Lecture 10 Cohomology of automorphic bundles and étale local systems

§1 Langlands dual group

Assume for the rest of today: $Sh_K(G)$ is compact/proper

($\Longleftrightarrow$ G is anisotropic, i.e. G^{ad} does not contain a $\mathbb{G}_m/\mathbb{Q}$)

- $T \subseteq B \subseteq G_{/F} \rightsquigarrow (X^*(T), \Phi, \Delta)$, $(X_*(T), \Phi^\vee, \Delta^\vee)$

E.g. $GL_{n,F} \supseteq \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \supseteq \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}_{/F}$ $\qquad A_{n-1}$ o—o— $\cdots$ —o—o

$\rightsquigarrow$ $\Big(X^*(T) = \mathbb{Z}^n$, $\Phi = \{\alpha_i - \alpha_j \,;\, i \neq j\}$, $\Delta = \{\alpha_1 - \alpha_2, \cdots, \alpha_{n-1} - \alpha_n\}$

$$\alpha_i = (0, \cdots, \overset{i^{th}}{1}, 0, \cdots 0)$$

$X_*(T) = \mathbb{Z}^n$, $\Phi^\vee = \{\alpha_i^\vee - \alpha_j^\vee \,;\, i \neq j\}$, $\Delta = \{\alpha_1^\vee - \alpha_2^\vee, \cdots, \alpha_{n-1}^\vee - \alpha_n^\vee\}\Big)$

$G := \mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}(GL_{n,\mathbb{Q}_{p^r}})$, $\quad X^*(T) = (\mathbb{Z}^n)^r$, $\Phi = \{\alpha_i^{(s)} - \alpha_j^{(s)} \,;\, i \neq j,\ s = 1, \cdots, r\}$

Frob. permutation $\qquad \Delta = \left\{\alpha_i^{(s)} - \alpha_{i+1}^{(s)} \,;\, \begin{smallmatrix} i = 1, \cdots, n-1 \\ s = 1, \cdots, r \end{smallmatrix}\right\}$

$X_*(T) = (\mathbb{Z}^n)^r$, $\Phi^\vee = \{\alpha_i^{(s),\vee} - \alpha_j^{(s),\vee} \,;\, i \neq j,\ s = 1, \cdots, r\}$, $\Delta^\vee = \left\{\alpha_i^{(s),\vee} - \alpha_{i+1}^{(s),\vee} \,;\, \begin{smallmatrix} i = 1, \cdots, n-1 \\ s = 1, \cdots, r \end{smallmatrix}\right\}$

$U_{n,\mathbb{Q}_p}$ unramified $\quad X^*(T) = \mathbb{Z}^n \qquad \Phi = \{\alpha_i - \alpha_j \,;\, i \neq j\}$

Frob_p acts by $(a_1, \cdots, a_n) \mapsto (-a_n, \cdots, -a_1)$

$$\Delta = \{\alpha_1 - \alpha_2,\ \alpha_2 - \alpha_3,\ \cdots,\ \alpha_{n-2} - \alpha_{n-1},\ \alpha_{n-1} - \alpha_n\}$$

Frob_p

$\rightsquigarrow$ dual group $(\hat{T} \subset \hat{B} \subset \hat{G})$

s.t. $(X^*(T), \Phi, X_*(T), \Phi^\vee, \Delta, \Delta^\vee) \cong (X_*(\hat{T}), \Phi^\vee(\hat{T}), X^*(\hat{T}), \Phi(\hat{T}), \Delta^\vee(\hat{T}), \Delta(\hat{T}))$

E.g.
- $G = GL_{n/F} \rightsquigarrow \hat{G} = GL_n(\mathbb{C}) \supseteq \hat{B} = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \supseteq \hat{T} = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$
- $G = \mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} GL_n$, $\quad \hat{G} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C})$

  Frob_p cyclic permute factors
- If $G = \mathrm{Res}_{F/\mathbb{Q}} G_0$, $\quad \hat{G} = \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_\mathbb{Q}} \hat{G}_0 \circlearrowleft \mathrm{Gal}_\mathbb{Q}$

  $\cong \hat{G}_0 \times \cdots \times \hat{G}_0 \quad [F:\mathbb{Q}]$ copies.
- $G = U_n/\mathbb{Q}_p$, $\quad \hat{G} = GL_{n,\mathbb{C}} \circlearrowleft$ $\quad \begin{pmatrix} & & 1 \\ & -1 & \\ \cdots & & \end{pmatrix} {}^t(\cdot)^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ \cdots & & \end{pmatrix}^{-1}$

$$\sigma : X = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix} {}^tX \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ & & & \end{pmatrix}$$

σ sends $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ to $\begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix} \begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}^{-1}$

$$= \begin{pmatrix} t_n^{-1} & & \\ & \ddots & \\ & & t_1^{-1} \end{pmatrix}$$

Why the signs? Preserves the "positive direction" in each simple root space

e.g. $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & -a \\ & 1 \end{pmatrix}$

but $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} {}^t\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$

The Langlands dual $G_{/F}$ is ${}^LG := \hat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$

often, we replace this by $\mathrm{Gal}(E/F)$, where E = an ext'n of F over which G splits.

§2 Expected Galois cohomology of étale local system

Satake isomorphism revisited (split version)

$$\begin{tikzcd} & B \arrow[dl, "\pi"'] \arrow[dr, "i"] & \\ T & & G \end{tikzcd}$$

$$\hat{T}/\!/W \simeq \hat{G}/\mathrm{Ad}(\hat{G})$$

$$\begin{tikzcd} \mathrm{Sat}_G : \mathcal{H}k_G \arrow[r, "i^*"] \arrow[d, equal] \arrow[drr, "\simeq"] & \mathcal{H}k_B \arrow[r, "\pi_!"] & \mathcal{H}k_T \\ \mathbb{C}_c\left[{}_{G(\mathbb{Z}_p)}\backslash^{G(\mathbb{Q}_p)}/_{G(\mathbb{Z}_p)}\right] & & \mathcal{H}k_T^W \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\mathcal{H}k_T^W \cong \mathbb{C}[X_*(T)]^W \simeq \mathbb{C}[X^*(\hat{T})]^W \cong \mathbb{C}[\hat{T}/\!/W] \simeq \mathbb{C}[\hat{G}]^{\mathrm{Ad}(\hat{G})}$$

So a character of $\mathcal{H}k_G \rightsquigarrow$ a conjugacy class in $\hat{G}$.

e.g. $\mathcal{H}k_G \circlearrowright (n\text{-}\mathrm{Ind}_B^G \chi)^{G(\mathbb{Z}_p)} \rightsquigarrow \gamma_p = \begin{pmatrix} \chi_1(p) & & \\ & \ddots & \\ & & \chi_n(p) \end{pmatrix}$

unram

Kottwitz: Given a Shimura datum $(G, X) \rightsquigarrow \mu : \mathbb{G}_m \to G_{\mathbb{C}}$ def'd over E. reflex field

Can conjugate so that μ is a dominant cocharacter in $X_*(T) = X^*(\hat{T})$

$\rightsquigarrow$ highest weight rep'n $r_\mu : \hat{G} \to GL(V_\mu)$

<u>Fact</u> r_μ extends to a rep'n $r_\mu : \hat{G} \rtimes \mathrm{Gal}_{\bar{\mathbb{Q}}/E} \to GL(V_\mu)$

Given an unramified rep'n $\pi_p \rightsquigarrow \gamma_{\pi_p}\sigma \in \hat{G} \rtimes \langle\sigma\rangle$ — element in the coset $\hat{G}\sigma$, unique up to conjugation by $\hat{G}$

$\rightsquigarrow$ For a p-adic place v of E, get an unram. rep'n.

$$\mathrm{Gal}_{E_v} \longrightarrow \hat{G} \rtimes \mathrm{Gal}_{\bar{\mathbb{Q}}_p/E_v} \longrightarrow GL(V_\mu)(\bar{\mathbb{Q}}_\ell)$$

$$\sigma_v = \sigma_p^m \longmapsto (\gamma_{\pi_p}\sigma)^m \longmapsto r_\mu\left((\gamma_{\pi_p}\sigma)^m\right)$$

<u>Twist</u>: $\chi : \mathrm{Gal}_{E_v} \longrightarrow \bar{\mathbb{Q}}_\ell^\times$

$$\sigma_v \longmapsto (\sqrt{p})^{\dim Sh_G \cdot [E_v : \mathbb{Q}_p]}$$

<u>Expectation</u>: Given a "nice" automorphic rep'n π,

$$H^{mid}_{ét}\left(Sh_G(K_f), \underline{V}(\lambda)\right) = \bigoplus_{\pi} (\pi_f^{K_f})^{\oplus m(\pi)} \otimes W(\pi) \quad \circlearrowleft \mathrm{Gal}_E$$

some ℓ-adic local system

$W(\pi) \simeq V_\mu$ is unramified at where K_p is "hyperspecial" ($= G(\mathbb{Z}_p)$) &

$$\mathrm{Gal}_{E_v} \longrightarrow GL(V_\mu)$$

$$\sigma_v \longmapsto r_\mu\left((\gamma_{\pi_p}\sigma)^m\right) \otimes \chi(\sigma_v).$$

§3 Cohomology of automorphic vector bundle

<u>Recall</u> Given a rep'n W of $Q \rightsquigarrow$ locally free coherent sheaf $\underline{W}$ of $Sh_K(G)$.

<u>Goal</u>: Compute $H^i(Sh_K(G), \underline{W})$ (e.g. $H^i(M_K^*, \omega^k)$)

<u>Fact</u>: $\Omega^1_{\check{D}} \cong \Omega^1_{G_{\mathbb{C}}/Q} = \underline{(\mathfrak{p}^+)^*}$ (for the adjoint Q-action)

$$\Rightarrow \Omega^1_{Sh_K(G)} = \underline{\mathfrak{p}^-}$$

Assume that $Sh_K(G)$ is compact.

Consider the resolution

d

$$0 \to \mathcal{O}^{hol}_{Sh_K(G)} \to \underline{C}^\infty \xrightarrow{\bar\partial} \bar\Omega^1 \xrightarrow{\bar\partial} \bar\Omega^2 \to \cdots \to \bar\Omega^{\dim Sh_G(K_f)} \to 0$$

$$\bar\Omega^1 = \underline{C}^\infty \otimes (\underline{\mathfrak{p}}^-)^*, \quad \bar\Omega^2 = \underline{C}^\infty \otimes \Lambda^2(\underline{\mathfrak{p}}^-)^*$$

b/c we've learned $\Omega^1 \cong (\underline{\mathfrak{p}}^+)^* \Rightarrow \bar\Omega^1 \cong (\underline{\mathfrak{p}}^-)^*$

$\rightsquigarrow$ tensor with $\underline{W}$ $\rightsquigarrow$ resolution of $\underline{W}^{hol}$

$$\Rightarrow H^*(Sh_G(K_f), \underline{W}) = H^*\left(\left(C^\infty\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K\right)\otimes W \to C^\infty\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K\right)\otimes(\mathfrak{p}^-)^*\otimes W \to \cdots\right)^{K_\infty}\right)$$

$$= H^*\left(\bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes \left(\pi_\infty \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{p}^-), W)\right)^{K_\infty}\right)$$

$$=: \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes \underbrace{H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes W)}_{(\mathfrak{g}, K_\infty)\text{-cohomology.}}$$

* <u>What is $(\mathfrak{g}, K_\infty)$-cohomology?</u> (E.g. [Borel-Wallach, Chap 1])

① Lie algebra cohomology

$\mathfrak{g}$ Lie algebra $\curvearrowright V$ vector space.

Define $C^q(\mathfrak{g}; V) := \mathrm{Hom}(\Lambda^q(\mathfrak{g}), V)$

$d: C^q(\mathfrak{g}; V) \to C^{q+1}(\mathfrak{g}; V)$ is given by

$$df(x_0, \dots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \dots, \hat{x}_i, \dots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_q)$$

The cohomology is $H^*(\mathfrak{g}; V)$ with $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}=0}$.

② Relative Lie algebra cohomology

$\mathfrak{k} \subseteq \mathfrak{g}$ Lie subalgebra $\curvearrowright V$ vector space

Define $C^q(\mathfrak{g}, \mathfrak{k}; V) := \mathrm{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V) \hookrightarrow C^q(\mathfrak{g}; V)$

$$\mathrm{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V) = \left\{ f: \mathfrak{g} \to V \text{ s.t. } \begin{array}{l} f(x_1, \dots, x_q) \text{ depends only on each } x_i \in \mathfrak{g}/\mathfrak{k} \\ \sum_i f(x_1, \dots, [x, x_i], \dots, x_q) = x \cdot f(x_1, \dots, x_q) \text{ for } x \in \mathfrak{k} \end{array} \right\}$$

Can show that d sends $C^q(\mathfrak{g}, \mathfrak{k}; V)$ into $C^{q+1}(\mathfrak{g}, \mathfrak{k}; V)$

$\rightsquigarrow$ The cohomology is $H^*(\mathfrak{g},\mathfrak{k};V)$.

③ $(\mathfrak{g},K)$-cohomology and $(\mathfrak{q},K)$-cohomology.

Let $\mathfrak{g}$ be a Lie alg, not necessarily reductive (so either $\mathfrak{g}$ or $\mathfrak{q}$).

$K :=$ max'l compact subgroup $(K = K_\infty) \supseteq K^\circ =$ connected component of K

Assume that K is reductive.

Define $C^q(\mathfrak{g},K;V) := \mathrm{Hom}_K(\Lambda^q(\mathfrak{g}/\mathfrak{k}),V) \cong C^q(\mathfrak{g},\mathfrak{k};V)^{K/K^\circ}$

So we have $H^*(\mathfrak{g},K;V) := H^*(\mathfrak{g},\mathfrak{k};V)^{K/K^\circ}$

<u>Theorem</u>. $H^*(Sh_G(K_f),\underline{W}) = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes H^*(\mathfrak{q},K_\infty;\pi_\infty\otimes W)$

<u>Example</u> $G = GL_2$, π_∞ discrete series $\pi_k \oplus \pi_{k+2} \oplus \cdots$

$\rightsquigarrow$ $\left\{\begin{array}{l} H^0(\mathfrak{q},K_\infty;\pi_\infty\otimes W_{-k}) = 1\text{-dim'l} \\ H^1(\mathfrak{q},K_\infty;\underbrace{\overline{\pi}_\infty\otimes W_{k-2}}_{\parallel}) = 1\text{-dim'l}\end{array}\right.$

$$\Big(\cdots\cdots \oplus \underbrace{\overline{\pi}_{k+2}\otimes\chi_{k-2}}_{wt=-4} \oplus \underbrace{\overline{\pi}_k\otimes\chi_{k-2}}_{wt=-2}\Big) \otimes \Big(\mathbb{C} \to \overset{\overset{wt=2}{\downarrow}}{\mathfrak{p}^+}\Big)$$

<u>Example</u>: F = totally real field. $G = \mathrm{Res}_{F/\mathbb{Q}} PGL_2$

weight $\underline{k} = (k_\tau)_{\tau\in\mathrm{Hom}(F,\mathbb{R})}$, k_τ all even.

$n_0 := \#\{\tau\in\mathrm{Hom}(F,\mathbb{R})\ ;\ k_\tau \le 0\}$

$$H^*(Sh_G(K_f),\omega^{k_\tau}) \underset{\substack{\uparrow \\ \text{multiplicity one} \\ \text{holds for } PGL_n}}{=} \bigoplus_\pi (\pi_f)^{K_f} \otimes \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{R})} \underbrace{H^*(\mathfrak{q}_\tau,K_{\infty,\tau};\pi_\tau\otimes\chi_{-k_\tau})}_{\substack{\uparrow \\ \text{Concentrated in one degree \& dim 1.} \\ \text{in degree 0 if } k_\tau\ge 2 \\ \text{in degree 1 if } k_\tau\le 0.}}$$

So $H^*(Sh_G(K_f),\omega^{k_\tau})$ is concentrated in degree n_0

When V is an algebraic $\mathbb{C}$-rep'n of G (def'd over a number field)

$$\rightsquigarrow \quad \underline{V} = \text{locally constant sheaf assoc. to } V$$

$$\downarrow$$

$$Sh_G(K_f) \qquad \bullet \quad \mathcal{V} := \underline{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_{Sh_G(K_f)} \leftarrow \text{de Rham local system}$$

$$(1 \otimes \nabla_{Sh_G(K_f)})$$

Get

$$H^*_{Betti}\left(Sh_G(K_f)(\mathbb{C}), \mathcal{L}^B(V)\right) = \mathbb{H}^*\left(Sh_G(K_f), \mathcal{V} \xrightarrow{1\otimes\nabla} \mathcal{V}\otimes\Omega^1_{Sh_G(K_f)} \to \cdots \to \mathcal{V}\otimes\Omega^d_{Sh_G(K_f)}\right)$$

$$= \mathbb{H}^*\left(Sh_G(K_f), \underbrace{\mathcal{V}\otimes\Omega^{\cdot,\cdot}_{Sh_G(K_f)}}_{C^\infty\text{- resolution}}\right)$$

$$= \bigoplus_\pi \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes H^*\left(\left(\pi_\infty \otimes \mathrm{Hom}\left(\wedge^\cdot(\mathfrak{p}^+)\otimes\wedge^\cdot(\mathfrak{p}^-), V\right)\right)^{K_\infty}\right)$$

$$= \bigoplus_\pi \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes H^*\left(\mathfrak{g}, K_\infty; \pi_\infty \otimes V\right)$$

<u>Langlands observation</u>: $\dim H^{mid}\left(\mathfrak{g}, K_\infty; \pi_\infty\otimes V(\lambda)\right) = \dim\left(\text{rep of } \hat{G} \text{ of highest wt } \mu\right)$

or rather this is how Langlands discovered the dual group.