Lecture 4 Moduli of elliptic curves, Geometric modular forms

§1 Moduli of elliptic curves

<u>Recall</u>: An elliptic curve E over $\mathbb{C}$ takes the form of $E(\mathbb{C}) = (\mathbb{C},+)/\mathbb{Z}\oplus\mathbb{Z}\tau$.

But $\tau \in \mathfrak{h}$ is uniquely determined up to the action of $SL_2(\mathbb{Z})$.

<u>Theorem 1</u> Assume $N \geq 4$. $Y_1(N)$ is the moduli space of elliptic curves with an N-torsion point,

that is, there exists $\mathcal{E}_{univ}$ = universal elliptic curve,

not important here, but will see μ_N instead for many references.

$$\text{zero section} \to s \Big(\Big\downarrow \pi \quad \text{together with} \quad i_{univ} : (\mathbb{Z}/N\mathbb{Z})_Y \hookrightarrow \mathcal{E}[N]$$

$$Y_1(N) = Y \qquad\qquad \text{an embedding of group scheme}$$

s.t. for every $\mathbb{C}$-scheme S together with an elliptic curve E/S & an embedding $i : (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$

there exists a <u>unique</u> morphism $\alpha : S \to Y$ s.t. $E = \alpha^* \mathcal{E}_{univ}$ and $i = \alpha^* i_{univ}$

Proof: Over $\tau \in \mathfrak{h}$, we define $E_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$, $i_{univ} : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E_\tau[N]$

$$1 \longmapsto \frac{1}{N}$$

* E_τ varies holomorphically as τ moves.

$$\rightsquigarrow \quad \begin{matrix} \mathcal{E} & \supseteq & E_\tau \\ \downarrow & & \downarrow \\ \mathfrak{h} & \ni & \tau \end{matrix} \qquad \circlearrowleft \Gamma_1(N)\text{-action} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \underset{\substack{\cup \\ \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau}}{z}) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} \right)$$

$$\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau = \mathbb{C}/\mathbb{Z}\langle a\tau+b, c\tau+d\rangle \xrightarrow{\cdot \frac{1}{c\tau+d}} \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\frac{a\tau+b}{c\tau+d}$$

$$z \longmapsto z \longmapsto \frac{z}{c\tau+d}$$

Taking the quotient by $\Gamma_1(N)$-action $\rightsquigarrow$

$$\begin{matrix} \mathcal{E}_{univ} := \Gamma_1(N)\backslash\mathcal{E} \\ \Big| \\ \Gamma_1(N)\backslash\mathfrak{h} \end{matrix}$$

One checks that this gives the moduli interpretation.

Now, we give a slightly different way to describe the level structure.

* $\hat{T}(E) :=$ Tate module of $E = \varprojlim_n E[n]$, $\quad \hat{V}(E) := \hat{T}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ free of rk 2 / $\mathbb{A}_f$

(If E is over a $\mathbb{Q}$-scheme S, $\hat{T}(E)$ is an étale $\hat{\mathbb{Z}}$-sheaf of rank 2.)

Claim: If $E/\mathbb{C}$ is an elliptic curve, giving an embedding $i : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$ is equivalent to a $\widehat{\Gamma_1(N)}$-orbit of isomorphisms $\hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E)$
$$\underset{\widehat{\Gamma_1(N)}}{\circlearrowleft}$$

Proof: Note: $\hat{T}(E) \twoheadrightarrow E[N]$. Stabilizer of i is precisely $\widehat{\Gamma_1(N)}$.

Remark: (Language issue) If E is over a local noetherian $\mathbb{Q}$-scheme S, we will need to take a $\pi_1(S,s)$-stable $\widehat{\Gamma_1(N)}$-orbit of isomorphisms $\hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E)$.

Or in a fancier language, $\underline{\mathrm{Isom}}(\hat{\mathbb{Z}}^{\oplus 2}, \hat{T}(E))$ is an étale $GL_2(\hat{\mathbb{Z}})$-torsor, a level-N-structure is a section of $\underline{\mathrm{Isom}}(\hat{\mathbb{Z}}^{\oplus 2}, \hat{T}(E))/\widehat{\Gamma_1(N)}$.

Theorem 2 Assume $N \geq 4$. The functor

$$\mathcal{M} = \mathcal{M}_{\widehat{\Gamma_1(N)}} : Sch^{loc.noe}/\mathbb{Q} \longrightarrow Sets$$

$$S \longmapsto \mathcal{M}_{\widehat{\Gamma_1(N)}}(S) := \left\{ \begin{array}{l} \text{isom. classes of } (E,\eta) : \ * \ E \text{ elliptic curve}/S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta : \hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E) \text{ is a } \pi_1(S,\bar{s})\text{-stable } \widehat{\Gamma_1(N)}\text{-orbit of isomorphisms} \end{array} \right\} \Big/ \simeq$$

is representable by a (geometrically connected) smooth curve $Y_1(N)$ over $\mathbb{Q}$.

Remark. This is equivalent to the earlier moduli problem, but we can easily modify $\widehat{\Gamma_1(N)}$ to any open compact subgroup $K \leq GL_2(\hat{\mathbb{Z}})$

Rational version:

$$\mathcal{M}' = \mathcal{M}'_{\widehat{\Gamma_1(N)}} : Sch^{loc.noe}/\mathbb{Q} \longrightarrow Sets$$

$$S \longmapsto \mathcal{M}'(S) = \left\{ \begin{array}{l} \underline{\underline{\text{equivalent}}} \text{ classes of } (E',\eta') : \ * \ E' \text{ elliptic curve}/S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta' : \mathbb{A}_f^{\oplus 2} \xrightarrow{\sim} \hat{V}(E) \text{ is a } \pi_1(S,\bar{s})\text{-stable } \widehat{\Gamma_1(N)}\text{-orbit of isoms} \end{array} \right\}$$

* Here two pairs $(E',\eta') \sim (E'',\eta'')$ are equivalent

if $\exists$ quasi-isogeny $\alpha: E' \dashrightarrow E''$ s.t. $\alpha \circ \eta' = \eta''$ (as $\widehat{\Gamma}_1(N)$-orbits)

i.e. we are classifying elliptic curves up to quasi-isogenies.

<u>Upshot</u>: Can replace $\widehat{\Gamma}_1(N)$ by an arbitrary $K \subseteq GL_2(\mathbb{A}_f)$, not necessarily in $GL_2(\hat{\mathbb{Z}})$

<u>Theorem 3</u> $\mathcal{M}_{\widehat{\Gamma}_1(N)} \xrightarrow{\simeq} \mathcal{M}'_{\widehat{\Gamma}_1(N)}$ is an equivalence.

$(E, \eta) \longmapsto (E, \eta)$

Proof: Conversely, given $(E', \eta') \in \mathcal{M}'(S)$, the issue is

$$\begin{array}{ccc} \eta' : \mathbb{A}_f^{\oplus 2} & \longrightarrow & \hat{V}(E') \\ \cup\!\shortmid & & \cup\!\shortmid \\ \hat{\mathbb{Z}}^{\oplus 2} & \dashrightarrow & \hat{T}(E') \end{array}$$

may not be compatible

<u>Lemma</u>. Fix an elliptic curve $E_0/\mathbb{C}$. There's a bijection

$$\begin{array}{ccc} \{\text{Elliptic curves } E \text{ with a quasi-isog. } \alpha: E \dashrightarrow E_0\} & & (E, \alpha) \\ \downarrow \cong & & \downarrow \\ \{\hat{\mathbb{Z}}\text{-lattices in } \hat{V}(E_0) := \hat{T}(E_0) \otimes_{\mathbb{Z}} \mathbb{Q}\} & & \alpha(\hat{T}(E)) \subseteq \hat{V}(E_0) \end{array}$$

Pf: Given a $\hat{\mathbb{Z}}$-lattice $\Lambda \subseteq \hat{V}(E_0)$,

we first assume that $\hat{T}(E_0) \subseteq \Lambda \subseteq \frac{1}{N} \hat{T}(E_0)$

then $\dfrac{\Lambda}{\hat{T}(E_0)} \hookrightarrow \dfrac{\frac{1}{N}\hat{T}(E_0)}{\hat{T}(E_0)} \cong E_0[N]$.

Define $E := E_0 / \left(\frac{\Lambda}{\hat{T}(E_0)}\right) \xleftarrow{\alpha^{-1}} E_0$.

Can check that the induced map is exactly $\hat{T}(E_0) \xrightarrow{\alpha^{-1}} \Lambda$

In general, $\exists N$ s.t. $N \cdot \hat{T}(E_0) \subseteq \Lambda \subseteq \frac{1}{N} \hat{T}(E_0)$

Define $E := E_0 / \left(\frac{\Lambda}{N \cdot \hat{T}(E_0)}\right)$ as subgp of $E_0[N^2]$

& $E \xleftarrow{\text{natural}} E_0 \xrightarrow{\times N} E_0$ $\square$

α.

<u>Variant of the lemma</u>: E_0/S, need to modify the target to

$\pi_1(S, s)$-stable lattice in $\hat{V}(E_0)$ (as $\mathbb{A}_f$-sheaf over S).

Back to the proof of the theorem:

$$\begin{array}{ccc} \eta' : \mathbb{A}_f^{\oplus 2} & \longrightarrow & \hat{V}(E') \\ \cup\!\shortmid & & \cup\!\shortmid \\ \hat{\mathbb{Z}}^{\oplus 2} & \dashrightarrow & \hat{T}(E') \end{array}$$

<u>Lemma</u> $\Rightarrow \exists\, E'' \xrightarrow{\alpha} E'$ quasi-isogeny s.t.

$$\mathbb{A}_f^{\oplus 2} \xrightarrow{\eta'} \hat{V}(E') \xleftarrow{\alpha} \hat{T}(E'')$$

$$\alpha(\hat{T}(E'')) \simeq \eta'(\hat{\mathbb{Z}}^{\oplus 2})$$

Then $(E'', \alpha^{-1}\circ\eta') \sim (E', \eta')$ & $\alpha^{-1}\circ\eta'$ gives an isom. $\hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E'')$. □

<u>Complex points</u> (another proof of the adelic description of modular curves)

<u>Theorem 4</u> For any open compact subgroup $K \subseteq GL_2(\mathbb{A}_f)$,

$$\mathcal{M}_K' : Sch^{loc.noe}_{/\mathbb{Q}} \longrightarrow Sets$$

$$S \longmapsto \mathcal{M}_K'(S) = \left\{ \begin{array}{l} \underline{\underline{\text{equivalent}}} \text{ classes of } (E,\eta) : * \; E \text{ elliptic curve}/S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta : \mathbb{A}_f^{\oplus 2} \xrightarrow{\sim} \hat{V}(E) \text{ is a } \pi_1(S,\bar{s})\text{-stable } K\text{-orbit of isoms} \end{array} \right\}$$

is representable (if K is "neat"), and

$$\mathcal{M}_K'(\mathbb{C}) \simeq {}_{GL_2(\mathbb{Q})}\backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) /_{K}$$

(In particular, this gives another proof of $\Gamma_1(N)\backslash\mathfrak{h} \simeq {}_{GL_2(\mathbb{Q})}\backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)/_{K}$.)

Proof: An elliptic curve $E/\mathbb{C}$ has three features:

- <u>Betti homology</u>: $H_1(E(\mathbb{C}),\mathbb{Q})$
- <u>Étale homology</u> $H_1^{et}(E,\mathbb{A}_f) \simeq \hat{V}(E)$

) <u>comparison</u>: $H_1(E(\mathbb{C}),\mathbb{Z})\otimes\hat{\mathbb{Z}} \cong H_1^{et}(E,\hat{\mathbb{Z}})$

- <u>de Rham filtration</u>: $0 \to H^0(E,\Omega^1_E) \to H^1_{dR}(E/\mathbb{C}) \longrightarrow H^1(E,\mathcal{O}_E) \to 0$

$\xrightarrow{\text{dualization}}$ $0 \to \underset{\substack{\parallel \\ H^0(E^\vee,\Omega^1_{E^\vee}) \\ \parallel \\ E}}{\omega_{E^\vee/\mathbb{C}}} \to \underset{\substack{\wr\parallel \\ H^1_{dR}(E^\vee/\mathbb{C})}}{H_1^{dR}(E/\mathbb{C})} \longrightarrow \mathrm{Lie}_{E/\mathbb{C}} \longrightarrow 0$

) <u>comparison</u>: $H_1(E(\mathbb{C}),\mathbb{Z})\otimes\mathbb{C} \simeq H_1^{dR}(E/\mathbb{C})$

Starting with. (E, η). We choose an isomorphism $\beta : H_1(E(\mathbb{C}), \mathbb{Q}) \simeq \mathbb{Q}^{\oplus 2}$.

Then we can construct elements in $GL_2(\mathbb{A}_f)$ and $\mathfrak{h}^{\pm}$ as follows:

* $$\mathbb{A}_f^{\oplus 2} \xrightarrow[\simeq]{\eta} H_1^{et}(E, \mathbb{A}_f) \xrightarrow[\sim]{\text{Comparison}} H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{A}_f) \xrightarrow[\sim]{\beta} \mathbb{A}_f^{\oplus 2}$$

  $\hookrightarrow$ gives an element $g_f \in GL_2(\mathbb{A}_f)$

* $$\omega_{E^\vee/\mathbb{C}} \subseteq H_1^{dR}(E/\mathbb{C}) \xrightarrow[\sim]{\text{comparison}} H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \overset{\beta}{\simeq} \mathbb{C}^{\oplus 2}$$

  $\hookrightarrow$ gives an element $\tau \in \mathbb{P}^1(\mathbb{C})$ (can prove that it does not belong to $\mathbb{R}$)

So, we get $(E, \eta) \rightsquigarrow (\tau, g_f) \in \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)$.

This association depends on
* choice of η in the K-orbit $\rightsquigarrow g_f \bmod \widehat{\Gamma_1(N)}$
* choice of isom. β, if $\beta' = h \circ \beta$, then
  $$(g_f', \tau') = (h \cdot g_f, h \cdot \tau)$$

Putting these together gives a map $Y_1(N)(\mathbb{C}) \to GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / \widehat{\Gamma_1(N)}$.

Exercise to check: This is independent of equivalent classes

This gives a bijection.

<u>Moduli over $\mathbb{Z}_{(p)}$</u>:

Let p be a prime number.

$\mathbb{Z}_{(p)}$ = localization at (p) ; $\hat{\mathbb{Z}}^{(p)} := \prod_{\ell \neq p} \mathbb{Z}_\ell$ & $\mathbb{A}_f^{(p)} := \hat{\mathbb{Z}}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Q}$

Let $K^p \subseteq GL_2(\hat{\mathbb{Z}}^{(p)})$ be an open compact subgroup.

$K := GL_2(\mathbb{Z}_p) \cdot K^p$ (so no level @ p).

Can define a functor $\mathcal{M}_{K^p} : Sch^{loc.\,noe.}_{/\mathbb{Z}_{(p)}} \longrightarrow Sets$

$$S \longmapsto \mathcal{M}(S) = \left\{ \begin{array}{l} \text{isom. classes of } (E, \eta) : \ * \ E \text{ elliptic curve } / S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta : \hat{\mathbb{Z}}^{(p)\oplus 2} \xrightarrow{\sim} \hat{T}(E)^{(p)} \text{ is a } \pi_1(S, \bar{s})\text{-stable } K^p\text{-orbit of isoms} \end{array} \right\}$$

It is represented by a scheme smooth over $\mathbb{Z}_{(p)}$.

§2 Geometric modular forms a la Katz

Algebraic point of view: $K \subseteq Gl_2(\hat{\mathbb{Z}})$ to make our life easier

$$\begin{tikzcd} E^{univ} \arrow[d, "\pi"] \\ \mathcal{M}_K \arrow[u, bend left, "s"] \end{tikzcd}$$

$\Omega^1_{E^{univ}/\mathcal{M}_K}$ is locally free of rank 1.

Define $\omega := s^* \Omega^1_{E^{univ}/\mathcal{M}_K}$ this is a line bundle.

<u>Next lecture</u>: ω extends naturally to the compactification $\mathcal{M}_K \subseteq \mathcal{M}_K^*$

Then
$$\underbrace{S_k(K)}_{\text{space of cusp forms.}} := H^0(\mathcal{M}_K^*, \omega^{\otimes k}(-D)) \subseteq M_k(K) := H^0(\mathcal{M}_K^*, \omega^{\otimes k})$$

$D := \mathcal{M}_K^* - \mathcal{M}_K = \text{cusps}.$

<u>Katz's new definition</u>: A <u>test object</u> over a $\mathbb{Z}[\frac{1}{N}]$-algebra R is a triple (E, η, ω), where

* (E,η) is an R-point of $\mathcal{M}_K^*$ (so E is a "generalized" elliptic curves)
* ω is a generator of the free rank one R-module $\omega_{E/R}$.

A <u>Katz modular form</u> of weight k is a rule to associate to

- every $\mathbb{Z}[\frac{1}{N}]$-algebra R. and
- every test object (E,η,ω)

} an element $f(E,\eta,\omega) \in R$

s.t. (1) This assignment depends only on isom. class of (E,η,ω)

(2) is compatible with base change in R,

i.e. for $\operatorname{Spec} R' \xrightarrow{\alpha} \operatorname{Spec} R$, $f(\alpha^* E, \alpha^*\eta, \alpha^*\omega) = \alpha^*(f(E,\eta,\omega)) \in R'$

(3) satisfies $f(E,\eta, a\cdot\omega) = a^{-k} f(E,\eta,\omega)$ for $a \in R^\times$

<u>Theorem</u>. The space of modular forms is the same as the space of Katz modular forms

Indeed, given a usual modular form $f \in H^0(\mathcal{M}_K^*, \omega^{\otimes k})$, we obtain a Katz modular form f^{Katz}:

for every test object (E,η,ω) over R,

$\exists!$ morphism $\alpha : \operatorname{Spec} R \to \mathcal{M}_K^*$ s.t. $(E,\eta) = \alpha^*(\mathcal{E}_{univ}, \eta_{univ})$

Then $\alpha^*(f)$ is a section of $H^0(\operatorname{Spec} R, \omega_{E/R}^{\otimes k})$

so $\alpha^*(f) = s \cdot \omega^{\otimes k}$ for some $s \in R$ $\rightsquigarrow$ set $f^{Katz}(E,\eta,\omega) := s$.

Properties (1) (2) (3) are easy to see. & the converse is also immediate.

- <u>Application I.</u> Describe Hecke operators T_p-action on Katz modular form f over $\mathbb{Q}$ $\quad p \nmid N$

Given a Katz modular form f, we define a new Katz modular form $T_p(f)$ as follows:

For each test object (E, i, ω) over a $\mathbb{Z}[\frac{1}{Np}]$-algebra R (assuming $\operatorname{Spec} R$ is connected)

there are exactly $p+1$ subgroup schemes $C \subset E[p]$ of rank p over $\operatorname{Spec} R$.

Define $$T_p(f)(E, i, \omega) := \underbrace{p^{k-1}}_{\text{normalizing factor}} \sum_{C \subset E[p]} f(E/C, i_C, \omega_C)$$

where ω_C is given as follows:
$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[rr, bend left, "\text{mult by } p"] & E/C \arrow[r, dashed, "\check{\pi}"'] & E \end{tikzcd}, \quad i_C : \mu_{N,S} \xrightarrow{i} E \xrightarrow{\pi} E/C$$

define $\omega_C := \check{\pi}^*(\omega)$

We can alternatively define this as follows:

$$\begin{tikzcd} & X(\Gamma_1(N) \cap \Gamma_0(p)) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1(N) & & X_1(N) \end{tikzcd}$$

(E, i)

$X(\Gamma_1(N) \cap \Gamma_0(p))$ — parametrizing (E, i, C) with C a subgroup of order p

equivalent to $(E \xrightarrow[\text{isog.}]{\pi} E', i)$

s.t. $\operatorname{Ker}(E \xrightarrow{\pi} E')$ has degree p.

$\pi_2(E \xrightarrow{\pi} E', i) = (E', i')$ where $i' : \mu_{N/S} \xrightarrow{i} E \xrightarrow{\pi} E'$

Note that there's a universal isogeny
$$\begin{tikzcd} \pi_1^* \mathcal{E} \arrow[r] \arrow[rr, bend right, "\text{mult}. p"'] & \pi_2^* \mathcal{E}' \arrow[r, dashed, "\check{\pi}"] & \pi_1^* \mathcal{E} \end{tikzcd}$$

Pulling back along $\check{\pi}$, get $\check{\pi}^* : \pi_1^* \omega \longrightarrow \pi_2^* \omega'$

We define T_p-operator as:

$$T_p' : \underbrace{H^0(X_1(N), \omega^{\otimes k})}_{\text{on the } \pi_1 \text{ side}} \longrightarrow H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \pi_1^* \omega^{\otimes k}) \cong H^0(\underbrace{X_1(N)}_{\text{on the } \pi_2 \text{ side}}, \pi_{2*} \pi_1^* \omega^{\otimes k})$$

$$\xrightarrow{\check{\pi}^*} H^0(X_1(N), \pi_{2*} \pi_2^* \omega'^{\otimes k}) \xrightarrow{\operatorname{Tr}_{\pi_2}} H^0(X_1(N), \omega'^{\otimes k})$$

Define $T_p := \frac{1}{p} \cdot T_p'$. The normalization factor $\frac{1}{p}$ is very important!