Congruence of automorphic forms and arithmetic of Shimura varieties

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$$\mathtt{f} = \mathtt{q} + a_2\mathtt{q}^2 + a_3\mathtt{q}^3 + \cdots \in O_L[[\mathtt{q}]]$$

of weight 2, level $\Gamma_0(\Sigma)$, and rationality field $L \subseteq \mathbb{C}$, together with an ℓ-adic prime λ of L.

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$$\mathbb{T}^{\Sigma p} \xrightarrow{\phi_{\mathtt{f}}} O_L \to O_L/\lambda,$$

where $\phi_{\mathtt{f}}$ is the Satake homomorphism determined by $\mathtt{f}$.

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$$\alpha \colon \mathbb{Z}_\lambda[S_0(\Sigma)] \to \mathrm{H}^2(Y_0(\Sigma) \otimes \mathbb{F}_{p^2}, \mathbb{Z}_\lambda(1)),$$

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Here, that $\mathtt{f} \mod \lambda$ is non-Eisenstein means that the Galois representation associated with $\mathtt{f}$ remains irreducible after modulo λ. When this is the case, $\alpha_{\mathfrak{m}_{\mathtt{f}}}$ is same as the map

$$\alpha_{\mathfrak{m}_{\mathtt{f}}}\colon \mathbb{Z}_\lambda[S_0(\Sigma)]^{\deg=0}_{\mathfrak{m}_{\mathtt{f}}} \to \mathrm{H}^1(\mathbb{F}_{p^2}, \mathrm{H}^1(Y_0(\Sigma) \otimes \overline{\mathbb{F}}_p, \mathbb{Z}_\lambda(1))_{\mathfrak{m}_{\mathtt{f}}}).$$

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Let B be the unique quaternion algebra over $\mathbb{Q}$ ramified at $\{\infty, p\}$. Then it is well-known that there is a canonical Hecke equivariant isomorphism

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of weight 2, level $\Gamma_0(\Sigma p)$, and rationality field $L' \subseteq \mathbb{C}$, satisfying

- $a_p' = \pm 1$;
- for a certain prime λ' of L' such that $O_{L'}/\lambda' \subseteq O_L/\lambda$, $a_v' \mod \lambda' = a_v \mod \lambda$ holds for every prime number $v \nmid \Sigma p$.

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- A_0 is an abelian scheme over $\mathbb{Z}_{q^2}$ of dimension $[F^+ : \mathbb{Q}]$;
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Under such simplification, we may define the Shimura variety associated with G over $\mathbb{Z}_{q^2}$ via a certain moduli interpretation, following Rapoport–Smithling–Zhang. Namely, for every neat open compact subgroup $K^p \subseteq G(\mathbb{A}^{\infty,p})$, we have a scheme $X(K^p)$, quasi-projective and smooth over $\mathbb{Z}_{q^2}$ of relative dimension $N-1$, such that

$$X(K^p)(\mathbb{C}) \simeq G(\mathbb{Q})\backslash D_\infty \times G(\mathbb{A}^\infty)/K^pK_p,$$

where D_∞ denotes the hermitian symmetric domain of negative complex lines in $V \otimes_{F,\tau} \mathbb{C}$.

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For every locally Noetherian scheme T over $\mathbb{Z}_{q^2}$, $X(K^p)(T)$ is the set of equivalence classes of quadruples (A, i, λ, η^p) where

- A is an abelian scheme over T of dimension $N[F^+ : \mathbb{Q}]$;
- $i\colon O_F \to \mathrm{End}(A)$ is an action of O_F such that for every $a \in O_F$, the characteristic polynomial for the action of $i(a)$ on the Lie algebra of A is given by

$$(X-a)^{N-1}(X-\overline{a}) \prod_{\tau' \in \Phi \setminus \{\tau\}} (X-\tau'(a))^N;$$

- $\lambda\colon A \to A^\vee$ is a p-principal polarization under which i turns the complex conjugation into the Rosati involution;
- η^p is a K^p-level structure, that is, for a chosen geometric point t on every connected component of T, a $\pi_1(T, t)$-invariant K^p-orbit of isometries

$$\eta^p\colon V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} \mathrm{Hom}^{\lambda_0,\lambda}_{F\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}}(\mathrm{H}_1(A_{0t}, \mathbb{A}^{\infty,p}), \mathrm{H}_1(A_t, \mathbb{A}^{\infty,p}))$$

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Put $Y(K^p) \coloneqq X(K^p) \otimes_{\mathbb{Z}_{q^2}} \mathbb{F}_{q^2}$.

Denote by $Y(K^p)^{\mathrm{b}}$ the **basic locus** of $Y(K^p)$, that is, the closed locus where the $O_{F_\mathfrak{p}}$-divisible group $A[\mathfrak{p}^\infty]$ is supersingular.

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We define a moduli problem $S(K^p)$ over $\mathbb{F}_{q^2}$, such that for every locally Noetherian scheme T over $\mathbb{F}_{q^2}$, $S(K^p)(T)$ is the set of equivalence classes of quadruples $(A', i', \lambda', \eta^{p\prime})$ where

- A' is an abelian scheme over T of dimension $N[F^+ : \mathbb{Q}]$;
- $i' \colon O_F \to \mathrm{End}(A')$ is an action of O_F "with the characteristic polynomial" $\prod_{\tau' \in \Phi}(X - \tau'(a))^N$;
- $\lambda' \colon A' \to A'^\vee$ is an "i'-compatible" polarization such that $\ker \lambda'[p^\infty]$ is trivial (resp. contained in $A'[\mathfrak{p}]$ of rank q^2) if N is odd (resp. even);
- $\eta^{p\prime} \colon V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} \mathrm{Hom}^{\varpi\lambda_0,\lambda'}_{F\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}}(\mathrm{H}_1(A_{0t}, \mathbb{A}^{\infty,p}), \mathrm{H}_1(A'_t, \mathbb{A}^{\infty,p}))$ is a K^p-level structure.

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To describe $Y(K^p)^{\mathrm{b}}$, we construct the so-called basic correspondence. We fix a totally positive element $\varpi \in O_{F^+}$ that has valuation 1 at $\mathfrak{p}$ and 0 at other p-adic places.

We define a moduli problem $S(K^p)$ over $\mathbb{F}_{q^2}$, such that for every locally Noetherian scheme T over $\mathbb{F}_{q^2}$, $S(K^p)(T)$ is the set of equivalence classes of quadruples $(A', i', \lambda', \eta^{p\prime})$ where

- A' is an abelian scheme over T of dimension $N[F^+ : \mathbb{Q}]$;
- $i' \colon O_F \to \mathrm{End}(A')$ is an action of O_F "with the characteristic polynomial" $\prod_{\tau' \in \Phi}(X - \tau'(a))^N$;
- $\lambda' \colon A' \to A'^{\vee}$ is an "i'-compatible" polarization such that $\ker \lambda'[p^\infty]$ is trivial (resp. contained in $A'[\mathfrak{p}]$ of rank q^2) if N is odd (resp. even);
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We then define a moduli problem $B(K^p)$ over $\mathbb{F}_{q^2}$ that parameterizes data $(A, i, \lambda, \eta^p; A', i', \lambda', \eta^{p\prime}; \alpha)$ where

- (A, i, λ, η^p) is an object of $Y(K^p)$;
- $(A', i', \lambda', \eta^{p\prime})$ is an object of $S(K^p)$;
- $\alpha \colon A \to A'$ is an O_F-linear isogeny such that
  - $\ker \alpha[p^\infty]$ is contained in $A[\mathfrak{p}]$;
  - $\varpi \cdot \lambda = \alpha^\vee \circ \lambda' \circ \alpha$; and
  - the K^p-orbit of maps $v \mapsto \alpha_* \circ \eta^p(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta^{p\prime}$.

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(4) $S(K^p)$ is a finite copy of $\operatorname{Spec} \mathbb{F}_{q^2}$ naturally indexed by the following double coset: Let V' be the totally positive definite hermitian space over F/F^+ such that $V' \otimes_{F^+} \mathbb{A}_{F^+}^{\infty,\mathfrak{p}} \simeq V \otimes_{F^+} \mathbb{A}_{F^+}^{\infty,\mathfrak{p}}$ (and fix such an isometry). Then the index set is

$$G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty) / K^p K'_p$$

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In particular, the absolute cycle classes give a map

$$\iota_! \circ \pi^* \colon \mathrm{H}^0(S(K^p), \Lambda) \to \mathrm{H}^{2(N-1-r)}(Y(K^p), \Lambda(N-1-r))$$

for any suitable coefficient ring Λ.

Cycle class maps

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By the Hoschchild–Serre sequence, we have a short exact sequence

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If we denote by $\mathrm{H}^0(S(K^p), \Lambda)^\diamondsuit$ the kernel of the composite map

$$\gamma_N \colon \mathrm{H}^0(S(K^p), \Lambda) \to \mathrm{H}^{2(N-1-r)}(Y(K^p), \Lambda(N-1-r)) \to \mathrm{H}^{2(N-1-r)}(Y(K^p)_{\overline{\mathbb{F}}_p}, \Lambda(N-1-r)),$$

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In many cases, we are interested in the middle-degree (geometric) cohomology. More precisely,

- when $N = 2r+1$, we are interested in the map $\gamma_N \colon \mathrm{H}^0(S(K^p), \Lambda) \to \mathrm{H}^{2r}(Y(K^p)_{\overline{\mathbb{F}}_p}, \Lambda(r))$, namely, Tate cycles given by basic locus (which has been extensively studied by Xiao–Zhu);

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The question of the surjective of α_N after certain localization will be our analogue of Ribet's level raising theorem for the unitary Shimura variety $X(K^p)$.

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$$\beta\colon \mathrm{H}^1(X_0(\Sigma)_{\overline{\mathbb{Q}}_p}, \Omega_\lambda) \hookrightarrow \mathrm{H}^1(X_0(\Sigma)_{\overline{\mathbb{Q}}_p}, \mathrm{Ind}_\mathcal{P}^\mathcal{K} \mathbb{Z}_\lambda)$$
$$= \mathrm{H}^1(X_0(\Sigma p)_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_\lambda) \xrightarrow{f_* \circ i_*} \mathrm{H}^1(X_0(\Sigma)_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_\lambda)$$

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Proposition

Suppose that $\ell \nmid q \prod_{i=1}^{N}(1-(-q)^i)$. We have a canonical decomposition

$$\mathbb{Z}_\lambda[\mathcal{P}\backslash\mathcal{K}] = \bigoplus_{j=0}^{r} \Omega_{N,\lambda}^{j}$$

of $\mathbb{Z}_\lambda[\mathcal{P}\backslash\mathcal{K}/\mathcal{P}]$-modules in which $\Omega_{N,\lambda}^{j}$ is the eigenspace of $\mathcal{Q}$ with eigenvalue $\frac{-(-q)^{N+1-j}-(-q)^j-q+1}{q^2-1}$ (the differences of these eigenvalues are all invertible in $\mathbb{Z}_\ell$).

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By the above proposition, one can see easily that $\Omega^j_{N,\lambda}$ is stable under the right translation of $\mathcal{K}$; and in particular, $\Omega^0_{N,\lambda} = \mathbb{Z}_\lambda$.

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It is the direct summand $\Omega^1_{N,\lambda}$ that will play the role of the "Steinberg component" Ω_λ in the modular curve case, if one wants to formulate the correct Ihara-type lemma for level raising for the unitary Shimura variety $X(K^p)$.

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It is the direct summand $\Omega^1_{N,\lambda}$ that will play the role of the "Steinberg component" Ω_λ in the modular curve case, if one wants to formulate the correct Ihara-type lemma for level raising for the unitary Shimura variety $X(K^p)$.

It is a good exercise to show that $\Omega^1_{N,\lambda}$ is a free $\mathbb{Z}_\lambda$-module of rank $q\frac{q^{N-1}+1}{q+1}$.

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By the previous proposition, we have the composite map

$$\begin{aligned}\beta_N\colon \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}}_p}, \Omega^1_{N,\lambda}) &\hookrightarrow \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}}_p}, \mathrm{Ind}_{\mathcal{P}}^{\mathcal{K}}\,\mathbb{Z}_\lambda) \\ &= \mathrm{H}^{N-1}(\widetilde{X}(K^p)_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_\lambda) \xrightarrow{f_*\circ i_*} \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_\lambda).\end{aligned}$$

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Denote by $\mathbb{T}_N^?$ the abstract spherical unitary Hecke algebra over F/F^+ of rank N away from ?. Fix a finite set Σ of prime numbers not containing p, away from which K^p is hyperspecial. Then $\mathbb{T}_N^{\Sigma\cup\{p\}}$ acts on $X(K^p)$ via Hecke correspondences which are finite étale. Put $\mathbb{T}_{N,\lambda}^? := \mathbb{T}_N^? \otimes \mathbb{Z}_\lambda$.

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Conjecture

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^\Sigma_{N,\lambda}$ that is "non-Eisenstein" such that the Satake parameters mod $\mathfrak{m}$ *at $\mathfrak{p}$ contain q at most once. Then the map β_N is surjective after localizing at $\mathfrak{m}\cap\mathbb{T}^{\Sigma\cup\{p\}}_{N,\lambda}$.*

Relation with level raising

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Suppose that p is odd and $q = p$. Then for every maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{\Sigma\cup\{p\}}_{N,\lambda}$, the surjectivity of $(\beta_N)_{\mathfrak{m}}$ implies the surjectivity of $(\alpha_N)_{\mathfrak{m}}$.

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Put $\mathfrak{m} := \mathfrak{m}^\dagger \cap \mathbb{T}^{\Sigma\cup\{p\}}_{N,\lambda}$. Then $(\beta_N)_{\mathfrak{m}}$ is surjective; hence $(\alpha_N)_{\mathfrak{m}}$ is surjective as well.

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Then there is a unique bijection between $\mathcal{S}$ and $\mathcal{S}'$ such that π and π' correspond if and only if $\mathrm{BC}(\pi) \simeq \mathrm{BC}(\pi')$.

Explicit reciprocity law

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Consider a hermitian space V_n over F/F^+ as before but of rank n. Put $V_{n+1} \coloneqq V_n \oplus F.e$ with e of length 1. We have corresponding unitary groups G_n and G_{n+1}, with a natural embedding $G_n \hookrightarrow G_{n+1}$ as the stabilizer of e.

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$$\mathbb{1}_{\Delta X(K_n^p)} \in \mathrm{H}^{2n}(X(K_n^p)_{\mathbb{Q}_{q^2}} \times X(K_{n+1}^p)_{\mathbb{Q}_{q^2}}, \mathbb{Z}_\lambda(n))$$

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Assume n **odd** from now on for simplicity. Then there is a natural map

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of Shimura sets as well, compatible with σ_X under basic correspondences. (When n is even, one has to replace σ_S by a finite correspondence.)

The surjectivity of $(\alpha_N)_{\mathfrak{m}}$ can provide a (second) explicit reciprocity law for the diagonal cycle on the Shimura variety associated with $\mathrm{U}_n \times \mathrm{U}_{n+1}$, which is the arithmetic avatar of the Rankin–Selberg integral.

Consider a hermitian space V_n over F/F^+ as before but of rank n. Put $V_{n+1} \coloneqq V_n \oplus F.e$ with e of length 1. We have corresponding unitary groups G_n and G_{n+1}, with a natural embedding $G_n \hookrightarrow G_{n+1}$ as the stabilizer of e. Fix a pair of open compact subgroups (K_n^p, K_{n+1}^p) satisfying $K_n^p \subseteq K_{n+1}^p \cap G_n(\mathbb{A}^{\infty,p})$. Then we have a natural morphism

$$\sigma_X \colon X(K_n^p) \to X(K_{n+1}^p)$$

over $\mathbb{Z}_{q^2}$, which is finite. Denote by $\Delta X(K_n^p)$ the graph of σ_X, and by

$$\mathbb{1}_{\Delta X(K_n^p)} \in \mathrm{H}^{2n}(X(K_n^p)_{\mathbb{Q}_{q^2}} \times X(K_{n+1}^p)_{\mathbb{Q}_{q^2}}, \mathbb{Z}_\lambda(n))$$

the absolute cycle class of $\Delta X(K_n^p)_{\mathbb{Q}_{q^2}}$.

Assume n **odd** from now on for simplicity. Then there is a natural map

$$\sigma_S \colon S(K_n^p) \to S(K_{n+1}^p)$$

of Shimura sets as well, compatible with σ_X under basic correspondences. (When n is even, one has to replace σ_S by a finite correspondence.) Denote by $\Delta S(K_n^p)$ the graph of σ_S, and by

$$\mathbb{1}_{\Delta S(K_n^p)} \in \mathbb{Z}_\lambda[S(K_n^p) \times S(K_{n+1}^p)]$$

the characteristic function of $\Delta S(K_n^p)$.

Explicit reciprocity law

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Consider maximal ideals $\mathfrak{m}_n$ and $\mathfrak{m}_{n+1}$ of $\mathbb{T}^{\Sigma}_{n,\lambda}$ and $\mathbb{T}^{\Sigma}_{n+1,\lambda}$, and ideals $\mathfrak{n}_n$ and $\mathfrak{n}_{n+1}$ of $\mathbb{T}^{\Sigma\cup\{p\}}_{n,\lambda}$ and $\mathbb{T}^{\Sigma\cup\{p\}}_{n+1,\lambda}$ containing some positive powers of $\mathfrak{m}_n \cap \mathbb{T}^{\Sigma\cup\{p\}}_{n,\lambda}$ and $\mathfrak{m}_{n+1} \cap \mathbb{T}^{\Sigma\cup\{p\}}_{n+1,\lambda}$, respectively.

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Theorem (Second explicit reciprocity law)

Suppose that

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Suppose that

- *p is odd and $q = p$;*
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- *the Satake parameters* mod $\mathfrak{m}_n$ *at $\mathfrak{p}$ contain* 1 *exactly once;*
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Then

$$\exp_\lambda\left(\mathbb{1}_{\Delta X(K_n^p)}, \mathrm{H}^{2n}(X(K_n^p)_{\mathbb{Q}_{q^2}} \times X(K_{n+1}^p)_{\mathbb{Q}_{q^2}}, \mathbb{Z}_\lambda(n))/(\mathfrak{n}_n, \mathfrak{n}_{n+1})\right)$$
$$\leqslant \exp_\lambda\left(\mathbb{1}_{\Delta S(K_n^p)}, \mathbb{Z}_\lambda[S(K_n^p) \times S(K_{n+1}^p)]/(\mathfrak{n}_n, \mathfrak{n}_{n+1})\right)$$

holds. Here, for a torsion $\mathbb{Z}_\lambda$-module M and $m \in M$, $\exp_\lambda(m, M)$ denotes the smallest nonnegative integer e such that $\lambda^e m = 0$.

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*Furthermore, if $\alpha_{n+1}/\mathfrak{n}_{n+1}$ is an **isomorphism**, then the above inequality is an equality.*

Thanks!