Lecture 2 Representations over nonarchimedean local fields

<u>Recall</u> $$\mathcal{A}_{cusp}(GL_2(\mathbb{Q});\omega) = \bigoplus_{\pi} \pi = \bigcup_{K_f} \bigoplus_{\pi} \pi_\infty \otimes \bigotimes_{p} \pi_p^{K_p}.$$

$K_f = \prod_p K_p$ with $K_p \subseteq GL_2(\mathbb{Q}_p)$ open compact & for all but finitely many p, $K_p = GL_2(\mathbb{Z}_p)$

A key step is to understand the "majority" case:

$$\pi_p^{GL_2(\mathbb{Z}_p)} \circlearrowleft \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p)).$$

§1. A digression on principal series

Let F_v be a finite ext'n of $\mathbb{Q}_p$, $F_v \supseteq \mathcal{O}_v \twoheadrightarrow \mathcal{O}_v/(\varpi_v) = k_v \simeq \mathbb{F}_{q_v}$

$G = GL_n(F_v)$ a reductive group $/F_v$

$\rotatebox{90}{$\subseteq$}$

$B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ a Borel subgroup $/F_v$ $\supseteq N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ = max'l unipotent subgp

$\rotatebox{90}{$\subseteq$}$

$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ a max'l torus $/F_v$

- Given a character $\chi : B(F_v) \twoheadrightarrow T(F_v) \longrightarrow \mathbb{C}^\times$

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \longmapsto \chi_1(t_1)\cdots\chi_n(t_n), \quad \text{for } \chi_i : F_v^\times \longrightarrow \mathbb{C}^\times.$$

Define $$\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi := \left\{ f : G(F_v) \longrightarrow \mathbb{C}, \begin{array}{l} f \text{ is locally constant} \\ f(bg) = \chi(b) f(g) \quad \forall b \in B(F_v) \end{array} \right\}$$

Its subquotients are called <u>principal series</u> for G.

Sometimes automorphic people like to keep unitarity.

- modulus character $\delta_B : T(F_v) \longrightarrow \mathbb{C}^\times$

$$t \longmapsto |\det(\mathrm{Ad}_t ; \mathfrak{n})|_v \qquad |\varpi_v|_v = q_v^{-1}$$

E.g. $G = GL_n(F_v)$, $$\delta_B \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = |t_1|^{n-1} \cdot |t_2|^{n-3} \cdot \cdots \cdot |t_n|^{1-n}$$

($n=2$, $\mathrm{Ad}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \circlearrowright \mathfrak{n}$ by mult by t_1/t_2 $\rightsquigarrow$ $\delta_B\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = |t_1/t_2|$

Define $n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)}\chi := \mathrm{Ind}_{B(F_v)}^{G(F_v)}\chi\cdot\delta_B^{\frac{1}{2}}$.

Then if χ is unitary, so is $n\text{-}\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi$.

Fact for $G = GL_n(F_v)$, if for every $i \neq j$, $\chi_i \neq \chi_j|\cdot|$

then $n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)}\chi$ is irreducible.

In this case, for every $w \in S_n$, ${}^{w}\chi := \chi_{w(1)}\otimes\cdots\otimes\chi_{w(n)}$

then $n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)}\chi \simeq n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)}{}^{w}\chi$

$n=2$: $n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)}(\mathbb{1}\otimes|\cdot|) \simeq \mathrm{Ind}_{B(F_v)}^{G(F_v)}(|\cdot|^{\frac{1}{2}}\otimes|\cdot|^{\frac{1}{2}})$

$\rightsquigarrow$ $0 \to \mathbb{1} \to \mathrm{Ind}_{B(F_v)}^{G(F_v)}\mathbb{1} \to \mathrm{St}_G \to 0$

$0 \to \mathrm{St}_G \to \mathrm{Ind}_{B(F_v)}^{G(F_v)}\delta_B \to \mathbb{1} \to 0$

§2 Unramified principal series

Assume that G is unramified over F_v, i.e. G is quasi-split (meaning admits a Borel$/F_v$) and G splits over an unramified ext'n of F_v.

In this case G extends to a reductive group $\mathcal{G}/\mathcal{O}_{F_v}$ (small issue with uniqueness in general)

Define $K_v := \mathcal{G}(\mathcal{O}_v) = GL_n(\mathcal{O}_v)$ = hyperspecial subgroup

$\mathcal{H}k_G := \mathcal{H}(G, K_v) = \mathbb{C}_c[K_v\backslash G/K_v]$ unramified Hecke algebra

$= \{ f : K_v\backslash G/K_v \longrightarrow \mathbb{C} \text{ compact support} \}$

↳ an algebra under convolution with $\mu(K_v) = 1$.

Theorem (Satake) $\mathcal{H}k_G$ is a commutative algebra (will give a description soon)

Cor: If π_v is an irred. adm. rep'n of $G(F_v)$ and if $\pi_v^{K_v} \neq 0$. (called spherical rep'ns)

then $\dim \pi_v^{K_v} = 1$ and $\mathcal{H}k_G$ acts on $\pi_v^{K_v}$ by a character.

Proof: Recall that $\mathcal{H}k_G$ acts on $\pi_v^{K_v}$ & $\pi_v^{K_v}$ is finite dim'l (by admissibility)

π_v irred $\Rightarrow$ $\pi_v^{K_v}$ as a (fin. dim'l) $\mathcal{H}k_G$-module is irreducible

$$\Rightarrow \dim \pi_v^{K_v} = 1 \text{ \& } \mathcal{H}k_G \text{ acts by a character.}$$

Fact: π_v an irreducible admissible rep'n of $G(F_v)$, then

$$\pi_v^{K_v} \neq 0 \iff \pi_v \text{ is a subrep'n of an } \underline{\text{unramified}} \text{ principal series}$$

$$\left(\text{usually, } \pi_v = n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi\right)$$

$\uparrow$ means $\chi = \chi_1 \otimes \cdots \otimes \chi_n$

$\chi_i : F_v^\times \twoheadrightarrow F_v^\times / \mathcal{O}_v^\times \longrightarrow \mathbb{C}^\times$

i.e. $\chi_i|_{\mathcal{O}_v^\times} = \text{triv}.$

Some explanation of "$\Leftarrow$": Cartan decomposition $G(F_v) = B(F_v) \cdot K_v$

$$\text{If } \varphi \in \left(n\text{-}\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi\right)^{K_v} \rightsquigarrow \varphi : G(F_v) \longrightarrow \mathbb{C}$$

$$\text{s.t. } \varphi(bk) = \chi(b)\, \delta_B^{\frac{1}{2}}(b)\, \underbrace{\varphi(k)}_{\varphi(1)}$$

So φ is uniquely determined by $\varphi(1)$

Moreover, $b \in B(F_v)$ above is well-def'd up to $B(\mathcal{O}_v)$

χ is trivial here as χ is unramified.

$$\text{So } \left(\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi\right)^{K_v} \xrightarrow{\sim} \mathbb{C} \quad \text{is 1-dim'l.}$$

$$\varphi \longmapsto \varphi(1)$$

Definition: When $\varphi(1) = 1$, this φ is called the *spherical vector*.

Back to lecture 1. $\mathcal{A}_{cusp}(GL_2(\mathbb{Q}), \omega) = \bigoplus_{\pi} \pi = \bigoplus_{\pi} \bigotimes_v{}' \pi_v$

Here the restricted product means:

① for all but finitely many v, π_v is unramified principal series

(yes b/c $\pi^K \neq 0 \rightsquigarrow \pi_p^{GL_2(\mathbb{Z}_p)} \neq 0$ for all but finitely many p.

② Fix spherical vectors $x_p^{*\circ} \in \pi_p^{GL_2(\mathbb{Z}_p)}$ for all but finitely many p.

$$\text{Then } \bigotimes_v{}' \pi_v = \bigcup_{I} \left(\bigotimes_{v \in I} \pi_v\right) \otimes \mathbb{C} \cdot \bigotimes_{v \notin I} x_v$$

I finite set of places containing all those p's not chosen x_p.

Property: For $K = \prod_p K_p \subseteq GL_2(\mathbb{A}_f)$ open compact, $\left(\bigotimes_v{}' \pi_v\right)^K = \bigotimes_v (\pi_v)^{K_v}$

Example: $G = GL_n(F_v)$, $\chi = \chi_1 \times \cdots \times \chi_n : T(F_v) \longrightarrow \mathbb{C}^\times$

$\alpha_i := \chi_i(\varpi_v)$ and $\chi_i|_{\mathcal{O}_v^\times} = \text{triv}.$

For $g_r = \begin{pmatrix} \overbrace{\varpi \ \ddots \ \varpi}^{r} & & \\ & 1 & \\ & & \ddots \ 1 \end{pmatrix}$ & $T_r = \mathbb{1}_{K_v g_r K_v}$ & φ the spherical vector

$$T_r(\varphi)(1) = \int_{GL_n(F_v)} \mathbb{1}_{K g_r K_v}(g) \underbrace{(\pi(g)\varphi)(1)}_{\substack{\shortparallel \\ \varphi(g)}}$$

in general

Need to compute for each coset $K_v g_r K_v / K_v$

$$\overset{\substack{r=2\\ n=3}}{=} \sum_{a,b\in\mathbb{F}_p} \varphi\begin{pmatrix}\varpi & & a\\ & \varpi & b\\ & & 1\end{pmatrix} + \sum_{a\in\mathbb{F}_p}\varphi\begin{pmatrix}\varpi & a & \\ & 1 & \\ & & \varpi\end{pmatrix} + \varphi\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix}$$

$$= q^2\,(q^{-1}\alpha_1\cdot\alpha_2) + q\cdot(q^{-1}\alpha_1\cdot q\alpha_3) + \alpha_2\cdot q\alpha_3$$

$$= q(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)$$

$$= q^{\frac{1}{2}r(n-r)}\sum_{a_1<\cdots<a_r}\alpha_{a_1}\cdots\alpha_{a_r}$$

Summary In this case, $\mathcal{H}k_G$ action on $\left(n\text{-}\mathrm{Ind}_{B_n(F_v)}^{GL_n(F_v)}\chi\right)^{K_v}$ is determined by T_r acts by $q^{\frac{1}{2}r(n-r)}\cdot$ (elementary r^{th} symmetric polynomial in $\chi_1(p),\dots,\chi_n(p)$).

§3 Satake isomorphism

Assume further that G splits over F_v

E.g. $G = GL_n(F_v) \supseteq B = \begin{pmatrix} * & * \\ 0 & \ddots \ * \end{pmatrix} \supseteq T.$ $\qquad U = \begin{pmatrix} 1 & * \\ 0 & \ddots \ 1 \end{pmatrix}$

$W := N_G(T)/T = \text{Weyl group} = S_n$

Satake isomorphism:

$$\begin{tikzcd} & B \arrow[ld, "\pi"'] \arrow[rd, "i"] & \\ T & & G \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sat}: C_c^\infty(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{C}) \arrow[r, "i^*"] \arrow[rrd, "\cong"'] & C_c^\infty(B(\mathcal{O}_v)\backslash B(F_v)/B(\mathcal{O}_v), \mathbb{C}) \arrow[r, "\pi_!"] & C_c^\infty(T(F_v)/\!/T(\mathcal{O}_v), \mathbb{C}) \\ f \arrow[u, phantom, sloped, "\in"] & & C_c^\infty(T(F_v)/\!/T(\mathcal{O}_v), \mathbb{C})^W \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

algebraic isom.
compatible w/ convolution

Explicitly, $(\mathrm{Sat}(f))(t) := \delta_B^{\frac{1}{2}}(t)\int_{U(F_v)} f(tu)\,du = \delta_B^{-\frac{1}{2}}(t)\int_{U(F_v)} f(ut)\,du$

<u>Example</u> : $G = GL_n$

$$C_c^\infty\left({}_{GL_n(\mathcal{O}_v)}\backslash^{GL_n(F_v)}/_{GL_n(\mathcal{O}_v)}, \mathbb{C}\right) \overset{\mathrm{Sat}}{\cong} C_c^\infty\left(T(F_v)/T(\mathcal{O}_v), \mathbb{C}\right)^W$$

$$\cong C_c^\infty\left((F_v^\times/\mathcal{O}_v^\times)^n, \mathbb{C}\right)^W$$

$$= \mathbb{C}[X_1^{\pm1}, \cdots, X_n^{\pm1}]^W$$

$$= \mathbb{C}[\sigma_1, \cdots, \sigma_{n-1}, \sigma_n^{\pm1}] \quad \text{where } \sigma_i = \sum_{a_1<\cdots<a_i} X_{a_1}\cdots X_{a_i}$$

$$\mathbb{1}_{G(\mathcal{O}_v)\begin{pmatrix}\overbrace{\varpi \ \ddots \ \varpi}^{r} & \\ & 1 \ \cdots \ 1\end{pmatrix}G(\mathcal{O}_v)} \longmapsto q^{\frac{1}{2}r(n-r)}\,\sigma_r$$

In terms of earlier computation, the eigenvalue of $\mathrm{Sat}^{-1}(\sigma_r)$ on the spherical vector is the r^{th} symmetric polynomial in $\chi_1(p), \cdots, \chi_n(p)$

§4 (local) Langlands correspondence for GL_n

<u>Langlands correspondence</u> :

$$\begin{tikzcd}
\left\{\text{(normalized) cuspidal eigennewforms of wt } k \geq 2\right\} \arrow[d, leftrightarrow] & \\
\left\{\begin{array}{l}\text{cuspidal automorphic representations of } GL_2(\mathbb{A}_{\mathbb{Q}}) \\ \text{that have regular algebraic weights}\end{array}\right\} \arrow[r, leftrightarrow, "\forall p"] \arrow[d, "\text{For every prime } \ell"', "\pi \mapsto \pi_\ell"] & \left\{\begin{array}{l}\text{cont. irred. rep'n } \rho : Gal_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p) \\ \text{unramified at all but finitely many } \ell. \\ \text{de Rham at } p \text{ of regular HT wts}\end{array}\right\} \arrow[d, "\ell \neq p", "\rho \mapsto \rho|_{Gal_{\mathbb{Q}_\ell}}"'] \\
\left\{\text{irreducible adm. rep'n of } GL_2(\mathbb{Q}_\ell)\right\} \arrow[r, leftrightarrow, "\text{Local Langlands correspondence}"] & \left\{\text{2-dim'l rep'n } \rho_\ell : Gal_{\mathbb{Q}_\ell} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)\right\}
\end{tikzcd}$$

(should be Weil-Deligne rep'ns instead.)

(1) Local Langlands known for GL_n.

(2) When π_ℓ is spherical, i.e. $\pi_\ell^{GL_n(\mathbb{Z}_\ell)} \neq 0$

$$\overset{\circlearrowright}{\mathcal{H}k_G} \overset{\mathrm{Sat}}{\cong} \mathbb{C}[\sigma_1, \dots, \sigma_{n-1}, \sigma_n^{\pm 1}]$$

If the $\mathrm{Sat}^{-1}(\sigma_i)$-eigenvalue is a_i, then

$$\exists\, \gamma_{\pi_\ell} \in Gl_n(\bar{\mathbb{Q}}_p) \text{ s.t. } \det(x \cdot I_n - \gamma_{\pi_\ell}) = x^n - a_1 x^{n-1} + \dots + (-1)^n a_n$$

Define $\rho_\ell : Gal_{\mathbb{Q}_\ell} \to Gl_n(\bar{\mathbb{Q}}_p)$ unramified

$$\begin{tikzcd} Gal_{\mathbb{Q}_\ell} \arrow[rr] \arrow[dr, two heads] & & Gl_n(\bar{\mathbb{Q}}_p) \\ & Gal_{\mathbb{F}_\ell} \arrow[ur] & \end{tikzcd}$$

$$\rho(\underbrace{\mathrm{Frob}_\ell}_{\uparrow \text{geom. Frob.}}) = \gamma_{\pi_\ell} \cdot \begin{pmatrix} \ell^{\frac{n-1}{2}} & & \\ & \ddots & \\ & & \ell^{\frac{n-1}{2}} \end{pmatrix}$$

(3) $\pi \longleftrightarrow \rho$. ρ is determined uniquely by ρ_ℓ's (up to conjugation) for all unramified ℓ's (by Chebotarov density)

Example: Galois rep'ns associated to modular forms

Fix an isom. $\mathbb{C} \simeq \bar{\mathbb{Q}}_p$

f weight k, level $\Gamma_1(N)$, character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

$\longleftrightarrow$ π autom. rep'n of $Gl_2(\mathbb{A})$. central character $\chi \cdot |\cdot|^{k-2} =: \omega$

$$\underset{\substack{\circlearrowright \\ T_\ell \quad \ell \nmid Np}}{S_k(\Gamma_1(N), \chi)} \longrightarrow \mathcal{A}_{cusp}(\underset{\circlearrowright}{Gl_2(\mathbb{Q})}; \omega)^{\widehat{\Gamma_1}(N)}$$

$$\mathbb{1}_{Gl_2(\mathbb{Z}_\ell)\begin{pmatrix} 1 & \\ & \ell^{-1} \end{pmatrix} Gl_2(\mathbb{Z}_\ell)}$$

$$\mathbb{1}_{Gl_2(\mathbb{Z}_\ell) \cdot \begin{pmatrix} \ell^{-1} & \\ & \ell^{-1} \end{pmatrix}} \text{ acts by } \omega(\ell) \cdot \ell^{k-2}$$

Associate Galois representations:

Normalization 1: $\rho_f^n : Gal_{\mathbb{Q}} \to Gl_2(\bar{\mathbb{Q}}_p)$

s.t. for $\ell \nmid Np$, $\mathrm{charpoly}(\rho_f^n(\underset{\uparrow \text{geom. Frob}}{\mathrm{Frob}_\ell})) = x^2 - a_\ell(f) x + \ell^{k-1} \chi(\ell)$

Normalization 2: $\mathrm{tr}(\gamma_{\pi_\ell}) = \ell^{-\frac{1}{2}} \mathrm{eval}\left(\mathbb{1}_{Gl_2(\mathbb{Z}_\ell)\begin{pmatrix} 1 & \\ & \ell \end{pmatrix} Gl_2(\mathbb{Z}_\ell)}\right) = a_\ell \cdot \omega(\ell)^{-1} \ell^{\frac{3}{2}-k}$

$\det(\gamma_{\pi_\ell}) = \omega(\ell)^{-1} \ell^{2-k}$

$$\rightsquigarrow \rho_f : Gal_{\mathbb{Q}} \to Gl_2(\bar{\mathbb{Q}}_p) \text{ s.t. } \rho_f(\mathrm{Frob}_\ell) \sim \gamma_{\pi_\ell} \cdot \underbrace{\begin{pmatrix} \ell^{\frac{1}{2}} & \\ & \ell^{\frac{1}{2}} \end{pmatrix}}$$

$$\rho_f^n = \rho_f \otimes \omega \cdot \chi_{cycl}^{k-1}$$

one must multiply with this, o/w ρ_f is not defined, as $\chi_{cycl}^{\frac{1}{2}}$ does not exist.

§5 old form / new form theory explained

$\ell \nmid N$

$$\begin{tikzcd}
S_k(\Gamma_0(N)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & S_k(\Gamma_0(\ell N)) \arrow[d] \\
f(z) \arrow[r, mapsto] & \begin{matrix} f(z) \\ f(\ell z) \end{matrix} \\
\mathcal{A}_{cusp}(GL_2(\mathbb{Q}),\mathbb{1})^{\widehat{\Gamma}_0(N)} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{A}_{cusp}(GL_2(\mathbb{Q}),\mathbb{1})^{\widehat{\Gamma}_0(N\ell)}
\end{tikzcd}$$

$$\mathcal{A}_{cusp}(GL_2(\mathbb{Q}),\mathbb{1})^{\widehat{\Gamma}_0(N)} = \bigoplus_{\pi} \pi^{\widehat{\Gamma}_0(N)} = \bigoplus_{\pi} \pi_\infty \otimes \bigotimes_{v \nmid \ell} \pi_v^{K_v^N} \otimes \pi_\ell^{GL_2(\mathbb{Z}_\ell)} \rightrightarrows \bigoplus_{\pi} \pi_\infty \otimes \bigotimes_{v \nmid \ell} \pi_v^{K_v^N} \otimes \pi_\ell^{Iw_\ell}$$

$$\mathcal{A}_{cusp}(GL_2(\mathbb{Q}),\mathbb{1})^{\widehat{\Gamma}_0(N\ell)} = \bigoplus_{\pi} \pi^{\widehat{\Gamma}_0(N\ell)}$$

$$Iw_\ell \leftarrow \begin{pmatrix} \mathbb{Z}_\ell^\times & \mathbb{Z}_\ell \\ \ell\mathbb{Z}_\ell & \mathbb{Z}_\ell^\times \end{pmatrix}$$

<u>Old forms</u> $\longleftrightarrow$ π s.t. π_ℓ unram. PS.

$$\dim \pi_\ell^{Iw_\ell} = 2 \quad \& \quad \left(\pi_\ell^{GL_2(\mathbb{Z}_\ell)}\right)^{\oplus 2} \xrightarrow{\cong} \pi_\ell^{Iw_\ell}$$

$$(x,y) \longmapsto \left(x - \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} y\right)$$

New forms $\longleftrightarrow$ π s.t. $\pi_\ell^{GL_2(\mathbb{Z}_\ell)} = 0$ but $\pi_\ell^{Iw_\ell} \neq 0$ (& central char = triv)

In this case, $\pi_\ell = St_{GL_2}$.

<u>Upshot</u> (special for GL_2)

For every irred. sm. adm. rep'n π_p of $GL_2(\mathbb{Q}_p)$,

$\exists!$ minimal n $\rightsquigarrow$ $Iw_{p^n} = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$

character $\chi : (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \to \mathbb{C}^\times$, viewed as a character of Iw_{p^n}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \chi(d)$$

s.t. $\pi_p^{Iw_{p^n},\chi} := \{ x \in \pi_p ,\ g(x) = \chi(g)\cdot x \quad \forall g \in Iw_{p^n} \} \neq 0$

& In this case, $\dim \pi_p^{Iw_{p^n},\chi} = 1$.

Then for each $\pi \rightsquigarrow K_f := \prod_p Iw_{p^{n_p}}$, χ char of $\widehat{\mathbb{Z}}^\times / (1 + \prod p^{n_p} \widehat{\mathbb{Z}})^\times$

$\pi_f := \bigotimes_p{}' \pi_p \rightsquigarrow \pi_f^{K_f,\chi}$ is 1-dim'l.

This is why newforms $\longleftrightarrow \pi$.