# Lecture 3 $(\mathfrak{g}, K)$-modules

## §1 Representation theory of $\mathfrak{gl}_2$

$$\mathfrak{gl}_{2,\mathbb{C}} = \mathbb{C}\left\langle E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$[H,E] = 2E, \quad [H,F] = 2F, \quad [E,F] = H$$

If $V$ is a rep'n of $\mathfrak{gl}_2$ on which $H$ and $Z$ act semisimply, then

(WLOG, after decomposing + twist, $Z$ acts trivially.)

$V = \bigoplus_a V_a$ direct sum over all characters of $H \longleftrightarrow a \in \mathbb{C}$

if $v_a \in V_a$, then $Hv = a \cdot v$

then $E : V_a \longrightarrow V_{a+2}$ (b/c $H(Ev_a) = [H,E]v_a + E(Hv_a) = 2Ev_a + a \cdot Ev_a = (a+2)Ev_a$

$F : V_a \longrightarrow V_{a-2}$

So we often write this as

$$\cdots \; V_{a-2} \underset{F}{\overset{E}{\rightleftarrows}} V_a \underset{F}{\overset{E}{\rightleftarrows}} V_{a+2} \; \cdots$$

Example: $\mathfrak{gl}_2 \curvearrowright \mathrm{Sym}^k \mathbb{C}^2$

$$\mathbb{C}v_{-k} \underset{F}{\overset{E}{\rightleftarrows}} \mathbb{C}v_{-k+2} \underset{F}{\overset{E}{\rightleftarrows}} \cdots \underset{F}{\overset{E}{\rightleftarrows}} \mathbb{C}v_k$$

## §2. Lie algebras and center

Let $\mathfrak{g} = \mathfrak{gl}_2$ = reductive Lie algebra $/ k$ $\quad k$ a field of char 0 (typically $\mathbb{R}$ or $\mathbb{C}$)

$\mathfrak{h} = \begin{pmatrix} * & \\ & * \end{pmatrix}$ = Cartan subalgebra ($\mathfrak{h} \subseteq \mathfrak{g}$)

Define $\mathcal{U}(\mathfrak{g}) := k\{X_1, \cdots, X_n\} \big/ (X_iX_j - X_jX_i - [X_i, X_j])$

($\uparrow$ non-commutative)

called "universal enveloping algebra"

So a $\mathfrak{g}$-module $V \longleftrightarrow \mathcal{U}(\mathfrak{g})$-module $V$.

Upshot: Even if center of $\mathfrak{g}$ is trivial, $Z(\mathcal{U}(\mathfrak{g}))$ is usually non-trivial.

$\Rightarrow$ On an irreducible $\mathfrak{g}$-module $V/\bar{k}$

$Z(\mathcal{U}(\mathfrak{g}))$ acts by a character $\chi : Z(\mathcal{U}(\mathfrak{g})) \longrightarrow \bar{k}$.

Fact: $Z(\mathcal{U}(\mathfrak{g})) \xrightarrow{\sim} \mathcal{U}(\mathfrak{h})^{W,\cdot}$

Example: $\mathfrak{g} = \mathfrak{sl}_{2,\mathbb{C}}$ $\quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ← split basis

Killing form on $\mathfrak{sl}_2$: $\quad B : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \longrightarrow k$

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

$$B = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix} \Rightarrow H^* = \frac{1}{8}H,\ E^* = \frac{1}{4}F,\ F^* = \frac{1}{4}E$$

$$\Omega = 2C = 2(EE^* + FF^* + HH^*)$$

$$= \frac{1}{2}EF + \frac{1}{2}FE + \frac{1}{4}H^2 = \frac{1}{4}H^2 + \frac{1}{2}H + FE$$

($[E,F] = H$)

$Z(\mathcal{U}(\mathfrak{sl}_2)) = \mathbb{C}[\Omega]$. $\quad$ ↖ Casimir operator

* For rep'n $\mathrm{Sym}^n \mathbb{C}^2$ of $\mathfrak{sl}_2$, its h.w. vector is killed by $E$.

$\Rightarrow \Omega$ acts by $\frac{1}{4}n^2 + \frac{1}{2}n$.

## §3. $(\mathfrak{g}, K)$-modules

Let $G$ be a reductive group $/\mathbb{R}$

$K \subseteq G$ is "almost" max'l compact mod center

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{k} = \mathrm{Lie}\, K$,

Definition A $(\mathfrak{g}, K)$-module is a vector space $V/\mathbb{C}$ equipped with

* a $\mathfrak{g}$-module structure
* a semisimple $K$-module structure, s.t. $\forall v \in V$ $\langle K \cdot v \rangle$ is finite dim'l

satisfying the following compatibility condition

(1) $\pi(k)\cdot(\pi(X)\cdot v) = \pi(\mathrm{Ad}_k(X))\cdot\pi(k)\cdot v \quad (k\in K, X\in U(\mathfrak{g}), v\in V)$

(2) If $F$ is a $K$-stable finite dim'l subspace of $V$, then the rep'n of $K$ on $F$ is differentiable, and has $\pi|_{\mathfrak{k}}$ as its differential.

Example: $\mathcal{A}_{cusp}(GL_2(\mathbb{Q}_p),\omega) \circlearrowleft GL_2(\mathbb{R})$

taking differentials $\rightsquigarrow (\mathfrak{g},K)$-action

has a $(\mathfrak{g},K)$-module structure

& $\mathcal{A}_{cusp}(GL_2(\mathbb{Q}_p),\omega) = \bigoplus_{\pi} \pi = \bigoplus_{\pi} \pi_\infty \otimes \bigotimes_p{}' \pi_p$

($\pi_\infty$: irreducible $(\mathfrak{g},K)$-module; $\pi_p$: irred. sm. adm rep'n of $GL_2(\mathbb{Q}_p)$)

Note: If $V$ is a $(\mathfrak{g},K)$-module, then $U(\mathfrak{g})$ acts on $V$

If $V$ is irreducible $\Rightarrow Z(U(\mathfrak{g}))$ must act by scalars

but there's a subtlety about real structures...

## §3 $(\mathfrak{g},K)$-module structure on $\mathcal{A}_{cusp}(GL_2(\mathbb{Q}),\omega)$

$G = \mathfrak{gl}_2/\mathbb{R}$. admits a Cartan involution $X \longmapsto \mathrm{Ad}_{\left(\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right)}(X)$

$K = G^\theta = \left\{\begin{pmatrix} a & -b\\ b & a\end{pmatrix},\ a,b\in\mathbb{R}\right\} = SO_2\cdot\mathbb{R}^\times$

$\theta$ also acts on $\mathfrak{g}$ & $\mathfrak{k}$.

$$\mathfrak{g} = \mathfrak{gl}_2 = \mathfrak{gl}_2^{\theta=1} \oplus \mathfrak{gl}_2^{\theta=-1}$$

$$i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$\mathfrak{gl}_2^{\theta=1} = \mathfrak{k} = \mathbb{R}\left\langle \begin{pmatrix} & 1\\ -1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1\end{pmatrix}\right\rangle$, where $\begin{pmatrix} & 1\\ -1 & \end{pmatrix} = \kappa$

$\mathfrak{gl}_2^{\theta=-1} = \mathfrak{p} = \left\{\begin{pmatrix} a & b\\ b & -a\end{pmatrix} \,\middle|\, a,b\in\mathbb{R}\right\}$ $\leftarrow$ compact basis of $\mathfrak{gl}_2(\mathbb{R})$

If we complexify this: $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\left\langle \begin{pmatrix} & 1\\ -1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1\end{pmatrix}\right\rangle \oplus \mathbb{C}\left\langle \frac{1}{2}\begin{pmatrix} 1 & i\\ i & -1\end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -i\\ -i & -1\end{pmatrix}\right\rangle$

(with $\kappa = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, $z = \begin{pmatrix} 1 & \\ & 1\end{pmatrix}$, $R = \frac{1}{2}\begin{pmatrix} 1 & i\\ i & -1\end{pmatrix}$, $L = \frac{1}{2}\begin{pmatrix} 1 & -i\\ -i & -1\end{pmatrix}$)

$\Big\downarrow$ Conjugate by $\begin{pmatrix} 1 & -i\\ 1 & i\end{pmatrix}$

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\langle H\cdot i,\ Z\rangle \oplus \mathbb{C}\langle E, F\rangle$$

Casimir operator $\Omega = \frac{1}{4}H^2 + \frac{1}{2}H + FE$

$$= -\frac{1}{4}\kappa^2 - \frac{1}{2}\kappa i + LR$$

Consider the Iwasawa decomposition $GL_2(\mathbb{R}) = B_2(\mathbb{R}) \times SO_2 \times \mathbb{R}^\times$

$$g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} r(\theta) \begin{pmatrix} z & \\ & z \end{pmatrix}$$

$\rightsquigarrow g_\infty \cdot i = iy + x = x + iy$

For an automorphic form $\phi : GL_2(\mathbb{A}) \longrightarrow \mathbb{C}$

with central char $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$

$$D_Z \phi = \frac{\partial}{\partial \log z}(\phi) = z \cdot \frac{\partial}{\partial z}(\phi). \rightsquigarrow \text{tells } \omega|_{\mathbb{R}^\times_{>0}}.$$

$$D_\kappa \phi = \frac{\partial}{\partial \theta}\Big(\phi(* \, r(\theta))\Big) = \frac{\partial}{\partial\theta}\left(e^{ik\theta}\phi(*)\right) = ik \cdot \phi$$

(↑ if $\phi$ has weight $k$)

$$D_R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta} - \frac{z}{2}\frac{\partial}{\partial z}\right)$$

$$D_L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta} - \frac{z}{2}\frac{\partial}{\partial z}\right)$$

If $\phi$ comes from a modular form $f$

$$\phi(g_\infty) = \det(g_\infty)^{k-1} j(g_\infty, i)^{-k} f(g_\infty)$$

$$= (yz^2)^{k-1} j\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, zr(\theta)\cdot i\right)^{-k} j(zr(\theta), i)^{-k} f(x+iy)$$

$$= (yz^2)^{k-1}(x+iy)^{-k} \cdot z^{-k} e^{ik\theta} f(x+iy)$$

$$\frac{\partial}{\partial \bar z} f = 0 \Rightarrow \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left((x+iy)^{-k} f(x+iy)\right) = 0$$

$$\Leftrightarrow D_L(\phi) = e^{-2i\theta}\left(-iy\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - \frac{1}{2i}\frac{\partial}{\partial\theta} - \frac{z}{2}\frac{\partial}{\partial z}\right)\underbrace{\left(y^{k-1} z^{k-2} e^{ik\theta}(x+iy)^{-k} f(x+iy)\right)}_{=\,\phi(g_\infty)}$$

$$= e^{-2i\theta} \cdot \left(y \cdot \frac{k-1}{y}\phi(g_\infty) - \frac{1}{2i}\cdot ik \cdot \phi(g_\infty) - \frac{k-2}{2}\phi(g_\infty)\right) = 0$$

So $\phi$ is "holomorphic" $\Leftrightarrow D_L(\phi) = 0$

<u>Remark</u>: One can check that, for $\Omega = -\frac{\kappa^2}{4} - \frac{1}{2}\kappa i + LR$,

$$D_\Omega \approx -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iy\frac{\partial^2}{\partial x\partial\theta}.$$

This is the usual Laplacian on $\mathfrak{h}$. (after proper twist)

By spectral theory of elliptic operators, $\mathcal{A}_{cusp}(GL_2,\omega)$ behaves well, (at least if we don't think about cusp issues.)

## §4 Classification of $(\mathfrak{g},K)$-modules for $GL_2(\mathbb{R})$

$\mathfrak{g}=\mathfrak{gl}_2$, $K=SO(2)\cdot\mathbb{R}^\times$, $(\mathfrak{g},K)$-module $V$

WLOG $Z=\begin{pmatrix}1&\\&1\end{pmatrix}$ acts on $V$ by scalar mult. by $\mu$

As $K$-representations, $V=\bigoplus_{m\in\mathbb{Z}} V_m$.

- for $v_m\in V_m$, $\kappa=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $\quad \kappa v_m = mi\cdot v_m$.
- $R=\frac{1}{2}\begin{pmatrix}1&i\\i&-1\end{pmatrix}$ $\quad R: V_m\longrightarrow V_{m+2}$

  $L=\frac{1}{2}\begin{pmatrix}1&-i\\-i&-1\end{pmatrix}$ $\quad L: V_m\longrightarrow V_{m-2}$.
- Assume the Casimir $\Omega=-\frac{1}{4}\kappa^2-\frac{1}{2}\kappa i+LR$ acts by $\gamma$ on $V$.

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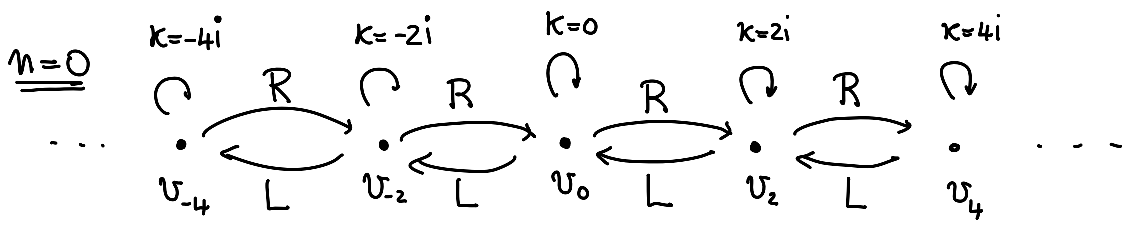
### Classification of irreducible $(\mathfrak{g},K)$-modules for $GL_2(\mathbb{R})$

Assume $Z$ acts by $\mu$

① Principal series $P_{\gamma,n}$ $\quad \underset{\mathbb{C}}{\gamma} \neq \frac{m^2-1}{4}$ for any $m\in\mathbb{Z}$. , $n=0$ or $1$.

or $\gamma=\frac{m^2-1}{4}$ but $m,n$ of same parity.

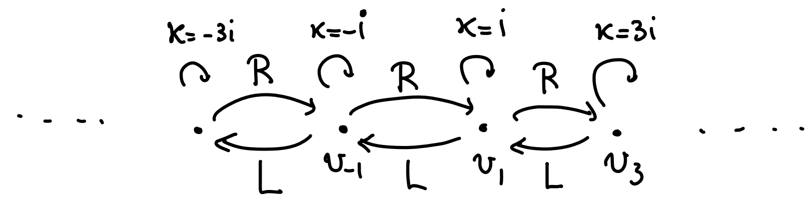



$v_{2k}=\pi(R^k)v_0$ , $v_{-2k}=\pi(L^k)v_0$ , $\pi(\kappa)v_\ell=\ell\cdot i\cdot v_\ell$

$$\Rightarrow \pi(L)\, v_{2k+2} = \pi(LR)\, v_{2k} = (\gamma - k^2 - k)\, v_{2k}$$

$$\pi(R)\, v_{-2k-2} = (\gamma - k^2 - k)\, v_{-2k}$$

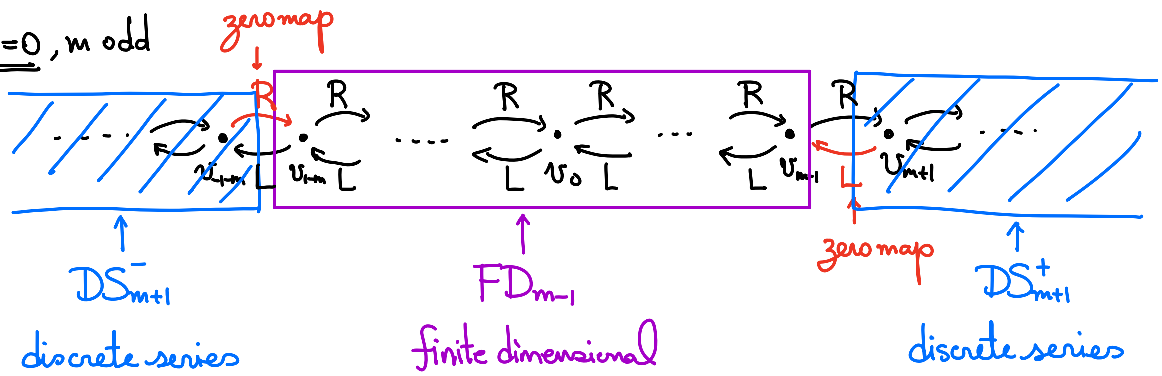
<u>$n = 1$</u>



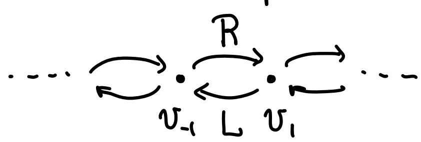

② When $\gamma = \frac{m^2-1}{4}$, $m$ & $n$ of different parity $n = 0, 1$. $m \geq 0$

the principal series "breaks up" into three parts

<u>$n=0$</u>, $m$ odd




<u>$n=1$</u> $m$ even, almost the same, except the middle part:




<u>Rmk</u>: When $m = 0$, $DS_1^{\pm}$ are limit of discrete series

When $\pi = DS_k^+$, there's a unique vector $v_k$ (up to scalar), s.t.

$$D_L(v_k) = 0$$

So

$$S_k(\Gamma_1(N), \chi)^{new} \cong \bigoplus_{\pi} \mathbb{C} \cdot v_k \otimes \bigotimes_p \underbrace{\pi_p^{\widehat{\Gamma_1}(N), \chi}}$$

($\pi$: for those $\pi_\infty = DS_k^+$, $\operatorname{cond}(\pi_p) = p^{v_p(N)}$; the last factor: 1-dim'l)

$\{$normalized cusp eigen newforms of weight $k$.$\}$ $\longleftrightarrow$ $\{$cusp. autom repr's $\pi$, s.t. $\pi_\infty \simeq DS_k^+\}$

<u>Rmk</u>: The principal series $\longleftrightarrow$ Maass forms.