Amplifications in Analytic Number Theory

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Inequalities as desired

Let $\mathcal{A} = \{a_n\}$ be a complex sequence indexed by n in a suitable family $\mathcal{F}$.

Question

What about the information of a_n as n varies in $\mathcal{F}$?

- lower bound
- upper bound
- vanishing or non-vanishing
- equidistribution

Typical instances:

- L-functions (Lindelöf Hypothesis, BSD conjecture, etc)
- Exponential sums (Riemann/Lindelöf Hypothesis, Weil conjecture, etc)

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Framework

- Examples (algebraic/analytic exponential sums, L-functions)
- Amplification from inside: van der Corput
- Amplification from inside: Vinogradov, Burgess
- Amplification from outside: Kloosterman
- Other amplifications

Example I: algebraic exponential sums

- $q \in \mathbf{Z}^+$
- $\mathrm{e}(z) = \exp(2\pi i z)$
- $K(m, n; q)$: the Kloosterman sum defined by

$$K(m, n; q) = \sum_{a \in (\mathbf{Z}/q\mathbf{Z})^\times} \mathrm{e}\Big(\frac{ma + na^{-1}}{q}\Big)$$

Theorem (Kloosterman, 1927)

For all primes $p \nmid mn$, we have

$$|K(m, n; p)| \leqslant 2p^{3/4}.$$

- Kloosterman's circle method and solvability of $N = ax^2 + by^2 + cz^2 + dt^2$
- Is 3/4 is best possible?

Example II: analytic exponential sums

- $\mathrm{e}(z) = \exp(2\pi i z)$, $t \in \mathbf{R}$
- zeta sum ($f(x) = \frac{t}{2\pi} \log x$)

$$\sum_{n \leqslant N} n^{it} = \sum_{n \leqslant N} \mathrm{e}\Big(\frac{t}{2\pi} \log n\Big) = \sum_{n \leqslant N} \mathrm{e}(f(n))$$

Theorem (van der Corput, 1921; Vinogradov, 1930's-50's)

For $t \geqslant N \geqslant 2$ we have

$$\sum_{n \leqslant N} n^{it} \ll N \cdot (N^{-1/2} t^{1/6} + t^{-1/6}) \log N,$$

$$\sum_{n \leqslant N} n^{it} \ll N \cdot \exp(-c (\log N)^3 (\log t)^{-2}).$$

- $\zeta(\frac{1}{2} + it) \ll t^{1/6+\varepsilon}$ (Weyl. Hardy–Littlewood, Landau)
- upper bound for $\zeta(\sigma + it)$ for σ close to 1, zero-free region of ζ (Vinogradov)

Example III: L-functions

- $q \in \mathbf{Z}^+$
- χ a Dirichlet character of conductor q
- $L(s,\chi)$ the Dirichlet L-function defined by

$$L(s,\chi) = \sum_{n \geqslant 1} \chi(n) n^{-s}, \quad \Re s > 1$$

- class number of $\mathbf{Q}(\sqrt{-q})$ and $L(1,\chi)$ with χ quadratic
- non-vanishing of $L(1,\chi)$ and primes in arithmetic progressions
- lower bound for $L(1,\chi)$ and Landau–Siegel zero
- upper bound for $L(\frac{1}{2},\chi)$ and Lindelöf Hypothesis

Amplification of van der Corput

$$S := \sum_{M<n\leqslant M+N} a_n$$

- For all integers h,

$$S = \sum_{M<n+h\leqslant M+N} a_{n+h}.$$

- Summing over $h \leqslant H$ gives

$$S = \frac{1}{H} \sum_{M-H<n\leqslant M+N-1} \sum_{\substack{h\leqslant H \\ M-n<h\leqslant M+N-n}} a_{n+h}.$$

- By Cauchy, we find

$$|S|^2 \leqslant \frac{N+H-1}{H^2} \sum_n \Big| \sum_h a_{n+h} \Big|^2 \leqslant \frac{N+H-1}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{M<n,n+h\leqslant M+N} a_n \overline{a_{n+h}}.$$

Amplification of van der Corput

$$S(f, I) := \sum_{M<n\leqslant M+N} \mathrm{e}(f(n))$$

- (Weyl differencing)

$$|S(f, I)|^2 \ll \frac{N+H-1}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{M<n,n+h\leqslant M+N} \mathrm{e}(f(n) - f(n+h)).$$

- In principle, the function $x \mapsto f(x) - f(x+h)$ has a slower growth than the original function $f(x)$.
- If f is a polynomial of degree k, one then arrives at a sum along with geometric progressions after $k-1$ iterations of the above process.
- One may optimize the choice of H in practice.

van der Corput method

There are two innovations of van der Corput:

- Weyl differencing (A-process):
  The parameter H is flexible: new exponential sum with new amplitude function
- Poisson summation (B-process):
  Dual sum with (essentially) the same amplitude function, but with longer/shorter length

The original van der Corput method relies on the key feature:

- The amplitude function f is "smooth" enough.

van der Corput in algebraic situations

- $q \geqslant 2$ an integer or even a prime power
- V a suitable algebraic variety over $\mathbf{Z}$, $\mathbf{Z}/q\mathbf{Z}$ or $\mathbf{F}_q$
- f a rational function over V

Algebraic Exponential Sums

$$\sum_{\mathbf{x} \in V(\mathbf{Z}/q\mathbf{Z})} \mathrm{e}\Big(\frac{f(\mathbf{x})}{q}\Big), \qquad \sum_{\mathbf{x} \in V(\mathbf{F}_q)} \psi(f(\mathbf{x}))$$

In general, we may consider the average of $W_q : \mathbf{Z}/q\mathbf{Z} \to \mathbf{C}$:

$$\sum_n W_q(n).$$

Typical examples (complete sums)

- Gauss sum

$$\tau(a,\chi)=\sum_{x\in\mathbf{Z}/q\mathbf{Z}}\chi(x)\mathrm{e}\Big(\frac{ax}{q}\Big)$$

- Kloosterman sum

$$\mathrm{Kl}(a,q)=\frac{1}{\sqrt{q}}\sum_{x\in(\mathbf{Z}/q\mathbf{Z})^\times}\mathrm{e}\Big(\frac{ax+x^{-1}}{q}\Big)$$

- hyper-Kloosterman sum ($k\geqslant 2$)

$$\mathrm{Kl}_k(a,q)=q^{\frac{1-k}{2}}\sum_{\substack{x_1,x_2,\cdots,x_k\in(\mathbf{Z}/q\mathbf{Z})^\times\\ x_1x_2\cdots x_k=a}}\mathrm{e}\Big(\frac{x_1+x_2+\cdots+x_k}{q}\Big)$$

Typical examples (incomplete sums)

- character sum

$$\sum_{n\in I}\chi(n)$$

- incomplete Kloosterman sum

$$\sum_{\substack{n\in I\\(n,q)=1}} \mathrm{e}\Big(\frac{an^{-1}}{q}\Big)$$

- bilinear form of Kloosterman sums

$$\sum_m\sum_n \alpha_m\beta_n \,\mathrm{Kl}(mn,q)$$

- sums of products of Kloosterman sums

$$\sum_{n\in I}\prod_{1\leqslant j\leqslant r}\mathrm{Kl}(n+h_j,q)$$

van der Corput in algebraic situations

$$S := \sum_{n \in I} W(n)$$

Lemma (Heath-Brown / Irving)

- *(A-process) Assume* $q = q_1 q_2$ *with* $(q_1, q_2) = 1$ *and* $W_i : \mathbf{Z}/q_i\mathbf{Z} \to \mathbf{C}$*. Define* $W = W_1 W_2$*, then for any* $1 \leqslant L \leqslant |I|/q_2$ *we have*

$$|S|^2 \leqslant \|W\|_\infty^2 L^{-1} |I| \Big(|I| + \|W_1\|_\infty^{-2} \sum_{0 < |\ell| \leqslant L} \Big| \sum_{n, n+\ell q_2 \in I} W_1(n) \overline{W_1(n + \ell q_2)} \Big| \Big).$$

- *(B-process) For* $W : \mathbf{Z}/q\mathbf{Z} \to \mathbf{C}$*, we have*

$$|S| \ll \frac{|I|}{\sqrt{q}} \Big(|\widehat{W}(0)| + (\log q) \Big| \sum_{h \in \mathcal{I}} \widehat{W}(h) \mathrm{e}\Big(\frac{ha}{q}\Big) \Big| \Big)$$

for certain $a \in \mathbf{Z}$ *and some interval* $\mathcal{I}$ *not containing* 0 *with* $|\mathcal{I}| \leqslant q/|I|$*, where* $\widehat{W}$ *denotes the (normalized) Fourier transform of* W*.*

van der Corput in algebraic situations

- New functions appear:

$$n \mapsto W_1(n)\overline{W_1(n+h)},$$

and

$$n \mapsto \widehat{W}(n)$$

with

$$\widehat{W}(n) = \frac{1}{\sqrt{q}} \sum_{x \in \mathbf{Z}/q\mathbf{Z}} W(x) \mathrm{e}\Big(\frac{-nx}{q}\Big).$$

van der Corput method and arithmetic exponent pairs

- $q \geqslant 3$ is a squarefree number with $P^+(q) < q^\eta$ for any small $\eta > 0$
- $K = \prod_{p|q} K_p$ is a compositely amiable trace function (ℓ-adic cohomology)
- We expect the following bound

$$\sum_{n\in I} K(n) \ll N^\varepsilon (q/N)^\kappa N^\lambda \qquad (\boldsymbol{\Omega})$$

holds for some (κ, λ), where $|I| = N < q$.

Proposition (Initial choices)

If K is compositely 1-amiable, then $(\boldsymbol{\Omega})$ holds for

$$(\kappa, \lambda) = (0, 1), \quad (\tfrac{1}{2}, \tfrac{1}{2}).$$

van der Corput method and arithmetic exponent pairs

For $J, L \geqslant 1$, put

$$\mathfrak{A}_q(J,L) = \{K \,(\mathrm{mod}\, q) : K \text{ compositely } J\text{-amiable, } \widehat{K} \text{ compositely } L\text{-amiable}\}$$

Definition (Exponent pairs)

Let $J, L \geqslant 1$ and $N \leqslant q$.

We say (κ, λ) satisfying $0 \leqslant \kappa \leqslant \frac{1}{2} \leqslant \lambda \leqslant 1$ is an exponent pair of width $(J; L)$, if $(\boldsymbol{\Omega})$ holds for all $K \in \mathfrak{A}_q(J,L)$.

An exponent pair of width $(\infty; L)$ with some $L \geqslant 1$ is called an arithmetic exponent pair.

van der Corput method and arithmetic exponent pairs

Theorem (Jie Wu & Xi[1], 2021)

Let $J \geqslant 1$. If (κ, λ) is an exponent pair of width $(J;1)$, then

$$A \cdot (\kappa, \lambda) = \left(\frac{\kappa}{2(\kappa+1)}, \ \frac{\kappa+\lambda+1}{2(\kappa+1)} \right).$$

is an exponent pair of width $(J+1;1)$.

Theorem (Jie Wu & Xi, 2021)

If (κ, λ) is an exponent pair of width $(1;1)$, then so is

$$B \cdot (\kappa, \lambda) = \left(\lambda - \frac{1}{2}, \ \kappa + \frac{1}{2} \right).$$

[1] J. Wu & P. Xi, Arithmetic exponent pairs for algebraic trace functions and applications, with an appendix by Will Sawin, *Algebra Number Theory* **15** (2021), 2123--2172.

van der Corput method and arithmetic exponent pairs

$$\sum_{M<n\leqslant M+N} K(n) \ll N^{\varepsilon}(q/N)^{\kappa}N^{\lambda}.$$

Processes	A	A^2	A^3	BA^2
(κ,λ)	$(\frac{1}{6},\frac{2}{3})$	$(\frac{1}{14},\frac{11}{14})$	$(\frac{1}{30},\frac{26}{30})$	$(\frac{2}{7},\frac{4}{7})$
Processes	BA^3	ABA^2	A^2BA^2	$BABA^2$
(κ,λ)	$(\frac{11}{30},\frac{16}{30})$	$(\frac{2}{18},\frac{13}{18})$	$(\frac{2}{40},\frac{33}{40})$	$(\frac{4}{18},\frac{11}{18})$

- estimates for short algebraic exponential sums
- applications to distribution of primes, L-functions, etc

Amplification of I. M. Vinogradov

$$\sum_{a<n\leqslant b} \mathrm{e}(f(n))$$

- Small shift & averaging

$$\sum_{a-xy<n\leqslant b-xy} \mathrm{e}(f(n+xy)) \approx \frac{1}{XY} \sum_{a<n\leqslant b} \sum_{x\sim X} \sum_{y\sim Y} \mathrm{e}(F_n(xy))$$

- Hölder & grouping variables (Take $F(t) = \sum_{1\leqslant i\leqslant k} \alpha_i t^i$)

$$\sum_{x\sim X} \Big| \sum_{y\sim Y} \mathrm{e}(F(xy)) \Big|^r = \sum_{x\sim X} \theta(x) \sum_{y_1,\cdots,y_r} \mathrm{e}(F(xy_1) + \cdots F(xy_r))$$

$$= \sum_{\lambda_1,\cdots,\lambda_k} \nu(\lambda_1,\cdots,\lambda_k) \sum_{x\sim X} \theta(x)\mathrm{e}(\alpha_1\lambda_1 x + \cdots + \alpha_k\lambda_k x^k),$$

where $\nu(\lambda_1,\cdots,\lambda_k)$ counts the solutions to the system of equations

$$y_1^i + \cdots + y_r^i = \lambda_i, \quad 1 \leqslant i \leqslant k.$$

Amplification of I. M. Vinogradov

$$\sum_{\lambda_1,\cdots,\lambda_k} \nu(\lambda_1,\cdots,\lambda_k) \sum_{x\sim X} \theta(x)\mathrm{e}(\alpha_1\lambda_1 x + \cdots + \alpha_k\lambda_k x^k)$$

- By Hölder, we need to consider

$$\sum_{\lambda_1,\cdots,\lambda_k} \nu(\lambda_1,\cdots,\lambda_k)^2, \quad \sum_{\lambda_1,\cdots,\lambda_k} \Big| \sum_{x\sim X} \theta(x)\mathrm{e}(\alpha_1\lambda_1 x + \cdots + \alpha_k\lambda_k x^k)\Big|^{2s}.$$

- The second moment of ν is equal to $J_{r,k}(Y)$, the number of solutions to

$$y_1^i + \cdots + y_r^i = y_{r+1}^i + \cdots + y_{2r}^i, \quad 1 \leqslant i \leqslant k.$$

- $J_{r,k}(Y)$ lies in the heart of Vinogradov's method, and it is exactly equal to

$$\int_0^1 \cdots \int_0^1 \Big| \sum_{x\sim X} \mathrm{e}(\alpha_1 x + \cdots + \alpha_k x^k)\Big|^{2r} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k.$$

- See the works of Vinogradov (1935), Hua (1949), Karatsuba (1973), Wooley (2016), Bourgain–Demeter–Guth (2016).

Burgess's method after Vinogradov

Let χ be a non-trivial Dirichlet character mod p. Let w be a smooth function mimicking the indicator function of $[1,2]$. Put

$$S := \sum_{n\in\mathbf{Z}} w\left(\frac{n}{N}\right)\chi(n).$$

- For all integers u, v, we have (Trivial!)

$$S = \sum_{n\in\mathbf{Z}} w\left(\frac{n+uv}{N}\right)\chi(n+uv).$$

- Summing over $u \sim U, v \sim V$, we have (Trivial!)

$$S \approx \frac{1}{UV}\sum_{u\sim U}\sum_{v\sim V}\sum_{n\in\mathbf{Z}} w\left(\frac{n+uv}{N}\right)\chi(n+uv).$$

- For $(u,p)=1$, observe that $\chi(n+uv) = \chi(u)\chi(n\overline{u}+v)$. (Closer to non-trivial!)
- How to group u, u and separate u, v?

Burgess's method after Vinogradov

- Put $\varrho(a) = |\{(n,u) : |n| \leqslant N, u \sim U, n \equiv ua \pmod{p}\}|$, so that

$$S \ll \frac{1}{UV} \sum_{a \pmod{p}} \varrho(a) \Big| \sum_{v \sim V} \theta_v \chi(a+v) \Big|, \quad \theta_v \in S^1.$$

- By Hölder, we are led to

$$\sum_{a \pmod{p}} \varrho(a)^2, \qquad \sum_{a \pmod{p}} \Big| \sum_{v \sim V} \theta_v \chi(a+v) \Big|^{2r}.$$

- The second average is at most

$$\sum_{\mathbf{v} \in [1,V]^{2r}} \Big| \sum_{x \pmod{p}} \chi\left(\frac{(x+v_1)\cdots(x+v_r)}{(x+v_{r+1})\cdots(x+v_{2r})} \right) \Big|.$$

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- complete character sums over finite fields: RH/$\mathbf{F}_q$

Burgess bound

Theorem (Burgess, 1960-70's)

Let p be a large prime and χ a non-trivial character modulo p. For all $r \in \mathbf{Z}^+$ we have

$$\sum_{M<n\leqslant M+N} \chi(n) \ll N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}+\varepsilon}$$

for any $\varepsilon > 0$.

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for any $\varepsilon > 0$.

- Trivial bound: $\min\{q, N\}$.
- Burgess is non-trivial as long as $N > p^{1/4+\varepsilon}$.
- Burgess's method requires the periodic and multiplicative feature.

Applications of Burgess's method to bilinear forms

$$\sum_m \sum_n \alpha_m \beta_n K(mn)$$

- Friedlander–Iwaniec, Fouvry–Michel, Kowalski–Michel–Sawin, et al.
- The aim is to beat the Pólya–Vinogradov barrier:

$$\sum_m \Big| \sum_n \beta_n K(mn) \Big|^2 = \sum_{n_1} \sum_{n_2} \beta_{n_1} \overline{\beta}_{n_2} \sum_m K(mn_1) \overline{K(mn_2)}$$

Theorem (Fouvry, Kowalski & Michel, 2014)

Let p be a large prime. For all "good" K and $1 \leqslant M, N \leqslant p$, we have

$$\sum_{m \leqslant M} \sum_{n \leqslant N} \alpha_m \beta_n K(mn, p) \ll \|\boldsymbol{\alpha}\| \|\boldsymbol{\beta}\| (MN)^{\frac{1}{2}} (N^{-\frac{1}{2}} + M^{-\frac{1}{2}} p^{\frac{1}{4}} \log p).$$

Applications of Burgess's method to bilinear forms

$$\mathrm{Kl}_k(n,p) := p^{\frac{1-k}{2}} \sum \cdots \sum_{\substack{x_1,\cdots,x_k \in \mathbf{F}_p^* \\ x_1\cdots x_k = n}} \mathrm{e}\Big(\frac{x_1+\cdots+x_k}{p}\Big).$$

Theorem (Kowalski–Michel–Sawin, 2017[2])

Let p be a large prime. For each fixed $k \geqslant 2$, we have

$$\sum_{m\leqslant M}\sum_{n\leqslant N} \alpha_m \beta_n \mathrm{Kl}_k(mn,p) \ll \|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\|(MN)^{\frac{1}{2}}(N^{-\frac{1}{2}} + (MN)^{-\frac{3}{16}}p^{\frac{11}{64}})p^{\varepsilon}$$

for

$$p^{4\varepsilon} < N < Mp^{\frac{1}{4}}, \ \ p^{\frac{11}{12}} < MN < p^{\frac{5}{4}}.$$

- This is non-trivial as long as $M = N > p^{\frac{11}{24}+\varepsilon}$.

[2] E. Kowalski, Ph. Michel & W. Sawin, Bilinear forms with Kloosterman sums and applications, *Annals of Math.* **186** (2017), 413–500.

Bilinear forms over arbitrary subsets

Let $\mathcal{M}, \mathcal{N} \subseteq \mathbf{F}_p$ be two arbitrary subsets and consider a function $K : \mathbf{F}_p \to \mathbf{C}$. Assume $\boldsymbol{\alpha} = (\alpha_m)$ and $\boldsymbol{\beta} = (\beta_n)$ are arbitrary coefficients with supports in $\mathcal{M}, \mathcal{N}$, respectively. Put

$$\mathcal{B}(\boldsymbol{\alpha}, \boldsymbol{\beta}; K) = \sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \alpha_m \beta_n K(mn). \tag{1}$$

- Neither of $\mathcal{M}, \mathcal{N}$ is obviously contained in suitable intervals, there is no hope to transform incomplete sums to complete sums directly by Fourier analysis (as in the Pólya–Vinogradov method).
- By raising powers in the application of Hölder's inequality, we are able to prove non-trivial bounds for some general K and $|\mathcal{M}| > p^{1/2+\varepsilon}$, $|\mathcal{N}| > p^{\varepsilon}$.

Bilinear forms over arbitrary subsets

Theorem (Xi[3], 2023)

Let $K = \mathrm{Kl}_k(a\cdot, p)$ *with* $a \in \mathbf{F}_p^\times$ *and* $k \geqslant 2$*. For all* $r \geqslant 2$ *and* $\mathcal{M}, \mathcal{N} \subseteq \mathbf{F}_p$ *satisfying*

$$|\mathcal{M}|, |\mathcal{N}| \leqslant p^{\frac{2}{3}}, \quad |\mathcal{N} + \mathcal{N}| \leqslant \lambda|\mathcal{N}| \ (\lambda \geqslant 1),$$

we have

$$\mathcal{B}(\boldsymbol{\alpha}, \boldsymbol{\beta}; K) \ll \|\boldsymbol{\alpha}\|_\infty \|\boldsymbol{\beta}\|_\infty |\mathcal{M}||\mathcal{N}| \left\{ |\mathcal{M}|^{-\frac{1}{2}} + \left(\frac{p^{3+\frac{9\lambda}{4r}}}{|\mathcal{M}|^4 |\mathcal{N}|^3} \right)^{\frac{1}{8r}} (\log p)^{\frac{1}{2r}} \right\},$$

where the implied constant depends only on (r, λ) *and polynomially on* k*.*

- This is non-trivial as long as

$$|\mathcal{M}| = |\mathcal{N}| > p^{\frac{3}{7}+\varepsilon}.$$

[3] P. Xi, Bilinear forms with trace functions over arbitrary sets, and applications to Sato–Tate, to appear in *SCIENCE CHINA Math.*, 2023

Bilinear forms over arbitrary subsets

We need the following Freiman type theorem from additive combinatorics.

Lemma (M.-C. Chang, 2002)

Assume that $\mathcal{A} \subseteq \mathbf{Z}$ is a finite set with $|\mathcal{A}+\mathcal{A}| \leqslant \lambda|\mathcal{A}|$ for some $\lambda \geqslant 1$. Then $\mathcal{A}$ is contained in a proper d-dimensional arithmetic progression $\mathcal{P}$ with

$$d \leqslant \lambda - 1, \quad \log(|\mathcal{P}|/|\mathcal{A}|) \leqslant C\lambda^2(\log\lambda)^3.$$

- While working with generalized arithmetic progressions, one is able to introduce the $+ab$ shift to amplify the original sum.

Multilinear character sums

- joint with É. Fouvry & I. E. Shparlinski (in progress, double Burgess)

$$\sum_{m\sim M}\sum_{n\sim N}\sum_{k\sim K}\alpha_m\beta_{n,k}\chi(mn+k)$$

$$\sum_{m\sim M}\sum_{n\sim N}\sum_{k\sim K}\sum_{l\sim L}\alpha_{m,l}\beta_{n,k}\chi(mn+kl)$$

Amplification of Kloosterman

$$K(m,n;q) = \sum_{a \in (\mathbf{Z}/q\mathbf{Z})^\times} \mathrm{e}\Big(\frac{ma + na^{-1}}{q}\Big)$$

$$\max_{(m,p)=1} |K(m,1;p)| \leqslant 2p^{3/4}.$$

- Kloosterman evaluated the fourth moment

$$\sum_{m \in (\mathbf{Z}/p\mathbf{Z})^\times} |K(m,1;p)|^4 = 2p^3 - 3p^2 - 3p - 1.$$

- The square root cancellation philosophy predicts that

$$\max_{(m,p)=1} |K(m,1;p)| \leqslant c\sqrt{p}.$$

- $c = 2$ is admissible thanks to Weil (1940's): Riemann Hypothesis over finite fields

A trivial amplification

Let $\mathcal{A} = \{a_n\}$ be a complex sequence indexed by n in a suitable family $\mathcal{F}$.

- Consider the moment ($k \in \mathbf{Z}^+$)

$$M_k(\mathcal{F}) = \sum_{n \in \mathcal{F}} |a_n|^k.$$

- Trivially, $\max_{n \in \mathcal{F}} |a_n|^k \leqslant M_k(\mathcal{F})$, which yields

$$\|\mathcal{A}\|_\infty \leqslant \|\mathcal{A}\|_{\ell^k}.$$

- This becomes stronger if one may bound M_k effectively for

larger k or smaller $\mathcal{F}$

The trivial amplification: second moment

Consider the second moment of Kloosterman sums

$$\sum_{m\in(\mathbf{Z}/p\mathbf{Z})^\times} |K(m,1;p)|^2 = p^2 - p - 1.$$

This can only produce the trivial bound by the trivial amplification.

The trivial amplification: second moment

- Let f be a holomorphic cusp form for $\mathrm{SL}_2(\mathbf{Z})$. Given a non-trivial Dirichlet character mod p, consider the second moment of twisted L-functions:

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, p)} |L(1/2, f \otimes \chi)|^2.$$

  We expect a non-trivial upper bound for $L(1/2, f \otimes \chi)$.

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- The Phragmen–Lindelöf principle implies $L(1/2, f \otimes \chi) \ll_f p^{1/2+\varepsilon}$, and Lindelöf Hypothesis predicts that $L(1/2, f \otimes \chi) \ll p^{\varepsilon}$.

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- The subconvexity problem aims to prove, for some constant $\delta > 0$, that

$$L(1/2, f \otimes \chi) \ll p^{1/2-\delta}.$$

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$$\sum_{\chi \pmod p}^{*} |L(1/2, f \otimes \chi)|^2.$$

  We expect a non-trivial upper bound for $L(1/2, f \otimes \chi)$.
- The Phragmen–Lindelöf principle implies $L(1/2, f \otimes \chi) \ll_f p^{1/2+\varepsilon}$, and Lindelöf Hypothesis predicts that $L(1/2, f \otimes \chi) \ll p^{\varepsilon}$.
- The subconvexity problem aims to prove, for some constant $\delta > 0$, that

$$L(1/2, f \otimes \chi) \ll p^{1/2-\delta}.$$

- (Kowalski, Michel & Sawin, 2017)

$$\sum_{\chi \pmod p}^{*} |L(1/2, f \otimes \chi)|^2 = c_1 p \log p + c_2 p + O(p^{1-\eta}).$$

  The trivial amplification gives a trivial bound.

A non-trivial amplification: twisted second moment

Given an arbitrary coefficient $\mathbf{c} = (c_\ell)_{\ell\sim L}$ with $\|\mathbf{c}\|_\infty \leqslant 1$, consider

$$M(\mathbf{c}, p) := \sum_{\chi \,(\mathrm{mod}\, p)}^{*} \Big|\sum_{\ell\sim L} c_\ell \chi(\ell)\Big|^2 |L(1/2, f\otimes\chi)|^2.$$

- Approximate functional equation yields that

$$M(\mathbf{c}, p) \approx \sum\sum_{\ell_1,\ell_2\sim L} c_{\ell_1}\bar{c}_{\ell_2} \sum\sum_{\substack{m_1,m_2\leqslant p \\ m_1\ell_1\equiv m_2\ell_2 \,(\mathrm{mod}\, p)}} \lambda_f(m_1)\overline{\lambda_f(m_1)} \ll pL + p^{1-2\delta}L^2.$$

- For each non-trivial character $\chi_1 \,(\mathrm{mod}\, p)$, choose $c_\ell = \overline{\chi}_1(\ell)$, so that

$$L^2|L(1/2, f\otimes\chi_1)|^2 \ll M(\mathbf{c}, p) \ll pL + p^{1-\delta}L^2.$$

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$$L^2 |L(1/2, f \otimes \chi_1)|^2 \ll M(\mathbf{c}, p) \ll pL + p^{1-\delta} L^2.$$

$$L(1/2, f \otimes \chi_1) \ll \sqrt{p/L} + p^{1/2-\delta}.$$

Well done!

A non-trivial amplification: twisted second moment

$$L(1/2, f), \qquad L(1/2, f\otimes\chi), \qquad L(1/2, f\otimes g)$$

[1] W. Duke, J. B. Friedlander, H. Iwaniec, Bounds for automorphic L-functions, *Invent. math.* **112** (1993), 1–8.

[2] W. Duke, J. B. Friedlander, H. Iwaniec, Bounds for automorphic L–functions II, *Invent. math.* **115** (1994), 219–239; erratum, ibid. 140 (2000), 227–242.

[3] W. Duke, J. Friedlander, H. Iwaniec, Class group L–functions, *Duke Math. J.* **79** (1995), 1–56.

[4] W. Duke, J. B. Friedlander, H. Iwaniec, Bounds for automorphic L–functions II, *Invent. math.* **143** (2001), 221–248.

[5] E. Kowalski, Ph. Michel, J. VanderKam, Rankin–Selberg L–functions in the level aspect, *Duke Math. J.* **114** (2002), 123–191.

[6] Ph. Michel, The subconvexity problem for Rankin–-Selberg L-functions and equidistribution of Heegner points. *Ann. Math.* **160** (2004), 185–236.

Another amplification: cubic moment

- Omniscient narration: $L(1/2, f \otimes \chi) \ll p^{\varepsilon}$ for all $\chi \pmod p$.
- Suppose we are able to prove the cubic moment

$$\sum_{\chi \pmod p} |L(1/2, f \otimes \chi)|^3 \ll p^n.$$

  A trivial amplification yields

$$L(1/2, f \otimes \chi) \ll p^{n/3}.$$

- An admissible choice $n = 3/2 - \delta$ is sufficient to produce subconvexity.

Cubic moment in the spirit of Motohashi [4]

THEOREM. *If* $0<\Delta<T(\log T)^{-1}$, *then there exist absolute constants* $c(a,b;k,l)$ *such that*

$$(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}\,dt$$

$$=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\mathrm{Re}\Big[\sum_{\substack{a,b,k,l\geqslant 0\\ ak+bl\leqslant 4}} c(a,b;k,l)\left(\frac{\Gamma^{(a)}}{\Gamma}\right)^k\left(\frac{\Gamma^{(b)}}{\Gamma}\right)^l\left(\tfrac{1}{2}+i(T+t)\right)\Big]e^{-(t/\Delta)^2}\,dt$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+it)|^6}{|\zeta(1+2it)|^2}\Theta(t;T,\Delta)\,dt+\sum_{j=1}^{\infty}\alpha_j H_j\left(\tfrac{1}{2}\right)^3\Theta(\varkappa_j;T,\Delta) \qquad (1.16)$$

$$+\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}H_{j,k}\left(\tfrac{1}{2}\right)^3\Theta\left(i\left(\tfrac{1}{2}-k\right);T,\Delta\right)+O(T^{-1}(\log T)^2).$$

[4] Y. Motohashi, An explicit formula for the fourth power mean of the Riemann zeta-function, *Acta Math.* **170** (1993), 181–220.

Cubic moment in the spirit of Motohashi

- (Motohashi, 1993)

$$\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it)|^4 w(t)\mathrm{d}t = \sum_f L(\tfrac{1}{2},f)^3 \check{w}(t_f) + (\text{Eisenstein part}),$$

where the sum runs over all holomorphic/Maass forms f with spectral parameter t_f for the group $\mathrm{SL}_2(\mathbb{Z})$.

- (Conrey–Iwaniec, 2000; Petrow–Young, 2020)

$$\sum_{f\in\mathcal{F}(T)} L(\tfrac{1}{2},f\otimes\chi)^3 + \int_{-T}^{T} L(\tfrac{1}{2}+it,\chi)^6 \mathrm{d}t \rightsquigarrow \text{fourth moment of } \mathrm{GL}_1\ L\text{-functions}$$

Cubic moment in the spirit of Motohashi

Theorem (Petrow–Young[5] [6], 2020/2023)

Let χ be a non-trivial character mod q. Then

$$L(\tfrac{1}{2}+it,\chi) \ll ((1+|t|)q)^{\frac{1}{6}+\varepsilon}.$$

- Conrey–Iwaniec[7] treated the case of quadratic χ with a larger exponent in t.
- Number Fields: Balkanova, Frolenkov & Han Wu[8] and Nelson[9]

[5] I. Petrow & M. Young, The Weyl bound for Dirichlet L-functions of cube-free conductor, *Ann. Math.* **192** (2020), 437–486.

[6] I. Petrow & M. Young, The fourth moment of Dirichlet L-functions along a coset and the Weyl bound, to appear in *Duke Math. J.*, DOI: 10.1215/00127094-2022-0069.

[7] J. B. Conrey & H. Iwaniec, The cubic moment of central values of automorphic L-functions, *Ann. Math.* **151** (2000), 1175–1216.

[8] O. Balkanova, D. Frolenkov & H. Wu, On Weyl's subconvex bound for cube-free Hecke characters: totally real case. arXiv: 2108.12283.

[9] P. Nelson, Eisenstein series and the cubic moment for PGL_2, arXiv:1911.06310.

From GL_1 to GL_2

Theorem (Petrow–Young[10], 2020)

Let p be an odd prime, and suppose F is a Hecke-Maass newform of level $q = p^2$, trivial central character, and spectral parameter t_F. If the local representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to F is not supercuspidal, then

$$L(\tfrac{1}{2}, F) \ll (q(1 + |t_F|^2))^{\frac{1}{6}+\varepsilon}.$$

[10] I. Petrow & M. Young, The Weyl bound for Dirichlet L-functions of cube-free conductor, *Ann. Math.* **192** (2020), 437–486.

[11] H. Wu & P. Xi, A uniform Weyl bound for L-functions of Hilbert modular forms, arXiv: 2302:14652.

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Theorem (Han Wu & Xi[11], 2023)

The above assumption on "supercuspidal" can be removed.

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Motohashi's formula in number fields

$$\mathrm{M}_3(\Psi) := \sum_{\pi \text{ cuspidal}} \mathrm{M}_3(\Psi \mid \pi) + \sum_{\chi \in \widehat{\mathbb{R}_+ \mathbf{F}^\times \backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} \mathrm{M}_3(\Psi \mid \chi, i\tau) \frac{\mathrm{d}\tau}{4\pi},$$

$$\mathrm{M}_4(\Psi) = \frac{1}{\zeta_{\mathbf{F}}^*} \sum_{\chi \in \widehat{\mathbb{R}_+ \mathbf{F}^\times \backslash \mathbb{A}^\times}} \int_{\Re s = \frac{1}{2}} \mathrm{M}_4(\Psi \mid \chi, s) \frac{\mathrm{d}s}{2\pi i}.$$

Theorem (Han Wu[12], 2022)

Let $\Psi \in \mathcal{S}(M_2(\mathbb{A}))$ be a Schwartz function. Then

$$\mathrm{M}_3(\Psi) + \frac{1}{\zeta_{\mathbf{F}}^*} \operatorname*{Res}_{s=\frac{1}{2}} \mathrm{M}_3(\Psi \mid 1, s) = \mathrm{M}_4(\Psi) + \frac{1}{\zeta_{\mathbf{F}}^*} \left\{ \operatorname*{Res}_{s=1} \mathrm{M}_4(\Psi \mid 1, s) - \operatorname*{Res}_{s=0} \mathrm{M}_4(\Psi \mid 1, s) \right\}.$$

[12] H. Wu, On Motohashi's formula, *Trans. Amer. Math. Soc.* **375** (2022), 8033–8081.

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Motohashi's formula in number fields

- $\mathrm{M}_3, \mathrm{M}_4$ can be expressed as products of local zeta integrals over all places.
- The novelty of this formulation is that one may construct the weight functions place by place; i.e., it suffices to work locally.

- We are led to several cases according to the sizes of conductor components $\mathfrak{a}(\pi)$ of $\pi = \pi_v$.
- For π supercuspidal with $\mathfrak{a}(\pi) = 2$, it has depth 0 and is constructed from a character of the group of invertible elements of the quadratic field extension of the residual field.
- Works of Deligne and Katz are employed to bound the double character sum

$$\sum_{\alpha \in \mathbb{F}_q} \rho(\alpha + \omega) \sum_{t \in \mathbb{F}_q} \chi(t) \eta(\alpha^2 - \omega^2 t) \overline{\eta}(1 - t),$$

where ω is a primitive element in $\mathbb{F}_{q^2}$ such that $\omega^2 \in \mathbb{F}_q$, χ, η are non-trivial multiplicative characters of $\mathbb{F}_q^\times$ and ρ is that of $\mathbb{F}_{q^2}^\times$.

Amplifications in other situations

- (Selberg sieve; upper bound)

$$\sum_{\substack{n\in\mathcal{A}\\(n,P(z))=1}} 1 \leqslant \sum_{n\in\mathcal{A}} \Big(\sum_{d|(n,P(z))} \lambda_d \Big)^2$$

- (prime gaps; GPY and Yitang Zhang)

$$\sum_{x<n\leqslant 2x} \Big(\sum_{1\leqslant j\leqslant k} 1_{n+h_j \text{ prime}} - 1 \Big) \Big(\sum_{d|(n+h_1)\cdots(n+h_k)} \lambda_d \Big)^2$$

- (Mollification; non-vanishing of L-functions)

$$\Big| \frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F}} L(\tfrac{1}{2},f)M(f) \Big|^2 \leqslant \Big(\frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F},\ L(\frac{1}{2},f)\neq 0} 1 \Big) \Big(\frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F}} |L(\tfrac{1}{2},f)M(f)|^2 \Big)$$

Thank You!