Unipotent categorical local Langlands Correspondence

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Satellite Conference in Number Theory of ICBS
July 12, 2023

Classical local Langlands

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For example, Kazhdan-Lusztig constructed (for G split) an injective map

$$\Big\{\text{Smooth irr. reps. of } G(F) \text{ with Iwahori fixed vectors}\Big\} \to \Big\{(\varphi\colon \mathbb{G}_a \rtimes W_F \to \hat{G}, \rho \in \mathrm{Rep}(Z_{\hat{G}}(\varphi))) \text{ up to } \hat{G} \text{ conjugation}\Big\}.$$

Categorical (arithmetic) local Langlands

Both sides of the classical correspondence have some geometric structure. E.g. Bernstein center, local deformation ring, etc.

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On the other hand, the appearing of $\rho \in \mathsf{Rep}(Z_{\hat{G}}(\varphi))$ in the work of Kazhdan-Lusztig suggests that there are stacks involved in the story. Namely, such ρ could be interpreted as a coherent sheaf on the stack

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Geometric Langlands suggests that the local Langlands correspondence can and probably needs to be lifted to an equivalence of categories.

The stack of local Langlands parameters

Theorem (Dat-Helm-Kurinczuk-Moss, Fargues-Scholze, Z.)

For every $\ell \neq p$, $\exists$ *an algebraic stack* $\mathrm{Loc}_{{}^cG} = \mathrm{Loc}^{\square}_{{}^cG}/\hat{G}$ *over* $\mathbb{Z}_\ell$, *where*

$$\mathrm{Loc}^{\square}_{{}^cG}(R) = \{\mathit{Continuous}\ \varphi : W_F \to {}^cG(R) + \cdots\},$$

is represented by disjoint of reduced finite type affine schemes, which are local complete intersection, flat of relative dim $\hat{G}$ *over* $\mathbb{Z}_\ell$.

Here "continuity" means that when ${}^cG \subset \mathrm{GL}_n$, $\varphi(I_F)v \subset R^n$ is a finite $\mathbb{Z}_\ell$-module for every $v \in R^n$, and the action of I_F on $\varphi(I_F)v$ is continuous for the usual ℓ-adic topology on finite $\mathbb{Z}_\ell$-modules.

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In categorical (arithmetic) local Langlands, the set of Langlands parameters is replaced by the (derived) category

$$\mathrm{Coh}(\mathrm{Loc}_{^cG})$$

of coherent sheaves on $\mathrm{Loc}_{^cG}$.

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Arithmetic geometry suggests to study the representation theory $G(F)$ together with its forms $\{J_b(F)\}$ arising from the Kottwitz set $B(G)$. In addition the categories $\mathrm{Rep}(J_b(F))$ can be glued together via the category of sheaves on certain geometric objects.

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There are two approaches to make the idea precise.

- $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ (Fargues-Scholze);
- $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$ (Xiao-Z., Hemo-Z., Gaitsgory).

Notations

- $\mathcal{O}_F \subset F$ ring of integers, $\varpi \in \mathcal{O}_F$ a uniformizer.
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- For a perfect k_F-algebra R, let

$$W_{\mathcal{O},n}(R) = W(R) \otimes_{\mathbb{Z}_p} (\mathcal{O}_F/\varpi^n), \quad W_{\mathcal{O}}(R) = \varprojlim_n W_{\mathcal{O},n}(R).$$

  If $R = K$ is a perfect field, then $L := W_{\mathcal{O}}(K)[1/\varpi]$ is a discrete valued non-archimedean field, with $\mathcal{O}_L := W_{\mathcal{O}}(K)$ is ring of integers, e.g. $W_{\mathcal{O}}(k_F) = \mathcal{O}$ and $W_{\mathcal{O}}(k_F)[1/\varpi] = F$.

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- Denote

$$D_R = \mathrm{Spec} W_{\mathcal{O}}(R), \quad D_R^* = \mathrm{Spec}(W_{\mathcal{O}}(R)[1/\varpi]),$$

  thought as family of (punctured) disks parameterized by Spec$R$.
- The q-Frobenius of R induces an automorphism σ_R (or denoted by σ for simplicity) of D_R (and D_R^*).

Isocrystals and the Kottwitz set

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When K is an algebraically closed field, isomorphism classes of G-isocrystals over K are bijective to elements in $G(L)$ up to σ-conjugacy

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For $b \in G(L)$, let

$$J_b(F) = \{h \in G(L) \mid h^{-1} b \sigma(h) = b\}.$$

E.g. When $b = 1$, $J_b = G$. In general J_b is a form of a Levi of G.

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$$\mathrm{Isoc}_G(R) = \left\{ (\mathcal{E}, \varphi) \middle| \ \mathcal{E} \text{ is a } G\text{-torsor on } D_R^*, \ \varphi : \mathcal{E} \simeq \sigma_R^* \mathcal{E} \right\}.$$

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But it is technically more convenient to take quotient in étale topology. Fortunately, the category of sheaves on Isoc_G we are going to define does not depend on the choice of topology.

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- $\mathrm{Isoc}_{G,b} \cong \mathbb{B}_{\mathrm{profet}} J_b(F)$;
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When $b \in B(G)$ is basic, $\mathrm{Isoc}_{G,b} = \mathrm{Isoc}_{G,\leq b}$ is closed in Isoc_G.

(Ind-)construcble sheaves

Let $\Lambda = \mathbb{Z}_\ell, \mathbb{F}_\ell, \mathbb{Q}_\ell$ or their finite algebraic extensions. One can make sense of the category of Λ-sheaves on objects like Isoc_G. Fix a field k of finite Λ-coh. dim (e.g. $k = \bar{k}$).

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- If S is a finite type k-scheme, let $\mathrm{Shv}_c(S, \Lambda) = \mathrm{D}_{\mathrm{ctf}}(S, \Lambda)$, $\mathrm{Shv}(S, \Lambda) = \mathrm{Ind}\big(\mathrm{Shv}_c(S, \Lambda)\big)$;

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- For every prestack $\mathcal{X}\colon \mathrm{Perf}_k \to \mathrm{Ani}$ (i.e. a(n accessible) functor), let $\mathrm{Shv}(\mathcal{X}, \Lambda) = \lim_{S \to \mathcal{X}} \mathrm{Shv}(S, \Lambda)$;

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- A natural transformation $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^! : \mathrm{Shv}(\mathcal{Y}, \Lambda) \to \mathrm{Shv}(\mathcal{X}, \Lambda)$;
- For a large class of morphisms (including representable fp morphisms) $f : \mathcal{X} \to \mathcal{Y}$, can define $f_* : \mathrm{Shv}(\mathcal{X}, \Lambda) \to \mathrm{Shv}(\mathcal{Y}, \Lambda)$ satisfying base change.

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- If $S = \mathrm{Spec} k(X)$ for a curve X over k, then $\mathrm{Shv}_c(S,\Lambda)$ is the (opposite) category of cont. reps. of $\Gamma_{k(X)}$ (on perfect Λ-modules) that are **unramified** almost everywhere, while $D_{\mathrm{ctf}}(S,\Lambda)$ is the category of all continuous representations of $\Gamma_{k(X)}$.

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Here are examples and features of Shv comparing with the usual theory of étale sheaves.

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- Pro-étale descent **fails** for Shv (e.g. $\overline{k(X)}/k(X)$) but h-descent holds;
- One can define "motivic sheaves" via this procedure.

The category of sheaves on the Kottwitz stack

We base change everything to $k = \overline{k_F}$. Now, we have categories $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$, $\mathrm{Shv}(\mathrm{Isoc}_{G,b}, \Lambda)$, etc, and functors $i_{b,*}, i_b^!$ and $j_{b,*}, j_b^!$, etc.

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Theorem (Hemo-Z.)

(1) *For every $b \in B(G)$ there is a canonical equivalence*

$$\mathrm{Shv}(\mathrm{Isoc}_{G,b}, \Lambda) \cong \mathrm{Rep}(J_b(F), \Lambda).$$

(2) *There are adjoint functors*

$$\mathrm{Isoc}_{G,b} \underset{j_*}{\overset{j_!}{\overset{\xleftarrow{j^!}}{\rightrightarrows}}} \mathrm{Isoc}_{G,\le b} \underset{i^!}{\overset{i^*}{\overset{\xleftarrow{i_*}}{\rightrightarrows}}} \mathrm{Isoc}_{G,<b}.$$

inducing semi-orthogonal decomposition of $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$ *in terms of* $\{\mathrm{Shv}(\mathrm{Isoc}_{G,b}, \Lambda)\}_b$.

The category of sheaves on the Kottwitz stack

Theorem (Hemo-Z., cont'd)

(3) *The category* $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$ *is compactly generated, and compact objects are those whose restriction to each* $\mathrm{Isoc}_{G,b}$ *is compact and is zero for almost all* b*'s.*

(4) *There is a self-duality* $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$

$$\mathbb{D}^{\mathrm{coh}}\colon \mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)^\omega \simeq (\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)^\omega)^{\mathrm{op}}$$

such that for every $b \in B(G)$

$$\mathbb{D}^{\mathrm{coh}} i_{b,*} \simeq i_{b,!}\mathbb{D}^{\mathrm{coh}}_{\mathrm{Rep}(J_b(F),\Lambda)}[\langle 2\rho, \nu_b\rangle].$$

(5) *There is a natural perverse t-structure obtained by gluing (shifted) standard t-structures on various* $\mathrm{Shv}(\mathrm{Isoc}_{G,b}, \Lambda)$*, preserved by* $\mathbb{D}^{\mathrm{coh}}$ *if* Λ *is a field.*

Categorical Arithmetic Local Langlands Correspondence

Conjecture

Let $\Lambda = \overline{\mathbb{Q}}_\ell$. Assume G is quasi-split with a pinning (B, T, e) and fix $\psi : F \to \Lambda^\times$. There is a canonical equivalence of ∞-categories

$$\mathbb{L}_G : \mathrm{Coh}(\mathrm{Loc}_{^cG,\Lambda}) \simeq \mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)^\omega,$$

compatible with parabolic induction, intertwining duality, and (after ind-completion) sending $\mathcal{O}_{\mathrm{Loc}_{^cG,\Lambda}}$ to $i_{1,*}\mathcal{W}$ where $\mathcal{W} := c\text{-}\mathrm{ind}_{U(F)}^{G(F)}(\psi \circ e)$.

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- The spectral action of $\mathrm{Perf}(\mathrm{Loc}_{^cG})$ on $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$ is currently unknown. Fargues-Scholze have such action in their version.
- There is, however, some convincing evidence that the above version should also be true.

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consisting of those $\mathcal{F}$ such that for all $b \in B(G)$, the cohomologies of

$$i_b^*\mathcal{F} \in \mathrm{Rep}(J_b(F), \overline{\mathbb{Q}}_\ell)$$

are unipotent in the sense of Lusztig. I.e. they are quotient of (direct sums of) $c\text{-}\mathrm{ind}_{\mathcal{P}}^{J_b(F)}\pi$, for some parahoric subgroup $\mathcal{P} \subset J_b(F)$ and some unipotent cuspidal representation π of the Levi quotient $L_{\mathcal{P}}$ of $\mathcal{P}$ (which is a finite group of Lie type).

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We mention one can similarly define $\mathsf{Shv}^{\mathsf{tame}}(\mathrm{Isoc}_G, \overline{\mathbb{Q}}_\ell)$ consisting of $\mathcal{F}$ such that cohomologies of $i_b^* \mathcal{F}$ are depth zero representations of $J_b(F)$.

The Stack of Unipotent Langlands Parameters

On the other hand, consider the open and closed substacks

$$\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell} \subseteq \mathrm{Loc}^{\mathrm{tame}}_{{}^cG,\overline{\mathbb{Q}}_\ell} \subset \mathrm{Loc}_{{}^cG,\overline{\mathbb{Q}}_\ell}$$

classifying representations which factor through the tame quotient of W_F (resp. carry unipotent monodromy).

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Fix a topological generator τ of tame inertia and a lifting of the Frobenius σ, one has a presentation:

$$\mathrm{Loc}^{\mathrm{tame}}_{{}^cG} \simeq \left\{(g,h) \in \hat{G} \times \hat{G}\sigma \subset {}^cG \times {}^cG \mid hgh^{-1} = g^q\right\}/\hat{G}$$

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The stack $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell}$ is connected, with irreducible components parameterized by unipotent conjugacy classes of $\hat{G}$.

Main theorem

Let $I \subset G(F)$ be the Iwahori (determined by the pinning).

Theorem (Hemo-Z.)

There is a canonical equivalence of ∞-categories

$$\mathbb{L}_G : \mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell}) \simeq \mathrm{Shv}^{\mathrm{unip}}(\mathrm{Isoc}_G, \overline{\mathbb{Q}}_\ell)^\omega$$

sending $\mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG}}$ *to* $c\text{-}\mathrm{ind}_I^{G(F)}(\overline{\mathbb{Q}}_\ell) \otimes_{H_I} \mathcal{W}^I$.

For every $c\text{-}\mathrm{ind}_{\mathcal{P}}^{G(F)}\pi$, *with* $\mathcal{P}$ *a parahoric of* $G(F)$ *and* π *a cusipdal irreducible unipotent representation of* $L_{\mathcal{P}}$ *as before,*

$$\mathfrak{A}_\pi := \mathbb{L}_G^{-1}\big(i_{1,*}(c\text{-}\mathrm{ind}_{\mathcal{P}}^{G(F)}\pi)\big)$$

is a maximal Cohen-Macauly coherent sheaf (rather than a complex of sheaves) on $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG}$.

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classifying triples $(g, h, \hat{B}')$ consisting of a unipotent parameter (g, h) and a Borel $\hat{B}' \subset \hat{G}$ containing g.

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Via the projection

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG} \to \widetilde{\mathcal{U}}_{\hat{G}}/\hat{G} = \hat{U}/\hat{B} \to \mathbb{B}\hat{T}$$

every character of $\hat{T}$ gives a line bundle $\mathcal{O}(\lambda)$ on $\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG}$.

Spectral Deligne-Lusztig stacks II

There is also a map

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For $w \in W$ in the finite Weyl group of $\hat{G}$, there is a (derived) closed substack, which I call the (unipotent) spectral Deligne-Lusztig stack

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which, roughly speaking classifying those $(g, h, \hat{B}')$ such that $\hat{B}'$ and $h\hat{B}'h^{-1}$ has relative position $\leq w^{-1}$.

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The natural map
$$\pi_w^{\mathrm{unip}}\colon \widetilde{\mathrm{Loc}}_{{}^cG,w}^{\mathrm{unip}} \to \mathrm{Loc}_{{}^cG}^{\mathrm{unip}}$$
which is a proper and (derived) schematic morphism.

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Theorem (Hemo-Z.)

We have

$$\mathbb{L}_G\big(\pi^{\mathrm{unip}}_{w_{b,f},*}(\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,w_{b,f}}}(\lambda_b))\big) = i_{b,*}\big(c\text{-}\mathrm{ind}_{I_b}^{J_b(F)}\overline{\mathbb{Q}}_\ell\big),$$

where $I_b \subset J_b(F)$ is an Iwahori.

There is a generalization for every b using some work of X. He.

Example: Coherent Springer sheaf

If $w = 1$, then

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,1} \cong \mathrm{Loc}^{\mathrm{unip}}_{{}^cB} \simeq \left\{ (g,h) \in \hat{\mathcal{U}} \times \hat{B}\sigma \subset {}^cB \times {}^cB \mid hgh^{-1} = g^q \right\} / \hat{B}.$$

We write $\mathrm{CohSpr}_{{}^cG} = \pi^{\mathrm{unip}}_{1,*} \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cB}}$.

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$$\mathbb{L}_G(\mathrm{CohSpr}_{{}^cG}) \cong i_{1,*}(c\text{-}\mathrm{ind}_I^{G(F)}\overline{\mathbb{Q}}_\ell).$$

Corollary

$$H_I \simeq R\mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell}}(\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG}).$$

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We mention one can similarly prove

$$H_K \simeq R\mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell}}(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{{}^cG,\overline{\mathbb{Q}}_\ell}}).$$

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Locally around the intersection, the stack looks like

$$(\mathrm{Spec}\overline{\mathbb{Q}}_\ell[x,y]/(xy))/\mathbb{G}_m$$

where x has $\mathbb{G}_m$-weight 2 and y has weight 0.

Example: Steinberg and trivial representations

Now let $G = \mathrm{PGL}_2$. Then

$$\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,\overline{\mathbb{Q}}_\ell} = \mathrm{Loc}^{\mathrm{st}}_{{}^cG,\overline{\mathbb{Q}}_\ell} \cup \mathrm{Loc}^{\mathrm{ur}}_{{}^cG,\overline{\mathbb{Q}}_\ell}, \quad \mathrm{CohSpr}_{{}^cG} \cong \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG}} \oplus \mathcal{O}_{\mathrm{Loc}^{\mathrm{ur}}_{{}^cG}}.$$

Locally around the intersection, the stack looks like

$$(\mathrm{Spec}\overline{\mathbb{Q}}_\ell[x,y]/(xy))/\mathbb{G}_m$$

where x has $\mathbb{G}_m$-weight 2 and y has weight 0.

Recall the short exact sequence of smooth representations of $G(F)$

$$0 \to \mathrm{St} \to \mathrm{Measure}(\mathbb{P}^1(F)) \xrightarrow{\int} \mathrm{triv} \to 0.$$

It corresponds to

$$\mathcal{O}_{\mathrm{Loc}^{\mathrm{st}}_{{}^cG}} \to \mathcal{O}_{\mathrm{Loc}^{\mathrm{st}}_{{}^cG} \cap \mathrm{Loc}^{\mathrm{ur}}_{{}^cG}} \to \mathcal{O}_{\mathrm{Loc}^{\mathrm{st}}_{{}^cG}}[-1](-2).$$

Example: quaternion algebra

For $G = \mathrm{PGL}_2$ and $w = s$ is the unique simple reflection,

$$\pi_s^{\mathrm{unip}} : \widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,s} \to \mathrm{Loc}^{\mathrm{unip}}_{{}^cG}$$

- is birational over the Steinberg component of $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG}$, and
- is a generic $\mathbb{P}^1$-fibration over the unramified component.

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For $b = \begin{pmatrix} & 1 \\ p & \end{pmatrix}$, so $J_b = D^\times / F^\times$, the line bundle $\mathcal{O}(\lambda_b)$ restricted to $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$.

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For $b = \begin{pmatrix} & 1 \\ p & \end{pmatrix}$, so $J_b = D^\times/F^\times$, the line bundle $\mathcal{O}(\lambda_b)$ restricted to $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$. So

$$\mathbb{L}_G^{-1}(c\text{-}\mathrm{ind}_{I_b}^{J_b(F)}\overline{\mathbb{Q}}_\ell) = \pi_{s,*}^{\mathrm{unip}}\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,s}}(\lambda_b) \cong \mathcal{O}_{\mathrm{Loc}^{\mathrm{st}}_{{}^cG}}(-1)$$

is a self-dual Cohen-Macaulay sheaf fully supported on the Steinberg component. An integral version of the sheaf appears in Manning's work.

A sample global application

Let (G, X) be an adjoint Shimura datum of abelian type unramified at p, and $K \subset G(\mathbb{A}_f)$ with K_p Iwahori. Let G' be the inner form of G that is trivial outside $\{p, \infty\}$ and $G'_{\mathbb{R}}$ is compact. Choose $\theta : G_{\mathbb{A}_f^p} \cong G'_{\mathbb{A}_f^p}$ compatible with inner twist and let $K' = K'_p K'^p$ where K'_p is Iwahori and $K'^p = \theta(K^p)$.

Theorem

There is a natural global Jacquet-Langlands transfer map

$$\mathrm{Hom}_{\mathrm{Coh}(\mathrm{Loc}_p^{\mathrm{unip}})}(\mathfrak{A}_{J_b, I_b}, \widetilde{V} \otimes \mathrm{CohSpr}) \to \mathrm{Hom}_{H_{K^p}}(C(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K', \overline{\mathbb{Q}}_\ell), C_c(\mathrm{Sh}_\mu, \overline{\mathbb{Q}}_\ell)),$$

- $\mathrm{Loc}_p^{\mathrm{unip}}$ *the stack of unipotent parameters for* $G_{\mathbb{Q}_p}$;
- $\widetilde{V}$ *the vector bundle on it given by the Shimura cocharacter* $-\mu$;
- $\mathfrak{A}_{J_b, I_b} = \pi^{\mathrm{unip}}_{w_{b,f},*}(\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{cG, w_{b,f}}}(\lambda_b))$, *where* $b \in B(G, -\mu)$ *is basic.*

Thank You!