Zeta functions of Shimura varieties.
past, present, and near future

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<u>Starting point</u>

* Eichler–Shimura 1950s:

  $X_0(N)$ modular curve

  $\rightsquigarrow \zeta(X_0(N), s) = \dfrac{\zeta(s)\cdot\zeta(s-1)}{\prod_i L(f_i, s)}$

  where $\{f_i\}$ eigenbasis of $S_2(\Gamma_0(N))$.

* Generalize to Shimura varieties:

  (G, X) Shimura data: G red gp $/\mathbb{Q}$,

  X $G(\mathbb{R})$-conj class of $h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$

  satisfying some axioms.

  For any small $K \subset G(\mathbb{A}_f)$ open cpt,

  $$\begin{array}{l} Sh_K \quad \text{s.t. } Sh_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K. \\ \big\downarrow \\ \mathrm{Spec}\, E, \quad E/\mathbb{Q} \text{ reflex field.} \end{array}$$

  $\rightsquigarrow H^i_{et}(\overline{Sh_{K,\bar{E}}}, IC(\bar{\mathbb{Q}}_\ell)) \circlearrowleft \mathrm{Gal}_E \times \mathcal{H}_K$.

  Canonical Baily-Borel compactification (proper & normal)

  (yet usually not smooth).

  For $IC(\bar{\mathbb{Q}}_\ell)$: more generally,

  can replace $\bar{\mathbb{Q}}_\ell$ (constant) with local system

  coming from rep of $G_{\mathbb{Q}_p}$.

<u>Known</u> As $\mathcal{H}_K$-mod, IH^i is semisimple & automorphic

$$IH^i = \bigoplus_{\pi_f} \underset{\mathcal{H}_K}{\underset{\circlearrowleft}{\boxed{\pi_f^K}}} \otimes \underset{Gal_E}{\underset{\circlearrowleft}{W^i(\pi_f)}}$$

$\uparrow$ fin part of autom rep of G

where $W(\pi_f) = \sum_i (-1)^i W^i(\pi_f) \in Groth(Gal_E)$

<u>Q</u> How to understand $W(\pi_f)$?

<u>Rough guess</u> $W(\pi_f)$ "$\approx$" $\left(Gal_E \xrightarrow{\rho_\pi} {}^LG_E \xrightarrow{r} GL_N(\bar{\mathbb{Q}}_\ell)\right)$

$\uparrow$ by global Langlands $\rightsquigarrow$ global L-para for π.

where $r : {}^LG_E \to GL_N(\bar{\mathbb{Q}}_\ell)$

$r|_{\hat{G}}$ is the highest wt mod of highest wt $-\mu_X$.

$\Leftrightarrow$ Hasse-Weil zeta function of Sh_K "$\approx$" $L(\pi, s, r)$.

Not always correct! (at least we must consider endoscopy.)

<u>Problem</u> $r \circ \rho_\pi$ may have different irred subreps and they may NOT show up with Serre multiplicities in $W(\pi_f)$

Suppose $W(\pi_f) \neq 0 \Rightarrow \pi_f$ is the finite part of $\pi \subset L^2_{disc}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

$\Rightarrow \pi_f \in \Pi_\psi\, G(\mathbb{A}_f)$

A-packet for a global A-para

$\psi : \mathcal{L}_{\mathbb{Q}} \times SL_2 \longrightarrow {}^LG$.

Let S_ψ = "centralizer of ψ"

For simplicity, assume S_ψ is finite abelian.

$\pi_f \rightsquigarrow \chi_{\pi_f} : S_\psi \longrightarrow \mathbb{C}^\times$ which appears in the multi formula for L^2_{disc}

$(\chi_{\pi_f} = \prod_{v\nmid\infty} \chi_{\pi_v}.)$

One uses (G, X) to canonically modify χ_{π_f}

$\rightsquigarrow$ get $\tilde{\chi}_{\pi_f} : S_\psi \longrightarrow \mathbb{C}^\times$.

Now $\mathrm{Gal}_{\mathbb{Q}} \xrightarrow{\rho_\psi} {}^L G_E \xrightarrow{r} GL_N(\bar{\mathbb{Q}}_\ell)$

Denote then by $$\underset{\mathrm{Gal}_E \times S_\psi}{\overset{}{V_\psi \circlearrowleft}} = \bigoplus_{\chi : S_\psi \to \mathbb{C}^\times} \underset{\mathrm{Gal}_E}{V_{\psi,\chi} \circlearrowleft}.$$

<u>Recipe of Langlands-Kottwitz (1970s)</u>

Conj (L-K) $$W(\pi_f) = \sum_{\substack{\psi \\ \text{s.t. } \pi_f \in \Pi_\psi(G(\mathbb{A}_f^p))}} \sum_{\chi : S_\psi \to \mathbb{C}^\times} (\pm 1) \cdot (\text{multi of } \tilde{\chi}_{\pi_f} \text{ in } \chi) \cdot [V_{\psi,\chi}]$$

± 1 determined by (ψ, χ).

<u>Thm</u> (Kisin-Shin-Zhu, in progress)

Conj (LK) true for unitary Shimura vars ass to $G = \mathrm{Res}_{F/\mathbb{Q}} U$

where $F/\mathbb{Q}$ tot real, U unitary gp w.r.t. CM $\tilde{F}/F$

of arbitrary signature.

(assuming $F \neq \mathbb{Q}$, based on [KMSW], [KMS], etc.)

Based on [KMSW] (etc.), have A-packets ass to A-parameters.

* Here a (square-integrable) A-parameter is
$$\psi = \boxplus_{i=1}^m \pi_i[d_i]$$

with π_i : conjugate self-dual cusp autom rep of $GL_{N_i}, \tilde{F}$.

$d_i \in \mathbb{Z}_{\geq 1}$ s.t. $\sum m_i d_i = \mathrm{rk}(U)$ for some m_i, $(\mathcal{L}_F \to {}^L U)$.

& (π_i, d_i) distinct.

Further know: only need ψ s.t. each π_i is regular algebraic up to a twist.

Thm (Kottwitz, Clozel, Harris-Taylor, Shin, Chenevier-Harris)

Attached to π_i, we have a rep of $\mathrm{Gal}_{\tilde F}$.

Using Bellaïche-Chenevier, we can use these Gal rep's of $\mathrm{Gal}_{\tilde F}$ attached to π_i's & d_i's to hold

$$\rho_\psi : \mathrm{Gal}_F \longrightarrow {}^c\mathcal{U} \quad (\text{arising from } {}^L\mathcal{U}.)$$

$$\rightsquigarrow \rho_\psi : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow {}^cG$$

$$\rightsquigarrow \mathrm{Gal}_E \xrightarrow{\rho_{\psi,E}} {}^cG_E \xrightarrow{r} GL_N(\bar{\mathbb{Q}}_\ell)$$

Upshot Get $V_\psi = \bigoplus_{x : S_\psi \to \mathbb{C}^\times} V_{\psi,x}$ $\circlearrowleft$ Gal_E

Thm ([KSZ], in progress) $F \neq \mathbb{Q}$.

(1) Define $W_c(\pi_f)$ using $H^i_c(Sh_K)$.

compact (monodromy) Weil gp

Then $$W_c(\pi_f) = \sum_\psi \sum_x m(\psi, x, \pi_f) \cdot [V_{\psi,x}].$$

Here $m(\psi, x, \pi_f)$ is given as predicted by L-K

$\in \{0, \pm 1\}$, only $\neq 0$ for at most one ψ.

(2) Have $$W(\pi_f) = W_c(\pi_f) \in \mathrm{Groth}(\mathrm{Gal}_E).$$

Rmks
- For PEL Shimura var not of type D,
  - Kottwitz: compact case.
  - Morel: non-compact case.
- Below all S.V. are of ab type & not PEL:
  - $F \neq \mathbb{Q}$: Similar results for $G = \mathrm{Res}_{F/\mathbb{Q}} SO$.

- $F = \mathbb{Q}$ & $G = SO(n,2)$: $W(\pi_f)$, $[Z]$ are known.
- $F = \mathbb{Q}$ & $G = U$ (unitary): open.

<u>Pf steps of (1)</u>

(A) Count points on mod p special fiber:

$K = K_p K^p$, $K_p \subset G(\mathbb{Q}_p)$ hyperspecial.

Non-PEL: need to prove a weak version of LR-Conjecture.

$S_K(\bar{\mathbb{F}}_p) \cong \coprod_{\varphi} I_\varphi(\mathbb{Q}) \backslash X_{\varphi,K}$ ($\mathrm{Gal}_{\mathbb{Q}} \times \mathcal{H}_K$-equivariant)

I_φ red gp $/\mathbb{Q}$ + $I_\varphi(\mathbb{A}_f) \curvearrowright X_{\varphi,K}$.

<u>Thm</u> (KSZ. 21) (G, X) abelian type, p as above.

Prove LR but $I_\varphi(\mathbb{Q}) \xrightarrow{\mathrm{Int}(\tau_\varphi)} I_\varphi(\mathbb{A}_f) \curvearrowright X_{\varphi,K}$

$\tau_\varphi \in I_\varphi^{ad}(\mathbb{A}_f)$

& control of τ_φ's.

Moreover, show that this is enough for Step (B).

(B) Compare results in (A) with stable trace formula on endoscopic gps of G.

(c) Use endoscopic results to spectral expansion of stable trace formula.