Arithmetic of Markoff surfaces

Fei XU

School of Mathematical Sciences

Capital Normal University

I. Introduction

- Let X be a smooth and projective curve over $\mathbb{Q}$.

Theorem (Fermat): When the genus of X is 0, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad |X(\mathbb{Q})| = \infty$$

Theorem (Mordell): When the genus of X is 1, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad X(\mathbb{Q}) \text{ is a finitely generated abelian group}$$

Theorem (Faltings-Vojta-Lawrence-Venkatesh): When the genus of X is bigger than 1, then

$$|X(\mathbb{Q})| < \infty.$$

I. Introduction

- Let X be a smooth and projective curve over $\mathbb{Q}$.

Theorem (Fermat): When the genus of X is 0, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad |X(\mathbb{Q})| = \infty$$

Theorem (Mordell): When the genus of X is 1, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad X(\mathbb{Q}) \text{ is a finitely generated abelian group}$$

Theorem (Faltings-Vojta-Lawrence-Venkatesh): When the genus of X is bigger than 1, then

$$|X(\mathbb{Q})| < \infty.$$

I. Introduction

- Let X be a smooth and projective curve over $\mathbb{Q}$.

Theorem (Fermat): When the genus of X is 0, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad |X(\mathbb{Q})| = \infty$$

Theorem (Mordell): When the genus of X is 1, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad X(\mathbb{Q}) \text{ is a finitely generated abelian group}$$

Theorem (Faltings-Vojta-Lawrence-Venkatesh): When the genus of X is bigger than 1, then

$$|X(\mathbb{Q})| < \infty.$$

- Let X be a smooth and projective curve over $\mathbb{Q}$.

Theorem (Fermat): When the genus of X is 0, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad |X(\mathbb{Q})| = \infty$$

Theorem (Mordell): When the genus of X is 1, then

$$X(\mathbb{Q}) = \emptyset \quad \text{or} \quad X(\mathbb{Q}) \text{ is a finitely generated abelian group}$$

Theorem (Faltings-Vojta-Lawrence-Venkatesh): When the genus of X is bigger than 1, then

$$|X(\mathbb{Q})| < \infty.$$

• For higher dimensional smooth and projective variety X over $\mathbb{Q}$, one also expect that $|X(\mathbb{Q})|$ is close related to the geometry of $\overline{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

• Let $\Omega_{\overline{X}}$ be the sheaf of differentials of $\overline{X}$ and

$$\omega_{\overline{X}} = \bigwedge^{\dim(X)} \Omega_{\overline{X}}$$

be the canonical divisor of $\overline{X}$

Definition.

1) X is Fano if $\omega_{\overline{X}}^{-1}$ is ample $(g = 0)$.

2) X is Calabi-Yau if $\omega_{\overline{X}}$ is trivial $(g = 1)$.

3) X is of general type if $\omega_{\overline{X}}$ is ample $(g > 1)$.

I. Introduction

• For higher dimensional smooth and projective variety X over $\mathbb{Q}$, one also expect that $|X(\mathbb{Q})|$ is close related to the geometry of $\overline{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

• Let $\Omega_{\overline{X}}$ be the sheaf of differentials of $\overline{X}$ and

$$\omega_{\overline{X}} = \bigwedge^{\dim(X)} \Omega_{\overline{X}}$$

be the canonical divisor of $\overline{X}$

Definition.

1) X is Fano if $\omega_{\overline{X}}^{-1}$ is ample $(g = 0)$.

2) X is Calabi-Yau if $\omega_{\overline{X}}$ is trivial $(g = 1)$.

3) X is of general type if $\omega_{\overline{X}}$ is ample $(g > 1)$.

• For higher dimensional smooth and projective variety X over $\mathbb{Q}$, one also expect that $|X(\mathbb{Q})|$ is close related to the geometry of $\overline{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

• Let $\Omega_{\overline{X}}$ be the sheaf of differentials of $\overline{X}$ and

$$\omega_{\overline{X}} = \bigwedge^{\dim(X)} \Omega_{\overline{X}}$$

be the canonical divisor of $\overline{X}$

Definition.

1) X is Fano if $\omega_{\overline{X}}^{-1}$ is ample ($g = 0$).

2) X is Calabi-Yau if $\omega_{\overline{X}}$ is trivial ($g = 1$).

3) X is of general type if $\omega_{\overline{X}}$ is ample ($g > 1$).

• For higher dimensional smooth and projective variety X over $\mathbb{Q}$, one also expect that $|X(\mathbb{Q})|$ is close related to the geometry of $\overline{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

• Let $\Omega_{\overline{X}}$ be the sheaf of differentials of $\overline{X}$ and

$$\omega_{\overline{X}} = \bigwedge^{\dim(X)} \Omega_{\overline{X}}$$

be the canonical divisor of $\overline{X}$

Definition.

1) X is Fano if $\omega_{\overline{X}}^{-1}$ is ample ($g = 0$).

2) X is Calabi-Yau if $\omega_{\overline{X}}$ is trivial ($g = 1$).

3) X is of general type if $\omega_{\overline{X}}$ is ample ($g > 1$).

• For higher dimensional smooth and projective variety X over $\mathbb{Q}$, one also expect that $|X(\mathbb{Q})|$ is close related to the geometry of $\overline{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

• Let $\Omega_{\overline{X}}$ be the sheaf of differentials of $\overline{X}$ and

$$\omega_{\overline{X}} = \bigwedge^{\dim(X)} \Omega_{\overline{X}}$$

be the canonical divisor of $\overline{X}$

Definition.

1) X is Fano if $\omega_{\overline{X}}^{-1}$ is ample $(g = 0)$.

2) X is Calabi-Yau if $\omega_{\overline{X}}$ is trivial $(g = 1)$.

3) X is of general type if $\omega_{\overline{X}}$ is ample $(g > 1)$.

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

• Case of Calabi-Yau varieties.

• For abelian varieties, there is a generalization of BSD conjecture.

• There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

• For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

I. Introduction

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

• Case of Calabi-Yau varieties.

• For abelian varieties, there is a generalization of BSD conjecture.

• There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

• For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

- Case of Calabi-Yau varieties.

- For abelian varieties, there is a generalization of BSD conjecture.

- There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

- For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

I. Introduction

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

• Case of Calabi-Yau varieties.

• For abelian varieties, there is a generalization of BSD conjecture.

• There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

• For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

• Case of Calabi-Yau varieties.

• For abelian varieties, there is a generalization of BSD conjecture.

• There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

• For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

I. Introduction

Conjecture (Colliot-Thelene). If X is Fano, then either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Conjecture (Bombieri-Lang). If X is of general type, then $X(\mathbb{Q})$ is not Zariski dense in X.

• Case of Calabi-Yau varieties.

• For abelian varieties, there is a generalization of BSD conjecture.

• There are two classes of 2-dimensional Calabi-Yau varieties: abelian surfaces and K3 surfaces which are simply connected.

• For a K3 surface X, it is conjectured that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Example. Euler generalized the Fermat equation and conjectured

$$X: \quad x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n$$

has no solution of positive integers for $n > 2$.

- X is a Calabi-Yau variety.
- Lander and Parkin (1966) found a counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

by a direct computer searching.

- When $n = 4$, X is a K3 surface.

I. Introduction

Example. Euler generalized the Fermat equation and conjectured

$$X: \quad x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n$$

has no solution of positive integers for $n > 2$.

- X is a Calabi-Yau variety.

- Lander and Parkin (1966) found a counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

by a direct computer searching.

- When $n = 4$, X is a K3 surface.

Example. Euler generalized the Fermat equation and conjectured

$$X : \quad x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n$$

has no solution of positive integers for $n > 2$.

- X is a Calabi-Yau variety.

- Lander and Parkin (1966) found a counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

by a direct computer searching.

- When $n = 4$, X is a K3 surface.

Example. Euler generalized the Fermat equation and conjectured

$$X: \quad x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n$$

has no solution of positive integers for $n > 2$.

- X is a Calabi-Yau variety.

- Lander and Parkin (1966) found a counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

by a direct computer searching.

- When $n = 4$, X is a K3 surface.

- Elkies (1988) provided a counterexample for $n = 4$

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

and $X(\mathbb{Q})$ is Zariski dense in X.

Sketch of Elkies' idea: There is a morphism ψ

$$Y : \begin{cases} (u^2+2)(\frac{r-s}{2})^2 = -(3u^2-8u+6)(\frac{r+s}{2})^2 - 2(u^2-2)(\frac{r+s}{2}) - 2u \\ \pm(u^2+2)t^2 = 4(u^2-2)(\frac{r+s}{2})^2 + 8u(\frac{r+s}{2}) + (2-u^2) \end{cases}$$

$$\longrightarrow Z : \{r^4 + s^4 + t^4 = 1\}$$

based on the work of Demjanenko (1974 in Russian).

- Elkies (1988) provided a counterexample for $n = 4$

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

and $X(\mathbb{Q})$ is Zariski dense in X.

Sketch of Elkies' idea: There is a morphism ψ

$$Y : \begin{cases} (u^2+2)(\frac{r-s}{2})^2 = -(3u^2-8u+6)(\frac{r+s}{2})^2 - 2(u^2-2)(\frac{r+s}{2}) - 2u \\ \pm(u^2+2)t^2 = 4(u^2-2)(\frac{r+s}{2})^2 + 8u(\frac{r+s}{2}) + (2-u^2) \end{cases}$$

$$\longrightarrow Z : \{r^4 + s^4 + t^4 = 1\}$$

based on the work of Demjanenko (1974 in Russian).

• One can partially compactify Y to Y^c with the fibration $Y^c \xrightarrow{\phi} \mathbb{P}^1$ by sending $(r,s,t,u) \mapsto [u,1]$. Moreover, the morphism ψ can be extended to

$$\psi : Y^c \to Z \quad \text{and} \quad \psi(Y^c(\mathbb{Q})) = Z(\mathbb{Q}).$$

• Each fiber of ϕ over $\mathbb{P}^1(\mathbb{Q})$ is an intersection of two conics over $\mathbb{Q}$. By using the Hasse principle for conics with several testing, one can choose $\phi^{-1}([16,-5])$ which is

$$153y^2 = -779x^2 - 206x + 80, \quad 153t^2 = 412x^2 - 320x - 103$$

where $x = \frac{r+s}{2}$ and $y = \frac{r-s}{2}$.

• One can partially compactify Y to Y^c with the fibration $Y^c \xrightarrow{\phi} \mathbb{P}^1$ by sending $(r, s, t, u) \mapsto [u, 1]$. Moreover, the morphism ψ can be extended to

$$\psi : Y^c \to Z \quad \text{and} \quad \psi(Y^c(\mathbb{Q})) = Z(\mathbb{Q}).$$

• Each fiber of ϕ over $\mathbb{P}^1(\mathbb{Q})$ is an intersection of two conics over $\mathbb{Q}$. By using the Hasse principle for conics with several testing, one can choose $\phi^{-1}([16, -5])$ which is

$$153y^2 = -779x^2 - 206x + 80, \quad 153t^2 = 412x^2 - 320x - 103$$

where $x = \frac{r+s}{2}$ and $y = \frac{r-s}{2}$.

- All rational points of first conic can parametrized by

$$x = \frac{51k^2 - 34k - 5221}{14(17k^2 + 779)}, \quad y = \frac{17k^2 + 7558k - 779}{42(17k^2 + 779)}$$

by Fermat's method. Substituting this x into the second conic and using the new coordinates $\xi = (k+2)/7$ and $\eta = 3(17k^2 + 779)t/14$, one has

$$\eta^2 = -31790\xi^4 + 36941\xi^3 - 56158\xi^2 + 28849\xi + 22030.$$

- By computer searching, one obtains

$$(\xi, \eta) = (-\frac{31}{467}, \frac{30731278}{467^2})$$

- All rational points of first conic can parametrized by

$$x = \frac{51k^2 - 34k - 5221}{14(17k^2 + 779)}, \quad y = \frac{17k^2 + 7558k - 779}{42(17k^2 + 779)}$$

by Fermat's method. Substituting this x into the second conic and using the new coordinates $\xi = (k+2)/7$ and $\eta = 3(17k^2 + 779)t/14$, one has

$$\eta^2 = -31790\xi^4 + 36941\xi^3 - 56158\xi^2 + 28849\xi + 22030.$$

- By computer searching, one obtains

$$(\xi, \eta) = (-\frac{31}{467}, \frac{30731278}{467^2})$$

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

• A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

• A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

- A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

• A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

- A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

Definition. Let X be a smooth and geometrically integral variety over $\mathbb{Q}$. Then X is called a log K3 surface if X satisfies

1) $\overline{X}$ is simply connected.

2) There is a smooth compactification $X \hookrightarrow X^c$ over $\mathbb{Q}$ such that $D = X^c \setminus X$ is a simple normal crossing divisor.

3) $[D] + \omega_{X^c} = 0$ in $\mathrm{Pic}(X^c)$.

• A log K3 surface is a generalization of K3 surface.

Conjecture (Vojta). If X be a log K3 surface over $\mathbb{Q}$ and $\mathfrak{X}$ be an integral model of X over $\mathbb{Z}$, then there is a number field F such that $\mathfrak{X}(\mathcal{O}_F)$ is Zariski dense in $\mathfrak{X}$ where $\mathcal{O}_F$ is the ring of integers of F.

II. Brauer-Manin obstruction

- Let X be a variety over $\mathbb{Q}$ and $\mathrm{Br}(X) = \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$ and

$$\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})).$$

- Let $\mathbf{A}_{\mathbb{Q}}$ be the adeles and $\mathbf{A}_{\mathbb{Q}}^f$ be the finite adeles of $\mathbb{Q}$. Write

$$X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} = \{(x_p) \in X(\mathbf{A}_{\mathbb{Q}}) : \sum_{p \leq \infty} inv_p(\xi(x_p)) = 0; \quad \forall\, \xi \in \mathrm{Br}(X)\}$$

where

$$inv_p : \mathrm{Br}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the invariant map from the local class field theory for $p \leq \infty$.

- The class field theory implies

$$X(\mathbb{Q}) \subseteq X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} \subseteq X(\mathbf{A}_{\mathbb{Q}}).$$

- Let X be a variety over $\mathbb{Q}$ and $\mathrm{Br}(X) = \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$ and

$$\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})).$$

- Let $\mathbf{A}_{\mathbb{Q}}$ be the adeles and $\mathbf{A}^f_{\mathbb{Q}}$ be the finite adeles of $\mathbb{Q}$. Write

$$X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} = \{(x_p) \in X(\mathbf{A}_{\mathbb{Q}}) : \sum_{p \leq \infty} inv_p(\xi(x_p)) = 0; \ \ \forall\, \xi \in \mathrm{Br}(X)\}$$

where

$$inv_p : \mathrm{Br}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the invariant map from the local class field theory for $p \leq \infty$.

- The class field theory implies

$$X(\mathbb{Q}) \subseteq X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} \subseteq X(\mathbf{A}_{\mathbb{Q}}).$$

II. Brauer-Manin obstruction

- Let X be a variety over $\mathbb{Q}$ and $\mathrm{Br}(X) = \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$ and

$$\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})).$$

- Let $\mathbf{A}_{\mathbb{Q}}$ be the adeles and $\mathbf{A}_{\mathbb{Q}}^f$ be the finite adeles of $\mathbb{Q}$. Write

$$X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} = \{(x_p) \in X(\mathbf{A}_{\mathbb{Q}}) : \sum_{p \leq \infty} inv_p(\xi(x_p)) = 0; \quad \forall\, \xi \in \mathrm{Br}(X)\}$$

where

$$inv_p : \mathrm{Br}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the invariant map from the local class field theory for $p \leq \infty$.

- The class field theory implies

$$X(\mathbb{Q}) \subseteq X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)} \subseteq X(\mathbf{A}_{\mathbb{Q}}).$$

Definition. We say X satisfies strong approximation with the Brauer-Manin obstruction if

$$X(\mathbb{Q}) \text{ is dense in } pr^f(X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)})$$

where

$$pr^f : X(\mathbf{A}_{\mathbb{Q}}) \to X(\mathbf{A}_{\mathbb{Q}}^f)$$

is the projection map.

Conjecture (Skorobogatov). If X is a K3 surface, then X satisfies strong approximation with the Brauer-Manin obstruction.

Proposition. Skorobogatov's conjecture implies that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

II. Brauer-Manin obstruction

Definition. We say X satisfies strong approximation with the Brauer-Manin obstruction if

$$X(\mathbb{Q}) \ \text{ is dense in } pr^f(X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)})$$

where

$$pr^f : X(\mathbf{A}_{\mathbb{Q}}) \to X(\mathbf{A}_{\mathbb{Q}}^f)$$

is the projection map.

Conjecture (Skorobogatov). If X is a K3 surface, then X satisfies strong approximation with the Brauer-Manin obstruction.

Proposition. Skorobogatov's conjecture implies that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

Definition. We say X satisfies strong approximation with the Brauer-Manin obstruction if

$$X(\mathbb{Q}) \text{ is dense in } pr^f(X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X)})$$

where

$$pr^f : X(\mathbf{A}_{\mathbb{Q}}) \to X(\mathbf{A}_{\mathbb{Q}}^f)$$

is the projection map.

Conjecture (Skorobogatov). If X is a K3 surface, then X satisfies strong approximation with the Brauer-Manin obstruction.

Proposition. Skorobogatov's conjecture implies that either $X(\mathbb{Q}) = \emptyset$ or $X(\mathbb{Q})$ is Zariski dense in X.

- Let

$$\mathfrak{U}_m : \quad x^2 + y^2 + z^2 - xyz = m$$

where m is a fixed integer. Then $U_m = \mathfrak{U}_m \times_{\mathbb{Z}} \mathbb{Q}$ is called a Markoff surface over $\mathbb{Q}$.

- When $m \neq 0$ and 4, then U_m is smooth.

- A smooth compactification X_m of U_m is

$$t(x^2 + y^2 + z^2) - xyz = mt^3$$

a cubic surfaces in $\mathbb{P}^3$ for $m \neq 0$ and 4.

- Let

$$\mathfrak{U}_m : \quad x^2 + y^2 + z^2 - xyz = m$$

where m is a fixed integer. Then $U_m = \mathfrak{U}_m \times_{\mathbb{Z}} \mathbb{Q}$ is called a Markoff surface over $\mathbb{Q}$.

- When $m \neq 0$ and 4, then U_m is smooth.

- A smooth compactification X_m of U_m is

$$t(x^2 + y^2 + z^2) - xyz = mt^3$$

a cubic surfaces in $\mathbb{P}^3$ for $m \neq 0$ and 4.

III. Markoff surfaces

- Let

$$\mathfrak{U}_m : \quad x^2 + y^2 + z^2 - xyz = m$$

where m is a fixed integer. Then $U_m = \mathfrak{U}_m \times_{\mathbb{Z}} \mathbb{Q}$ is called a Markoff surface over $\mathbb{Q}$.

- When $m \neq 0$ and 4, then U_m is smooth.

- A smooth compactification X_m of U_m is

$$t(x^2 + y^2 + z^2) - xyz = mt^3$$

a cubic surfaces in $\mathbb{P}^3$ for $m \neq 0$ and 4.

- Let $\overline{X}_m \xrightarrow{\pi} \mathbb{P}^2$ be a blowing down the following six lines in $\overline{X}_m$

$$\begin{cases} l_1: \ x = 2t, \ y - z = (\sqrt{m-4})t \\ l_2: \ y = 2t, \ z - x = (\sqrt{m-4})t \\ l_3: \ z = 2t, \ x - y = (\sqrt{m-4})t \\ l_4: \ x = (\sqrt{m})t, \ y = \frac{1}{2}(\sqrt{m} + \sqrt{m-4})z \\ l_5: \ y = (\sqrt{m})t, \ z = \frac{1}{2}(\sqrt{m} + \sqrt{m-4})x \\ l_6: \ z = (\sqrt{m})t, \ x = \frac{1}{2}(\sqrt{m} + \sqrt{m-4})y \end{cases}$$

and L is the inverse image of a line in $\mathbb{P}^2$. Then $\mathrm{Pic}(\overline{X}_m) \cong \mathbb{Z}^7$ is generated by $L, l_1, \cdots, l_6$ such that

$$\langle L, L \rangle = 1, \quad \langle L, l_i \rangle = 0 \quad \text{and} \quad \langle l_i, l_i \rangle = -1$$

for $1 \leq i \leq 6$ and $\langle l_i, l_j \rangle = 0$ for $i \neq j$.

- The canonical divisor $\omega_{\overline{X}_m}$ of $\overline{X}_m$ satisfies

$$\omega_{\overline{X}_m} = -3L + \sum_{i=1}^{6} l_i \text{ in } \mathrm{Pic}(\overline{X}_m).$$

- The divisor $D_m = X_m \setminus U_m$ consists of three lines

$$L_1 : \{t = x = 0\}, \quad L_2 : \{t = y = 0\}, \quad L_3 : \{t = z = 0\}$$

and

$$L_1 = L - l_1 - l_4, \quad L_2 = L - l_2 - l_5, \quad L_3 = L - l_3 - l_6 \text{ in } \mathrm{Pic}(\overline{X}_m)$$

- $\overline{U}_m$ is simply connected.

Conclusion. U_m is a log K3 surface.

- The canonical divisor $\omega_{\overline{X}_m}$ of $\overline{X}_m$ satisfies

$$\omega_{\overline{X}_m} = -3L + \sum_{i=1}^{6} l_i \text{ in } \mathrm{Pic}(\overline{X}_m).$$

- The divisor $D_m = X_m \setminus U_m$ consists of three lines

$$L_1 : \{t = x = 0\}, \quad L_2 : \{t = y = 0\}, \quad L_3 : \{t = z = 0\}$$

and

$$L_1 = L - l_1 - l_4, \quad L_2 = L - l_2 - l_5, \quad L_3 = L - l_3 - l_6 \text{ in } \mathrm{Pic}(\overline{X}_m)$$

- $\overline{U}_m$ is simply connected.

Conclusion. U_m is a log K3 surface.

III. Markoff surfaces

- The canonical divisor $\omega_{\overline{X}_m}$ of $\overline{X}_m$ satisfies

$$\omega_{\overline{X}_m} = -3L + \sum_{i=1}^{6} l_i \ \text{ in } \ \mathrm{Pic}(\overline{X}_m).$$

- The divisor $D_m = X_m \setminus U_m$ consists of three lines

$$L_1 : \{t = x = 0\}, \quad L_2 : \{t = y = 0\}, \quad L_3 : \{t = z = 0\}$$

and

$$L_1 = L - l_1 - l_4, \quad L_2 = L - l_2 - l_5, \quad L_3 = L - l_3 - l_6 \ \text{ in } \ \mathrm{Pic}(\overline{X}_m)$$

- $\overline{U}_m$ is simply connected.

Conclusion. U_m is a log K3 surface.

III. Markoff surfaces

- The canonical divisor $\omega_{\overline{X}_m}$ of $\overline{X}_m$ satisfies

$$\omega_{\overline{X}_m} = -3L + \sum_{i=1}^{6} l_i \text{ in } \operatorname{Pic}(\overline{X}_m).$$

- The divisor $D_m = X_m \setminus U_m$ consists of three lines

$$L_1 : \{t = x = 0\}, \quad L_2 : \{t = y = 0\}, \quad L_3 : \{t = z = 0\}$$

and

$$L_1 = L - l_1 - l_4, \quad L_2 = L - l_2 - l_5, \quad L_3 = L - l_3 - l_6 \text{ in } \operatorname{Pic}(\overline{X}_m)$$

- $\overline{U}_m$ is simply connected.

Conclusion. U_m is a log K3 surface.

- The canonical divisor $\omega_{\overline{X}_m}$ of $\overline{X}_m$ satisfies

$$\omega_{\overline{X}_m} = -3L + \sum_{i=1}^{6} l_i \text{ in } \mathrm{Pic}(\overline{X}_m).$$

- The divisor $D_m = X_m \setminus U_m$ consists of three lines

$$L_1 : \{t = x = 0\}, \quad L_2 : \{t = y = 0\}, \quad L_3 : \{t = z = 0\}$$

and

$$L_1 = L - l_1 - l_4, \quad L_2 = L - l_2 - l_5, \quad L_3 = L - l_3 - l_6 \text{ in } \mathrm{Pic}(\overline{X}_m)$$

- $\overline{U}_m$ is simply connected.

Conclusion. U_m is a log K3 surface.

- Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

- Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

.

• Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

• Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

- Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

- Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

• Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

• Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

- Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

- Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

- Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

- Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

- Basic question: For which m, $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$?

Proposition.

$$\prod_p \mathfrak{U}_m(\mathbb{Z}_p) = \emptyset$$

if and only if $m \equiv 3 \mod 4$ or $m \equiv \pm 3 \mod 9$.

Proposition. $\mathfrak{U}_m$ admits an automorphism group Γ generated by

(a) The Vieta involution: $(x, y, z) \mapsto (yz - x, y, z)$;

(b) The sign change: $(x, y, z) \mapsto (-x, -y, z)$;

(c) The permutations of x, y, z.

- Γ acts on $\mathfrak{U}_m(\mathbb{Z})$ naturally.

.

Theorem (Markoff, Hurwitz, Mordell, ..., Ghosh-Sarnak)

When $m < 0$, then each orbit of Γ contains a unique integral solution (x_0, y_0, z_0) such that

$$3 \leq x_0 \leq y_0 \leq z_0 \leq \frac{1}{2} x_0 y_0$$

and the number of Γ-orbits is finite.

When $m > 0$, then each orbit of Γ contains an integral solutions (x_0, y_0, z_0) such that

$$3 \leq x_0 \leq y_0 \leq -z_0 \quad \text{or} \quad x_0 = 0, 1, 2.$$

Moreover, if m is not form of

$$(i)\ u^2 + v^2\ (x_0 = 0); \quad (ii)\ u^2 - uv + v^2 + 1\ (x_0 = 1); \quad (iii)\ u^2 + 4\ (x_0 = 2),$$

then the integral solution is unique and the number of Γ-orbits is finite.

Theorem (Markoff, Hurwitz, Mordell, ..., Ghosh-Sarnak)

When $m < 0$, then each orbit of Γ contains a unique integral solution (x_0, y_0, z_0) such that

$$3 \leq x_0 \leq y_0 \leq z_0 \leq \frac{1}{2}x_0 y_0$$

and the number of Γ-orbits is finite.

When $m > 0$, then each orbit of Γ contains an integral solutions (x_0, y_0, z_0) such that

$$3 \leq x_0 \leq y_0 \leq -z_0 \quad \text{or} \quad x_0 = 0, 1, 2.$$

Moreover, if m is not form of

$(i)\ u^2 + v^2\ (x_0 = 0)$; $(ii)\ u^2 - uv + v^2 + 1\ (x_0 = 1)$; $(iii)\ u^2 + 4\ (x_0 = 2)$,

then the integral solution is unique and the number of Γ-orbits is finite.

Theorem (Ghosh-Sarnak).

$$\sharp\{m \in \mathbb{Z} : \ 0 < m \leq K, \ \prod_p \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

$$\sharp\{m \in \mathbb{Z} : \ -K \leq m < 0, \ \prod_p \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

Theorem (Ghosh-Sarnak).

$$\sharp\{m \in \mathbb{Z} : \ 0 < m \leq K, \ \prod_p \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

$$\sharp\{m \in \mathbb{Z} : \ -K \leq m < 0, \ \prod_p \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

IV. Computation of Brauer groups

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.

If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2$$

with a generator

$$\{((\frac{x}{t})^2 - 4, d) = ((\frac{y}{t})^2 - 4, d) = ((\frac{z}{t})^2 - 4, d)\} \quad \text{over} \quad t \neq 0.$$

If $d \notin (\mathbb{Q}^\times)^2$ and $m \in (\mathbb{Q}^\times)^2$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with two generators

$$\{((\frac{x}{t})^2 - 4, d),\ ((\sqrt{m} - \frac{x}{t})(\frac{x}{t} + 2), d)\} \quad \text{over} \quad t \neq 0.$$

If $d \in (\mathbb{Q}^\times)^2$ or $d \cdot m \in (\mathbb{Q}^\times)^2$, then $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.
If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2$$

with a generator

$$\{((\frac{x}{t})^2 - 4, d) = ((\frac{y}{t})^2 - 4, d) = ((\frac{z}{t})^2 - 4, d)\} \quad \text{over} \quad t \neq 0.$$

If $d \notin (\mathbb{Q}^\times)^2$ and $m \in (\mathbb{Q}^\times)^2$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with two generators

$$\{((\frac{x}{t})^2 - 4, d),\ ((\sqrt{m} - \frac{x}{t})(\frac{x}{t} + 2), d)\} \quad \text{over} \quad t \neq 0.$$

If $d \in (\mathbb{Q}^\times)^2$ or $d \cdot m \in (\mathbb{Q}^\times)^2$, then $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.
If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2$$

with a generator

$$\{((\frac{x}{t})^2 - 4, d) = ((\frac{y}{t})^2 - 4, d) = ((\frac{z}{t})^2 - 4, d)\} \quad \text{over} \quad t \neq 0.$$

If $d \notin (\mathbb{Q}^\times)^2$ and $m \in (\mathbb{Q}^\times)^2$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with two generators

$$\{((\frac{x}{t})^2 - 4, d),\ ((\sqrt{m} - \frac{x}{t})(\frac{x}{t} + 2), d)\} \quad \text{over} \quad t \neq 0.$$

If $d \in (\mathbb{Q}^\times)^2$ or $d \cdot m \in (\mathbb{Q}^\times)^2$, then $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.
If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2$$

with a generator

$$\{((\frac{x}{t})^2 - 4, d) = ((\frac{y}{t})^2 - 4, d) = ((\frac{z}{t})^2 - 4, d)\} \quad \text{over} \quad t \neq 0.$$

If $d \notin (\mathbb{Q}^\times)^2$ and $m \in (\mathbb{Q}^\times)^2$, then

$$\mathrm{Br}(X_m)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with two generators

$$\{((\frac{x}{t})^2 - 4, d),\ ((\sqrt{m} - \frac{x}{t})(\frac{x}{t} + 2), d)\} \quad \text{over} \quad t \neq 0.$$

If $d \in (\mathbb{Q}^\times)^2$ or $d \cdot m \in (\mathbb{Q}^\times)^2$, then $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m) = \mathrm{Br}(\mathbb{Q})$.

Sketch of proof. Since $\overline{X}_m$ is rational, one obtains $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m)$.

By $H^p(\mathbb{Q}, H^q(\overline{X}_m, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_m, \mathbb{G}_m)$, one has

$$\mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m))$$

$$H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m)) \cong \begin{cases} \mathbb{Z}/2 & m, d \text{ and } md \notin (\mathbb{Q}^\times)^2 \\ 0 & d \in (\mathbb{Q}^\times)^2 \ \text{ or } \ d \cdot m \in (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^2 & m \in (\mathbb{Q}^\times)^2 \text{ but } d \notin (\mathbb{Q}^\times)^2 \end{cases}$$

IV. Computation of Brauer groups

Sketch of proof. Since $\overline{X}_m$ is rational, one obtains $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m)$.

By $H^p(\mathbb{Q}, H^q(\overline{X}_m, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_m, \mathbb{G}_m)$, one has

$$\mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m))$$

$$H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m)) \cong \begin{cases} \mathbb{Z}/2 & m, d \text{ and } md \notin (\mathbb{Q}^\times)^2 \\ 0 & d \in (\mathbb{Q}^\times)^2 \text{ or } d \cdot m \in (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^2 & m \in (\mathbb{Q}^\times)^2 \text{ but } d \notin (\mathbb{Q}^\times)^2 \end{cases}$$

Sketch of proof. Since $\overline{X}_m$ is rational, one obtains $\mathrm{Br}_1(X_m) = \mathrm{Br}(X_m)$.

By $H^p(\mathbb{Q}, H^q(\overline{X}_m, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_m, \mathbb{G}_m)$, one has

$$\mathrm{Br}_1(X_m)/\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m))$$

$$H^1(\mathbb{Q}, \mathrm{Pic}(\overline{X}_m)) \cong \begin{cases} \mathbb{Z}/2 & m, d \text{ and } md \notin (\mathbb{Q}^\times)^2 \\ 0 & d \in (\mathbb{Q}^\times)^2 \ \text{ or } \ d \cdot m \in (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^2 & m \in (\mathbb{Q}^\times)^2 \text{ but } d \notin (\mathbb{Q}^\times)^2 \end{cases}$$

IV. Computation of Brauer groups

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.

If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^3$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $dm \in \mathbb{Q}^{\times 2}$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with the generators $\{(x-2, d),\ (y-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $m \in \mathbb{Q}^{\times 2}$, then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^4$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d),\ (x-\sqrt{m}, d)\}$.

If $d \in \mathbb{Q}^{\times 2}$, then $\mathrm{Br}_1(U_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.
If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^3$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $dm \in \mathbb{Q}^{\times 2}$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with the generators $\{(x-2, d),\ (y-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $m \in \mathbb{Q}^{\times 2}$, then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^4$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d),\ (x-\sqrt{m}, d)\}$.

If $d \in \mathbb{Q}^{\times 2}$, then $\mathrm{Br}_1(U_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.

If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^3$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $dm \in \mathbb{Q}^{\times 2}$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with the generators $\{(x-2, d),\ (y-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $m \in \mathbb{Q}^{\times 2}$, then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^4$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d),\ (x-\sqrt{m}, d)\}$.

If $d \in \mathbb{Q}^{\times 2}$, then $\mathrm{Br}_1(U_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.

If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^3$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $dm \in \mathbb{Q}^{\times 2}$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with the generators $\{(x-2, d),\ (y-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $m \in \mathbb{Q}^{\times 2}$, then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^4$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d),\ (x-\sqrt{m}, d)\}$.

If $d \in \mathbb{Q}^{\times 2}$, then $\mathrm{Br}_1(U_m) = \mathrm{Br}(\mathbb{Q})$.

Theorem (Colliot-Thelene, Wei and X.; Loughran and Mitankin).
Let $m \neq 0, 4$ and $d = m - 4$.

If $[\mathbb{Q}(\sqrt{m}, \sqrt{d}) : \mathbb{Q}] = 4$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^3$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $dm \in \mathbb{Q}^{\times 2}$ then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^2$$

with the generators $\{(x-2, d),\ (y-2, d)\}$.

If $d \notin \mathbb{Q}^{\times 2}$ and $m \in \mathbb{Q}^{\times 2}$, then

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2)^4$$

with the generators $\{(x-2, d),\ (y-2, d),\ (z-2, d),\ (x-\sqrt{m}, d)\}$.

If $d \in \mathbb{Q}^{\times 2}$, then $\mathrm{Br}_1(U_m) = \mathrm{Br}(\mathbb{Q})$.

IV. Computation of Brauer groups

Sketch of proof. By the exact sequence of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules

$$0 \to \overline{\mathbb{Q}}^{\times} \to \overline{\mathbb{Q}}[U_m]^{\times} \to \mathrm{Div}_{\overline{X}_m \setminus \overline{U}_m}(\overline{X}_m) \xrightarrow{\phi} \mathrm{Pic}(\overline{X}_m) \to \mathrm{Pic}(\overline{U}_m) \to 0$$

and $\mathrm{Div}_{\overline{X}_m \setminus \overline{U}_m}(\overline{X}_m) = \oplus_{i=1}^{3} \mathbb{Z} L_i$, one can show ϕ is injective. Indeed, if

$$aL_1 + bL_2 + cL_3 = a(L - l_1 - l_4) + b(L - l_2 - l_5) + c(L - l_3 - l_6) = 0,$$

then $a = b = c = 0$. This implies that $\overline{\mathbb{Q}}^{\times} = \overline{\mathbb{Q}}[U_m]^{\times}$ and the sub-lattice

$$\mathrm{Div}_{\overline{X}_m \setminus \overline{U}_m}(\overline{X}_m) = \oplus_{i=1}^{3} \mathbb{Z} L_i$$

is primitive in $\mathrm{Pic}(\overline{X}_m)$. Therefore $\mathrm{Pic}(\overline{U}_m)$ is torsion free.

IV. Computation of Brauer groups

By $H^p(\mathbb{Q}, H^q(\overline{U}_m, \mathbb{G}_m)) \Rightarrow H^{p+q}(U_m, \mathbb{G}_m)$, one has

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(\overline{U}_m))$$

which is finite.

$$H^1(\mathbb{Q}, \mathrm{Pic}(\overline{U}_m)) \cong \begin{cases} (\mathbb{Z}/2)^3 & m, d \text{ and } md \notin (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^2 & d \notin (\mathbb{Q}^\times)^2 \text{ and } md \in (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^4 & d \notin (\mathbb{Q}^\times)^2 \text{ and } m \in (\mathbb{Q}^\times)^2 \\ 0 & d \in (\mathbb{Q}^\times)^2 \end{cases}$$

By $H^p(\mathbb{Q}, H^q(\overline{U}_m, \mathbb{G}_m)) \Rightarrow H^{p+q}(U_m, \mathbb{G}_m)$, one has

$$\mathrm{Br}_1(U_m)/\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(\overline{U}_m))$$

which is finite.

$$H^1(\mathbb{Q}, \mathrm{Pic}(\overline{U}_m)) \cong \begin{cases} (\mathbb{Z}/2)^3 & m, d \text{ and } md \notin (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^2 & d \notin (\mathbb{Q}^\times)^2 \text{ and } md \in (\mathbb{Q}^\times)^2 \\ (\mathbb{Z}/2)^4 & d \notin (\mathbb{Q}^\times)^2 \text{ and } m \in (\mathbb{Q}^\times)^2 \\ 0 & d \in (\mathbb{Q}^\times)^2 \end{cases}$$

Theorem (Colliot-Thelene, Wei and X.).

Let $m \neq 0, 4$; $d = m - 4$ and $K = \mathbb{Q}(\sqrt{d}, \sqrt{m})$.

If -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$, then $\mathrm{Br}(U_m)/\mathrm{Br}_1(U_m) \cong \mathbb{Z}/2$.

Otherwise $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Sketch of proof. Recall $\pi : \overline{X}_m \to \mathbb{P}^2$ is a contraction of $\{l_1, \cdots, l_6\}$ to $\mathbb{P}^2$ and

$$\overline{X}_m = \overline{U}_m \cup \{L_1, L_2, L_3\}.$$

This implies that π induces an isomorphism

$$\overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{P}^2 \setminus \{\pi(L_1), \pi(L_2), \pi(L_3)\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$$

over $\overline{\mathbb{Q}}$.

IV. Computation of Brauer groups

Theorem (Colliot-Thelene, Wei and X.).

Let $m \neq 0, 4$; $d = m - 4$ and $K = \mathbb{Q}(\sqrt{d}, \sqrt{m})$.

If -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$, then $\mathrm{Br}(U_m)/\mathrm{Br}_1(U_m) \cong \mathbb{Z}/2$.

Otherwise $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Sketch of proof. Recall $\pi : \overline{X}_m \to \mathbb{P}^2$ is a contraction of $\{l_1, \cdots, l_6\}$ to $\mathbb{P}^2$ and

$$\overline{X}_m = \overline{U}_m \cup \{L_1, L_2, L_3\}.$$

This implies that π induces an isomorphism

$$\overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{P}^2 \setminus \{\pi(L_1), \pi(L_2), \pi(L_3)\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$$

over $\overline{\mathbb{Q}}$.

Theorem (Colliot-Thelene, Wei and X.).

Let $m \neq 0, 4$; $d = m - 4$ and $K = \mathbb{Q}(\sqrt{d}, \sqrt{m})$.

If -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$, then $\mathrm{Br}(U_m)/\mathrm{Br}_1(U_m) \cong \mathbb{Z}/2$.

Otherwise $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Sketch of proof. Recall $\pi : \overline{X}_m \to \mathbb{P}^2$ is a contraction of $\{l_1, \cdots, l_6\}$ to $\mathbb{P}^2$ and

$$\overline{X}_m = \overline{U}_m \cup \{L_1, L_2, L_3\}.$$

This implies that π induces an isomorphism

$$\overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{P}^2 \setminus \{\pi(L_1), \pi(L_2), \pi(L_3)\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$$

over $\overline{\mathbb{Q}}$.

Theorem (Colliot-Thelene, Wei and X.).

Let $m \neq 0, 4$; $d = m - 4$ and $K = \mathbb{Q}(\sqrt{d}, \sqrt{m})$.

If -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$, then $\mathrm{Br}(U_m)/\mathrm{Br}_1(U_m) \cong \mathbb{Z}/2$.

Otherwise $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Sketch of proof. Recall $\pi : \overline{X}_m \to \mathbb{P}^2$ is a contraction of $\{l_1, \cdots, l_6\}$ to $\mathbb{P}^2$ and

$$\overline{X}_m = \overline{U}_m \cup \{L_1, L_2, L_3\}.$$

This implies that π induces an isomorphism

$$\overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{P}^2 \setminus \{\pi(L_1), \pi(L_2), \pi(L_3)\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$$

over $\overline{\mathbb{Q}}$.

Theorem (Colliot-Thelene, Wei and X.).

Let $m \neq 0, 4$; $d = m - 4$ and $K = \mathbb{Q}(\sqrt{d}, \sqrt{m})$.

If -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$, then $\mathrm{Br}(U_m)/\mathrm{Br}_1(U_m) \cong \mathbb{Z}/2$.

Otherwise $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Sketch of proof. Recall $\pi : \overline{X}_m \to \mathbb{P}^2$ is a contraction of $\{l_1, \cdots, l_6\}$ to $\mathbb{P}^2$ and

$$\overline{X}_m = \overline{U}_m \cup \{L_1, L_2, L_3\}.$$

This implies that π induces an isomorphism

$$\overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{P}^2 \setminus \{\pi(L_1), \pi(L_2), \pi(L_3)\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$$

over $\overline{\mathbb{Q}}$.

IV. Computation of Brauer groups

Let $V = \overline{U}_m \setminus \{l_1, \cdots, l_6\} \cong \mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m$. Then

$$0 \to \mathrm{Br}(\overline{U}_m) \to \mathrm{Br}(V) \xrightarrow{res} \bigoplus_{i=1}^{6} H^1_{et}(D_i, \mathbb{Q}/\mathbb{Z})$$

where

$$D_i = l_i \setminus \{\text{the intersection point of } l_i \text{ with } L_1, L_2, L_3\} \cong \mathbb{A}^1$$

over $\overline{\mathbb{Q}}$. Since $H^1_{et}(\mathbb{A}^1_{\overline{\mathbb{Q}}}, \mathbb{Q}/\mathbb{Z}) = 0$, one concludes that

$$\mathrm{Br}(\overline{U}_m) \cong \mathrm{Br}(V) \cong \mathrm{Br}(\mathbb{G}_m \times_{\overline{\mathbb{Q}}} \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.$$

Example. If $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, then $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Proof. Suppose that -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$. Since $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, one has $K \neq \mathbb{Q}(\sqrt{d})$ and $\sqrt{m} \notin \mathbb{Q}(\sqrt{d})$. Then

$$Norm_{K/\mathbb{Q}(\sqrt{d})}(\frac{\sqrt{d}-\sqrt{m}}{2}) = (\frac{\sqrt{d}-\sqrt{m}}{2})(\frac{\sqrt{d}+\sqrt{m}}{2}) = -1$$

On the other hand, $\frac{\sqrt{d}-\sqrt{m}}{2} \in (K^\times)^2$. This implies that $\sqrt{-1} \in \mathbb{Q}(\sqrt{d})$. A contradiction is derived.

Example. If $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, then $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Proof. Suppose that -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$. Since $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, one has $K \neq \mathbb{Q}(\sqrt{d})$ and $\sqrt{m} \notin \mathbb{Q}(\sqrt{d})$. Then

$$Norm_{K/\mathbb{Q}(\sqrt{d})}(\frac{\sqrt{d}-\sqrt{m}}{2}) = (\frac{\sqrt{d}-\sqrt{m}}{2})(\frac{\sqrt{d}+\sqrt{m}}{2}) = -1$$

On the other hand, $\frac{\sqrt{d}-\sqrt{m}}{2} \in (K^\times)^2$. This implies that $\sqrt{-1} \in \mathbb{Q}(\sqrt{d})$. A contradiction is derived.

Example. If $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, then $\mathrm{Br}(U_m) = \mathrm{Br}_1(U_m)$.

Proof. Suppose that -1 and $\frac{\sqrt{d}-\sqrt{m}}{2}$ are in $(K^\times)^2$. Since $\sqrt{-1} \notin \mathbb{Q}(\sqrt{d})$, one has $K \neq \mathbb{Q}(\sqrt{d})$ and $\sqrt{m} \notin \mathbb{Q}(\sqrt{d})$. Then

$$Norm_{K/\mathbb{Q}(\sqrt{d})}(\frac{\sqrt{d}-\sqrt{m}}{2}) = (\frac{\sqrt{d}-\sqrt{m}}{2})(\frac{\sqrt{d}+\sqrt{m}}{2}) = -1$$

On the other hand, $\frac{\sqrt{d}-\sqrt{m}}{2} \in (K^\times)^2$. This implies that $\sqrt{-1} \in \mathbb{Q}(\sqrt{d})$. A contradiction is derived.

Example. If $m = 4 + rv^2$ where $r \in \mathbb{Z}$ is one of $2, -2, -3, 12, -12$ and all prime factors of v are congruent to

$$\begin{cases} \pm 1 \mod 8 & \text{when } r = 2 \\ \pm 1 \mod 12 \text{ and } v^2 \equiv 25 \mod 32 & \text{when } r = 12 \\ 1 \text{ or } 3 \mod 8 & \text{when } r = -2 \text{ and } m < 0 \\ 1 \mod 3 & \text{when } r = -3 \text{ and } m < 0 \\ 1 \mod 3 & \text{when } r = -12, \end{cases}$$

then

$$\Big(\prod_{p \le \infty} \mathfrak{U}_m(\mathbb{Z}_p)\Big)^{\mathrm{Br}_1(U_m)} = \emptyset$$

In particular, $\mathfrak{U}_m(\mathbb{Z}) = \emptyset$.

Example. If $m = 4 + 20v^2$ where all prime factors of v are congruent to $\pm 1 \mod 5$, then

$$(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}_1(U_m)} = \emptyset$$

In particular, $\mathfrak{U}_m(\mathbb{Z}) = \emptyset$.

Theorem (Loughran and Mitankin). Let $d = m - 4$. If

$$(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p)) \neq \emptyset \quad \text{but} \quad (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}_1(U_m)} = \emptyset$$

then d belongs to a finite subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ generated by

$$\{\pm 1, 2, 3, 5\}.$$

Example. If $m = 4 + 20v^2$ where all prime factors of v are congruent to $\pm 1 \mod 5$, then

$$\Big(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p)\Big)^{\mathrm{Br}_1(U_m)} = \emptyset$$

In particular, $\mathfrak{U}_m(\mathbb{Z}) = \emptyset$.

Theorem (Loughran and Mitankin). Let $d = m - 4$. If

$$\Big(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p)\Big) \neq \emptyset \quad \text{but} \quad \Big(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p)\Big)^{\mathrm{Br}_1(U_m)} = \emptyset$$

then d belongs to a finite subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ generated by

$$\{\pm 1, 2, 3, 5\}.$$

V. Integral Brauer-Manin obstruction

Theorem (Loughran and Mitankin).

$$\sharp\{m \in \mathbb{Z} : \ |m| \leq K, \ \prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} = \emptyset\}$$

$$\asymp \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

Theorem (Loughran and Mitankin).

$$\sharp\{m \in \mathbb{Z} : \ |m| \leq K, \ (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-1}$$

as $K \to \infty$.

Theorem (Loughran and Mitankin).

$$\sharp\{m \in \mathbb{Z} : \ |m| \leq K, \ \ \prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \ \text{ but } \ (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} = \emptyset\}$$

$$\asymp \sqrt{K}(log K)^{-\frac{1}{2}}$$

as $K \to \infty$.

Theorem (Loughran and Mitankin).

$$\sharp\{m \in \mathbb{Z} : \ |m| \leq K, \ \ (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(log K)^{-1}$$

as $K \to \infty$.

Proposition (Ghosh-Sarnak).

If $m = 4 + 2l^2$ where $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$, then

$$\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \quad \text{but} \quad \mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

Let $m = 4 + 2l^2w^2$ where w is an odd integer and l is a prime with $l \equiv \pm 3 \mod 8$.

If $lw \equiv \pm 4 \mod 9$, then

$$(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} \neq \emptyset.$$

Proposition (Ghosh-Sarnak).

If $m = 4 + 2l^2$ where $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$, then

$$\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p) \neq \emptyset \quad \text{but} \quad \mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

Let $m = 4 + 2l^2 w^2$ where w is an odd integer and l is a prime with $l \equiv \pm 3 \mod 8$.

If $lw \equiv \pm 4 \mod 9$, then

$$\Big(\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p)\Big)^{\mathrm{Br}(U_m)} \neq \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

VI. Brauer-Manin obstruction + reduction

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, ...$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Proposition (Colliot-Thelene, Wei and X.).

If $m = 4 + rl^2$ such that one of the following conditions holds

i) $r = 2$ and $l \geq 13$ is a prime with $l \equiv \pm 4 \mod 9$,

ii) $r = 12$ and $l \geq 37$ is a prime, $l^2 \equiv 25 \mod 32$ and $1 + 3l^2$ is not a sum of two squares (e.g. $l = 37, 43, \ldots$),

iii) $r = -2$ and $l \geq 13$ is a prime,

iv) $r = -3$ and $l \geq 17$ is a prime,

v) $r = -12$ and $l \geq 37$ is a prime,

then

$$\mathfrak{U}_m(\mathbb{Z}) = \emptyset.$$

Sketch of proof for case i).

Suppose $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$. By the reduction theory, there is $(x_0, y_0, z_0) \in \mathfrak{U}_m(\mathbb{Z})$ such that

$$3 \leq x_0 \leq y_0 \leq -z_0 \quad \text{or} \quad x_0 = 0, 1, 2 \text{ (which can be excluded)}$$

This implies that $x_0 \leq (4 + 2l^2)^{\frac{1}{3}}$. Then $x_0 < l - 2$.

One concludes $(l, x_0^2 - 4) = 1$ and the Hilbert symbol $(x_0^2 - 4, 2)_l = 1$. Therefore

$$(x_0^2 - 4, 2)_p = \begin{cases} 1 & p \neq 2 \\ -1 & p = 2 \end{cases}$$

This contradicts to the Hilbert (quadratic) reciprocity law.

Sketch of proof for case i).

Suppose $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$. By the reduction theory, there is $(x_0, y_0, z_0) \in \mathfrak{U}_m(\mathbb{Z})$ such that

$$3 \leq x_0 \leq y_0 \leq -z_0 \quad \text{or} \quad x_0 = 0, 1, 2 \text{ (which can be excluded)}$$

This implies that $x_0 \leq (4 + 2l^2)^{\frac{1}{3}}$. Then $x_0 < l - 2$.

One concludes $(l, x_0^2 - 4) = 1$ and the Hilbert symbol $(x_0^2 - 4, 2)_l = 1$. Therefore

$$(x_0^2 - 4, 2)_p = \begin{cases} 1 & p \neq 2 \\ -1 & p = 2 \end{cases}$$

This contradicts to the Hilbert (quadratic) reciprocity law.

Sketch of proof for case i).

Suppose $\mathfrak{U}_m(\mathbb{Z}) \neq \emptyset$. By the reduction theory, there is $(x_0, y_0, z_0) \in \mathfrak{U}_m(\mathbb{Z})$ such that

$$3 \leq x_0 \leq y_0 \leq -z_0 \quad \text{or} \quad x_0 = 0, 1, 2 \text{ (which can be excluded)}$$

This implies that $x_0 \leq (4 + 2l^2)^{\frac{1}{3}}$. Then $x_0 < l - 2$.

One concludes $(l, x_0^2 - 4) = 1$ and the Hilbert symbol $(x_0^2 - 4, 2)_l = 1$. Therefore

$$(x_0^2 - 4, 2)_p = \begin{cases} 1 & p \neq 2 \\ -1 & p = 2 \end{cases}$$

This contradicts to the Hilbert (quadratic) reciprocity law.

VI. Brauer-Manin obstruction + reduction

By applying this idea, we can also improve the result of Loughran and Mitankin as follows

Theorem (Colliot-Thelene, Wei and X.).

$$\sharp\{m \in \mathbb{Z} : \ |m| \leq K, \ \ (\prod_{p \leq \infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)} \neq \emptyset \ \text{ but } \ \mathfrak{U}_m(\mathbb{Z}) = \emptyset\}$$

$$\gg \sqrt{K}(logK)^{-\frac{1}{2}}$$

as $K \to \infty$.

VII. Failure of strong approximation

Theorem (Colliot-Thelene, Wei and X.).

For any finite set S of primes, the image of the natural map

$$\mathfrak{U}_m(\mathbb{Z}) \to \prod_{p \notin S} \mathfrak{U}_m(\mathbb{Z}_p)$$

is not dense.

Corollary. $\mathfrak{U}_m(\mathbb{Z})$ is not dense in $pr^f((\prod_{p\leq\infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)})$.

On the other hand, $\mathfrak{U}_m(\mathbb{Z})$ is Zariski dense in $\mathfrak{U}_m$ if

$$\begin{cases} m > 4 \text{ is not a square} \\ m \text{ is a square with a prime } p|m \text{ and } p \equiv 1 \mod 4 \\ m < 0 \end{cases}$$

VII. Failure of strong approximation

Theorem (Colliot-Thelene, Wei and X.).

For any finite set S of primes, the image of the natural map

$$\mathfrak{U}_m(\mathbb{Z}) \to \prod_{p \notin S} \mathfrak{U}_m(\mathbb{Z}_p)$$

is not dense.

Corollary. $\mathfrak{U}_m(\mathbb{Z})$ is not dense in $pr^f((\prod_{p\leq\infty} \mathfrak{U}_m(\mathbb{Z}_p))^{\mathrm{Br}(U_m)})$.

On the other hand, $\mathfrak{U}_m(\mathbb{Z})$ is Zariski dense in $\mathfrak{U}_m$ if

$$\begin{cases} m > 4 \text{ is not a square} \\ m \text{ is a square with a prime } p | m \text{ and } p \equiv 1 \mod 4 \\ m < 0 \end{cases}$$

Thank you for your attention !