Lecture 5 Tate curves, Gauss-Manin connection

§1. Tate curve & q-expansion explained. (following Katz)

Over $\mathbb{C}$: Given a lattice $\Lambda_\tau = \mathbb{Z}\oplus\mathbb{Z}\tau$, the quotient $\mathbb{C}/\Lambda_\tau$ is the elliptic curve $Y^2 = 4X^3 - \frac{E_4}{12}X + \frac{E_6}{216}$

$$\mathbb{C}/\Lambda_\tau \ni z \longmapsto x = \wp(2\pi i z, 2\pi i \Lambda_\tau),\ y = \wp'(2\pi i z, 2\pi i\Lambda_\tau)$$

where $\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\ell\in\Lambda-\{0\}}\left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}\right)$

$$\wp'(z,\Lambda) = \frac{d\wp(z;\Lambda)}{dz} = \sum_{\ell\in\Lambda}\frac{-2}{(z-\ell)^3}$$

$$E_4 = \frac{45}{\pi^4}\sum_{\ell\in\Lambda_\tau-\{0\}}\frac{1}{\ell^4} = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n \in \mathbb{Z}[\![q]\!]$$

$$E_6 = \frac{945}{2\pi^6}\sum_{\ell\in\Lambda_\tau-\{0\}}\frac{1}{\ell^6} = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n \in \mathbb{Z}[\![q]\!]$$

When $\Lambda_\tau = \mathbb{Z}\oplus\mathbb{Z}\tau$, we can view elliptic curve as

$$\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau \xrightarrow[\cong]{\exp(2\pi i\,\cdot)} \mathbb{C}^\times/q^{\mathbb{Z}} \quad \text{for } q = e^{2\pi i\tau}$$

when $q=0$, $Y^2 = 4X^3 - \frac{1}{12}X + \frac{1}{216}$ is singular.
$\overset{\textstyle (X-\frac{1}{6})(X-\frac{1}{12})^2}{\shortparallel}$

As $E_4, E_6 \in \mathbb{Z}[\![q]\!]$, so the <u>Tate curve</u> $\mathrm{Tate}_q := \mathbb{C}^\times/q^{\mathbb{Z}}$ is def'd over $\mathbb{Z}[\frac{1}{6}]((q))$

<u>Rmk</u>: can change coordinates to be def'd over $\mathbb{Z}((q))$

<u>Remark</u>: This analytic construction also works over $\mathbb{C}_p$: $\mathbb{C}_p^\times/q^{\mathbb{Z}}$ $\qquad |q|_p = r < 1$.

mult. by q folds the annulus with radius in $[1, r^{-1}]$ into a rigid analytic space over $\mathbb{Q}_p$

or more generally over disc $|q|\in(0,1)$ over $\mathbb{Q}_p$

By rigid GAGA, this defines an elliptic curve Tate_q over $\mathbb{Q}_p(q)$

This Tate curve is equipped with a natural level structure

$$i_N: \mu_N \hookrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}} = \mathrm{Tate}_q$$

There's a natural basis $\frac{dX}{Y}$ of $\omega_{\mathrm{Tate}_q/\mathbb{Z}[\frac{1}{6}]((q))}$

Using the parametrization above $\frac{dX}{Y} = 2\pi i\, dz = \frac{dz^\times}{z^\times} =: \omega_{can}$ where $z^\times := \exp(2\pi i z)$

invariant differential on $\mathbb{C}^\times$.

In terms of the moduli problem, we get a morphism (& a Cartesian pullback diagram)

$$\begin{tikzcd} \mathrm{Tate}_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d, dash] \\ \mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]((q)) \arrow[r] & X_1(N) \end{tikzcd}$$

If f is a modular form of wt k, its evaluation on the object $(\mathrm{Tate}_q, i_N, \omega_{can})$

is $\quad f(\mathrm{Tate}_q, i_N, \omega_{can}) \in \mathbb{Z}[\frac{1}{6N}]((q))$

This is the q-expansion of f.

<u>We compute</u>: $T_p(f)(\mathrm{Tate}_q, i_N, \omega_{can}) = p^{k-1} \sum_{C \subset E_q[p]} f(\mathrm{Tate}_q/C, i_N', \check{\pi}^* \omega_{can})$

* <u>Case 1</u>. $C = \mu_p$, $\mathrm{Tate}_q/\mu_p \cong \mathbb{C}^\times/q^{\mathbb{Z}} \cdot \mu_p \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{p\mathbb{Z}}$

  $\check{\pi}: \mathbb{C}^\times/q^{p\mathbb{Z}} \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ is the natural quotient

  so $\check{\pi}^* \frac{dz^\times}{z^\times} = \frac{dz^\times}{z^\times}$.

  Diamond operator

  $i_N' : \mu_N \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}} \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{p\mathbb{Z}}$ is the $\langle p \rangle \cdot i_N$

* <u>Case 2</u> $C = \langle \zeta_p^i q^{\frac{1}{p}} \rangle$ for $i = 0, 1, \cdots, p-1$

  $\mathrm{Tate}_q/C \cong \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}} = \mathrm{Tate}_{\zeta_p^i q^{\frac{1}{p}}}$

  $\check{\pi}: \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}} \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ is raising to p^{th} power

  $\Rightarrow \check{\pi}^* \frac{dz^\times}{z^\times} = p \cdot \frac{dz^\times}{z^\times}$ so $\check{\pi}^* \omega_{can} = p \cdot \omega_{can}$

  $i_N' : \mu_N \to \mathbb{C}^\times/q^{\mathbb{Z}} \longrightarrow \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}}$ is the natural one.

So we have $T_p(f) = p^{k-1} \langle p \rangle f(q^p) + p^{k-1} \cdot \sum_{i=0}^{p-1} \underbrace{p^{-k}}_{\text{from } \check{\pi}^*\omega_{can} = p \cdot \omega_{can}} \cdot f(\zeta_p^i q^{\frac{1}{p}})$

This is exactly the usual formula on q-expansions.

§2 <u>Modular curve at cusps</u>

$$\begin{array}{ccc} \mathcal{E}^{univ} & \hookrightarrow & \mathcal{E}^* \\ \downarrow & & \downarrow \\ \mathcal{M}_K & \hookrightarrow & \mathcal{M}_K^* \end{array}$$

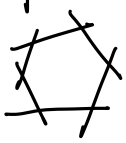
<u>Definition</u> A <u>generalized elliptic curve</u> over a scheme S is a proper flat scheme $p: E \to S$ together with a morphism

$$+ : E^{sm} \times_S E \longrightarrow E$$

and a section $e: S \to E$ s.t.

(1) $+$ with e gives E^{sm} a structure of commutative group scheme.

(2) the geometric fibers of E are elliptic curves or Néron n-gons.

where <u>Néron n-gon</u> is  ← each irred. component is $\mathbb{P}^1$ & identifing 0 & ∞'s of each $\mathbb{P}^1$ locally looks like $k[x,y]/(xy)$

$\Rightarrow$ If E_x is a Néron n-gon then $E_x^{sm} = \mathbb{G}_m \times (\mathbb{Z}/n\mathbb{Z})$ as group scheme.

<u>M_K^* in the case of $K = \widehat{\Gamma}_0(N)$</u> (okay for $\widehat{\Gamma}_1(N)$ as well)

A <u>level-N subgroup</u> of a generalized elliptic curve $E \to S$ $/\mathbb{Z}[\frac{1}{N}]$ is a subgroup $C \hookrightarrow E^{sm}[N]$ s.t. at each geometric pt $\bar{s} \in S$ $C_{\bar{s}}$ is cyclic of order N & <u>$C_{\bar{s}}$ meets every irred. component of E^{sm}</u>

$$\mathcal{M}^*_{\widehat{\Gamma}_0(N)} : \mathrm{Sch}^{\text{loc. noe.}}/\mathbb{Z}[\tfrac{1}{N}] \longrightarrow \mathcal{S}ets$$

$$S \longmapsto \mathcal{M}^*_{\widehat{\Gamma}_0(N)}(S) = \left\{ \begin{array}{l} \text{generalized elliptic curve } E \to S \\ \text{\& a level structure } C \text{ of } E^{sm}[N] \end{array} \right\}$$

$$\begin{tikzcd} \mathcal{E}^{univ*} \arrow[d, "\pi"'] & \mathcal{E}^{univ} \arrow[l, hook'] \arrow[d] \\ \mathcal{M}^*_{\widehat{\Gamma}_0(N)} \arrow[u, bend left, "s"] & \mathcal{M}_{\widehat{\Gamma}_0(N)} \arrow[l, hook'] \end{tikzcd}$$

Note image of s belongs to $\mathcal{E}^{univ,*,sm}$

$\rightsquigarrow \omega = s^* \Omega^1_{\mathcal{E}^{univ,*,sm}/\mathcal{M}^*_{\widehat{\Gamma}_0(N)}}$

§3. Gauss-Manin connections

X proper smooth variety. <u>Note</u>: We emphasize that X is defined over a subfield $E \subseteq \mathbb{C}$ b/c the de Rham cohomology $H^n_{dR}(X/E)$ is canonically defined $/E \subseteq \mathbb{C}$

$\downarrow$

$\mathrm{Spec}\, E$, $\mathrm{char}\, E = 0$ de Rham cohomology $H^n_{dR}(X/E) := \mathbb{H}^n(X, \Omega^\bullet_{X/E})$

Here $\Omega^\bullet_{X/E} = \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/E} \xrightarrow{d} \Omega^2_{X/E} := \wedge^2 \Omega^1_{X/E} \xrightarrow{d} \cdots$ is the de Rham complex

$\rightsquigarrow$ spectral sequence to compute it:

filtration $\rightarrow$ this way   filtration $\rightarrow$ this way

$\vdots$

$H^2(X, \mathcal{O}_X) \to H^2(X, \Omega^1_X)$

d^0, d^1, d^2

$E^{p,q}$

$$E_1 = \boxed{\begin{array}{ccccc} H^1(X,\mathcal{O}_X) & \to & H^1(X,\Omega^1_X) & \to & H^1(X,\Omega^2_X) \\ H^0(X,\mathcal{O}_X) & \to & H^0(X,\Omega^1_X) & \to & H^0(X,\Omega^2_X) \end{array}} \quad \cdots \quad \xRightarrow{\text{Converges to}} H^*_{dR}(X/E)$$

Fact: This spectral sequence degenerates at E_1, i.e. all maps d^1 are zero

But there are additional information, e.g. look at $H^2_{dR}(X/E)$

$\rightsquigarrow$ get Hodge filtration $F^2H^2_{dR}(X/E) \subseteq F^1H^2_{dR}(X/E) \subseteq F^0H^2_{dR}(X/E) = H^2_{dR}(X/E)$

$F^2H^2_{dR}(X/E) = H^0(X,\Omega^2_X)$; subquot is $H^1(X,\Omega^1_X)$; subquot is $H^2(X,\mathcal{O}_X)$

decreasing filtration!

In general, we write

$$H^n_{dR}(X/E) = \left(H^0(X,\Omega^n_X) - H^1(X,\Omega^{n-1}_X) - H^2(X,\Omega^{n-2}_X) - \cdots - H^n(X,\mathcal{O}_X) \right)$$

Now, let X vary in family S is a smooth E-scheme. $\operatorname{char} E = 0$

$$\begin{array}{l} X \\ \pi \downarrow \text{ proper smooth.} \\ S/\operatorname{Spec} E \end{array}$$

$\rightsquigarrow \Omega^\bullet_{X/S} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} := \wedge^2\Omega^1_{X/S} \xrightarrow{d} \cdots \to \Omega^{\dim_S X}_{X/S}$

$\mathcal{H}^\bullet_{dR}(X/S) := R\pi_*(\Omega^\bullet_{X/S})$

comes from $0 \to \pi^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0$, $\mathcal{O}_X \xrightarrow{d} \Omega^1_X$

There's a family of Hodge filtration

$$\mathcal{H}^n_{dR}(X/S) = \left(\pi_*\Omega^n_{X/S} - R^1\pi_*\Omega^{n-1}_{X/S} - R^2\pi_*\Omega^{n-2}_{X/S} - \cdots - R^n\pi_*\mathcal{O}_X\right)$$

Facts: (1) $\mathcal{H}^n_{dR}(X/S)$ is a vector bundle over S, equipped with a Gauss-Manin connection

$$\nabla_{GM}: \mathcal{H}^n_{dR}(X/S) \longrightarrow \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E}$$

$$\nabla_{GM}(a\cdot x) = x \otimes da + a\cdot\nabla(x) \quad \text{for } a \in \mathcal{O}_S,\ x \text{ a section of } \mathcal{H}^n_{dR}(X/S)$$

(2) The Gauss-Manin connection is integrable:

$$\mathcal{H}^n_{dR}(X/S) \xrightarrow{\nabla_{GM}} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E} \xrightarrow{\nabla_{GM}} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^2_{S/E}$$

$$x \otimes \xi \longmapsto \nabla_{GM}(x) \wedge \xi + x \otimes d\xi$$

satisfies $\nabla^2_{GM} = 0 \Rightarrow \left(\mathcal{H}^n_{dR}(X/S) \otimes \Omega^\bullet_{S/E}, \nabla_{GM}\right)$ is a complex of sheaves on S.

Construction by example: Say relative dim of X/S is 2 & $\dim S = 1$.

Then $0 \to f^*\Omega^1_{S/E} \to \Omega^1_{X/E} \to \Omega^1_{X/S} \to 0$

rank 1   rank 3   rank 2

Consider the de Rham complex of X/E:

$$\begin{array}{rl}
 & \mathcal{O}_X \to \Omega^1_{X/E} \to \Omega^2_{X/E} \longrightarrow \Omega^3_{X/E} \\
= & \boxed{f^*\Omega^1_{S/E} \to f^*\Omega^1_{S/E}\otimes\Omega^1_{X/S} \longrightarrow f^*\Omega^1_{S/E}\otimes\Omega^2_{X/S}} \\
 & \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S}
\end{array}$$

$$\Longrightarrow \quad 0 \to f^*\Omega^1_{S/E}\otimes\Omega^\bullet_{X/S}[-1] \longrightarrow \Omega^\bullet_{X/E} \longrightarrow \Omega^\bullet_{X/S} \longrightarrow 0$$

Taking R^if_* : $\quad R^if_*\Omega^\bullet_{X/S} = \mathcal{H}^i_{dR}(X/S)$

$$\to R^{i+1}f_*\left(f^*\Omega^1_{S/E}\otimes\Omega^\bullet_{X/S}[-1]\right)$$
$$\simeq R^if_*\Omega^\bullet_{X/S}\otimes\Omega^1_{S/E} = \mathcal{H}^i_{dR}(X/S)\otimes\Omega^1_{S/E} \qquad \checkmark$$

<u>Griffith transversality</u>: $\mathcal{H}^n_{dR}(X/S)$ carries a decreasing filtration

$$\mathrm{Fil}^i\mathcal{H}^n_{dR}(X/S) = R^nf_*\left(\Omega^{\geq i}_{X/S}\right) \subseteq R^nf_*(\Omega^\bullet_{X/S}) = \mathcal{H}^n_{dR}(X/S)$$

The Gauss-Manin connection does not preserve this filtration but "almost"

$$\begin{array}{ccc}
\nabla_{GM}: \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \mathcal{H}^n_{dR}(X/S)\otimes\Omega^1_{S/E} \\
\cup\!\shortmid & & \cup\!\shortmid \\
\mathrm{Fil}^i\mathcal{H}^n_{dR}(X/S) & \dashrightarrow & \mathrm{Fil}^{i-1}(\mathcal{H}^n_{dR}(X/S))\otimes\Omega^1_{S/E}
\end{array}$$

Prove by example:

$$\begin{array}{rl}
 & \mathcal{O}_X \to \Omega^1_{X/E} \to \Omega^2_{X/E} \longrightarrow \Omega^3_{X/E} \\
= & \boxed{f^*\Omega^1_{S/E} \to f^*\Omega^1_{S/E}\otimes\Omega^1_{X/S} \longrightarrow f^*\Omega^1_{S/E}\otimes\Omega^2_{X/S}} \\
 & \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S}
\end{array}$$

$$\Longrightarrow \quad 0 \to f^*\Omega^1_{S/E}\otimes\mathrm{Fil}^1\Omega^\bullet_{X/S}[-1] \to \mathrm{Fil}^2\Omega^\bullet_{X/E} \longrightarrow \mathrm{Fil}^2\,\Omega^\bullet_{X/S} \longrightarrow 0$$

Taking cohomology $\quad R^nf_*(\mathrm{Fil}^2\Omega^\bullet_{X/S})$

$$\to R^{n+1}f_*\left(f^*\Omega^1_{S/E}\otimes\mathrm{Fil}^1\Omega^\bullet_{X/S}[-1]\right)$$

$$= \Omega^1_{S/E} \otimes R^n f_* (\mathrm{Fil}^1 \Omega^\bullet_{X/S}) \qquad \square$$

Over $\mathbb{C}$, by which we meant $X_{\mathbb{C}} := X \otimes_E \mathbb{C}$, $S_{\mathbb{C}} := S \otimes_E \mathbb{C}$

we have Betti cohomology $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q}) := R^n \pi_* \underline{\mathbb{Q}}_{X^{an}_{\mathbb{C}}}$

Betti-de Rham comparison: $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}} \xrightarrow{\sim} \mathcal{H}^n_{dR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}}$

$$(1 \otimes \nabla_{S^{an}_{\mathbb{C}}}) \longleftrightarrow \nabla_{GM}$$

In particular, all sections of $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q})$ are horizontal

(and $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{C}) \longleftrightarrow (\mathcal{H}^n_{dR}(X^{an}/S^{an}), \nabla_{GM})$ is Riemann-Hilbert correspondence)

& In analytic topology,

$$0 \longrightarrow \mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{C}) \longrightarrow \mathcal{H}^n_{dR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}) \longrightarrow \mathcal{H}^n_{dR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}) \otimes \Omega^1_{S^{an}_{\mathbb{C}}} \longrightarrow \cdots$$

is a resolution.