odd}}$: Ishimoto 2023 (tempered)

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Why care?

- fundamental results of intrinsic interest
- arithmetic applications: cohomology of Shimura varieties, Langlands reciprocity, Beilinson–Bloch–Kato conjecture, Euler systems, Gross–Zagier type formulas, ... (e.g., Liu–Tian–Xiao–Zhang–Zhu, Disegni–Liu, ...)

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Hence let's look into (1) and (2).

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- stabilization of the trace formula: conditional on (ii)
- local trace formula
- certain results on endsocopic transfer [A24]

(ii) Weighted Fundamental Lemma for Lie algebras (incl. "non-standard"): extend Chaudouard–Laumon beyond the split case

(iii) Twisted Fundamental Lemma for full unramified Hecke algebras

(iv) "Duality, Endoscopy, and Hecke operators" [A25]

(v) "A nontempered intertwining relation for $\mathrm{GL}(N)$" [A26]

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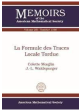

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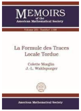

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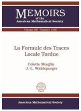

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(iv)–(vi) for **qs/nqs unitary groups**: in progress by AGIKMS/KMS, resp.

([A26], [A27] seem okay for U as well as Sp, SO.)

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- 2023: embarked on [A25] for **qs U**. cf. Arthur: [A25] for qs Sp, SO.
- It made little sense to clarify only [A25]. This led to AGIKMS:
  the goal is to make main theorems unconditional for **qs U** (mod WFL).

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“Intertwined Relations” by jaap schokkenkamp, available for purchase at $235

Statement of LIR

- $G = U(N)$ qs unitary group / F char 0 local field
- $M \subsetneq G$ **proper** Levi (induction hypothesis applies to M)
- $\phi_M \in \Phi_2(M) \mapsto \phi \in \Phi(G)$ L-parameters
- $x \in \mathscr{S}_\phi = \pi_0(S_\phi / Z(\widehat{G})^\Gamma)$

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Remark

Also stated and proved for

- $\psi_M \in \Psi_2(M) \mapsto \psi \in \Psi(G)$ A-parameters
- ϕ_M and ψ_M not square-integrable on M.

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$$f(\phi, x) := \sum_{\pi_M \in \Pi_{\phi_M}} \mathrm{tr}(R(u_x) \circ \mathrm{Ind}(\pi_M)),$$

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RHS (endoscopic): stable character on endoscopic group (for G)

$$f'(\phi, x) := f'(\phi'),$$

where $(\phi, x) \rightsquigarrow (G', \phi')$, and $f \rightsquigarrow f'$ is a transfer from G to G'.

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The same works for A-parameters and A-packets.

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Key tool for 1st arrow: Aubert–Zelevinsky (AZ) involution

Propagate LIR to some non-temp A-parameters via AZ involution.

LIR in the "seed case"

Here $\widehat{\phi}$, $\widehat{\pi}$ denote AZ dual.

"Seed" A-parameters for unitary groups, d'après Arthur–Mok

Consider only ψ such that $\boxed{\psi = \widehat{\phi}}$ for tempered ϕ of the form

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AZ duality is understood on endoscopic side of LIR (e.g., Hiraga).
So morally we'd obtain LIR for enough members of "seed" *A*-packets.
Then feed this into global machine to prove full LIR.

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Resolution? (AGIKMS, in progress)

[local thm for temp param.] $\xrightarrow[\text{aim less}]{\text{local}}$ [LIR for some A param. **"reduced seed case"**] $\xrightarrow[\text{do more}]{\text{global}}$ [LIR for all A param.]

E.g., we ensure small sc support ($m \leq 4$) in the "reduced seed case". □