# Lecture 8 General theory of Shimura varieties

<u>Ultimate goal</u>: Explain the meaning of a writing $Sh_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ we've seen many times.

## §1. Shimura data

<u>Definition</u>. A <u>Cartan involution</u> $\theta$ on a linear algebraic group $G$ over $\mathbb{R}$ is an automorphism $\theta$ of $G$ over $\mathbb{R}$ s.t.

(1) $\theta^2 = \mathrm{id}$ and

complex conjugation w.r.t. the $\mathbb{R}$-structure of $G$

(2) $G^{(\theta)}(\mathbb{R}) := \{ g \in G(\mathbb{C}) \mid \theta(g) = \bar{g} \}$ is compact.

<u>Blackbox Theorem</u> $G$ has a Cartan involution if and only if $G$ is reductive.

In this case, two Cartan involutions are differed by conjugation by an elt of $G(\mathbb{R})$

We now first give the definition of Shimura data & then discuss what it entails.

<u>Definition</u> A <u>Shimura datum</u> consists of a pair $(G, X)$.

* $G$ is a reductive group $/\mathbb{Q}$
* $X$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_\mathbb{R}$

s.t. for one (and thus every) $h$, the following condition holds

(SV1) the composition $\mathbb{S} \to G_\mathbb{R} \xrightarrow{\mathrm{Ad}} \mathfrak{g}_\mathbb{R}$ defines a Hodge structure of type $(-1,1), (0,0), (1,-1)$ (i.e. $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{g}_\mathbb{R}$ with eigenvalues $\frac{z}{\bar z}, 1, \frac{\bar z}{z}$)

(SV2) Conjugation by the image of $h(i)$ in $G^{ad} = G/\text{center of } G$ is a Cartan involution.

(SV3) For every $\mathbb{Q}$-simple factor $H$ of $G^{ad}$, $H(\mathbb{R})$ is not compact.

can be removed as well.
e.g. it excludes many important case.
$D/\mathbb{Q}$ quaternion alg, $D \otimes \mathbb{R} = \mathbb{H}$.
For $G = D^\times$, $\mathbb{H}^\times$ is cpt mod center

There are other "axioms" that simplify situations/discussions.

The question is which generality we will allow.

<u>Remark</u>: By definition, $X \cong G/K_\infty$ with $K_\infty$ = stabilizer of $h(\mathbb{S})$

in practice $\cong$ stabilizer of $h(i) \overset{(SV2)}{=\!=}$ a max'l compact subgp of $G(\mathbb{R})$ mod center

**Example**: $G = GL_2/\mathbb{Q}$ $\qquad h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow GL_2(\mathbb{R})$

$$z = x+iy \longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

The action of $Ad \circ h_0(x)$ acts trivially on $\mathfrak{gl}_2$, $Ad\circ h_0(yi)$ acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, eigenvalues are 2 copies of $1$ & $-1$

$$\mathrm{Stab}_{h_0}(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} = \underbrace{O(2)}_{\text{Compact mod center}} \cdot \mathbb{R}^\times$$

Compact mod center $\Rightarrow$ (SV2)

$$X = Ad_{GL_2(\mathbb{R})}(h_0) \longrightarrow \mathfrak{h}^\pm$$

$$Ad_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(h_0) \longmapsto \frac{ai+b}{ci+d}$$

**Example**: $G = GSp_{2g}/\mathbb{Q}$ $\quad$ w.r.t. symplectic form $\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$

$$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow GSp_{2g}(\mathbb{R})$$

$$x+iy \longmapsto \begin{pmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{pmatrix}$$

**Example**: $E$ imaginary quadratic field. $V$ a Hermitian space of dim $n/E$. signature $(a,b)$, $a+b=n$
$\mathbb{Q}$

The group $G = GU(V)$, for a $\mathbb{Q}$-algebra $R$,

$$G(R) = \{ (g,c) \in GL(V \otimes_{\mathbb{Q}} R) \times R^\times \mid \langle gx, gy \rangle = c \langle x, y \rangle \ \forall x, y \in V \otimes_{\mathbb{Q}} R \}$$

$$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G(\mathbb{R}) \subseteq GL_n(\mathbb{C}) \quad \leftarrow \text{ w.r.t. Hermitian matrix } \begin{pmatrix} \overbrace{1 \ \cdots \ 1}^{a} & \\ & \overbrace{-1 \ \cdots \ -1}^{b} \end{pmatrix}$$

$$z \longmapsto \begin{pmatrix} \overbrace{z \ \cdots \ z}^{a} & \\ & \overbrace{\bar z \ \cdots \ \bar z}^{b} \end{pmatrix}$$

$$X = Ad_{G(\mathbb{R})}(h_0) \qquad \mathrm{Stab}_{h_0}(G(\mathbb{R})) = G(U(a) \times U(b))$$

A typical $h_0$ for unitary group is $h_0' : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow U(V)(\mathbb{R})$

$$z \longmapsto \begin{pmatrix} \overbrace{1 \ \cdots \ 1}^{a} & \\ & \overbrace{z/\bar z \ \cdots \ z/\bar z}^{b} \end{pmatrix}$$

**Remark**: Somehow, $h_0$ & $h_0'$ are not quite the same. e.g. $h_0$ is not $\mathbb{S} \xrightarrow{h'} U(V)_\mathbb{R} \to GU(V)_\mathbb{R}$

So going from $\underbrace{GU(V)}_{\text{better moduli problem}}$ to $\underbrace{U(V)}_{\text{better automorphic stories}}$ is a bit tricky

* We now explain the conditions in the definition, especially how it's related to variation of Hodge structure

Recall : A Hodge structure on a $\mathbb{Q}$-vector space $V$ is the following equivalent structure :

① a homomorphism $h: \mathbb{S} \to GL_{\mathbb{R}}(V_{\mathbb{R}})$

② a bigrading decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} V^{pq}$ s.t. $\overline{V^{pq}} = V^{qp}$

We say $V$ has pure weight $n$ if $V^{pq} \neq 0 \Rightarrow p+q=n$

Let $S$ be a complex analytic manifold. A variation of Hodge structure consists of

(1) a $\mathbb{Q}$-local system $\underline{V}$ on $S$

(2) a decreasing filtration $F^p\mathcal{V}$ of $\mathcal{V} := \underline{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_S$ satisfying Griffiths transversality

i.e. $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$.

s.t. at each $s \in S$, this filtration gives a Hodge structure on $\underline{V}_s$.

**Theorem** Let $(G,X)$ be a Shimura datum, & let $G \to GL(V)$ be a faithful (= injective) $\mathbb{Q}$-rep'n of $G$.

(1) The part in (SV1) claiming that the Hodge types of $Ad \circ h: \mathbb{S} \to \mathfrak{g}_{\mathbb{R}}$ are of pure wt 0

$\Rightarrow$ there's a unique complex structure on $X$ s.t. the filtration on $V_X$ (as longs of pure wt)

defined by the $h: \mathbb{S} \to G_{\mathbb{R}} \to GL(V_{\mathbb{R}})$ varies holomorphically.

(2) Under (1), (SV1) (meaning the Hodge types $\in \{(-1,1), (0,0), (1,-1)\}$)

is equivalent to that the filtration for the above Hodge structure satisfies Griffiths transversality

(3) (SV2) implies (when $V$ has pure wt $n$) that there's a polarization

$\Psi : V \times V \to \underline{\mathbb{R}}_X(-n)$ for the Hodge structure

Proof: (1) Will only prove existence.

The condition in (1) $\Rightarrow$ $h|_{\mathbb{G}_m}$ acts trivially on $\mathfrak{g}_{\mathbb{R}}$

$\Rightarrow h(\mathbb{G}_m) \subseteq Z^\circ_{G_{\mathbb{R}}} \leftarrow$ connected component of center of $G_{\mathbb{R}}$.

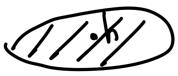
May assume that $V$ is irreducible $\Rightarrow$ $\mathbb{G}_m \xrightarrow{h} Z^\circ_{G_{\mathbb{R}}} \longrightarrow GL(V_{\mathbb{R}})$

acts by $x \mapsto x^{-n}$ for some $n$.

Each $h \in X$ defines a Hodge structure on $V = V_h$

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to GL(V_{\mathbb{R}})$$

$\Big| V_h$ i.e. $V_{h,\mathbb{C}} = \bigoplus_{p+q=n} V_h^{pq}$ s.t. $h(z)$ acts on $V_h^{pq}$ by $z^{-p}\bar{z}^{-q}$

 $X$ Define $F^pV_{h,\mathbb{C}} := \bigoplus_{p'\geq p} V_h^{p'q}$ so that $V_h^{pq} = F^pV_{h,\mathbb{C}} \cap \overline{F^qV_{h,\mathbb{C}}}$

Easy to see, $\dim F^pV_{h,\mathbb{C}}$ is independent of $h$. (b/c doesn't change by $G(\mathbb{R})$-conj)

As each $V_h$ is isomorphic to $V$ abstractly, get a natural map

$$\varphi : X \longrightarrow Gr(V_{\mathbb{C}}; \dim F^pV_{h,\mathbb{C}}) = \text{Grassmannian of filtrations on } V_{\mathbb{C}} \text{ with given dimension data.}$$
$$h \longmapsto F^pV_{h,\mathbb{C}} \subseteq V$$

Now, we verify that the image of $X$ is a sub-analytic variety of the Grassmanian

E.g. $h : \mathbb{S} \to GL_2(\mathbb{R})$ $\rightsquigarrow$ $\mathfrak{h}^{\pm} \subseteq G/B(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$

$z = x+iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$

* The tangent space of the Grassmannian is $\mathrm{End}(V_{\mathbb{C}}) / F^0\mathrm{End}(V_{\mathbb{C}})$

$\uparrow$ those endom. preserving the filtration on $V_{\mathbb{C}}$

The tangent space map of $\varphi$ factors as

note: for a Hodge structure $W$, $W^{00}$ is def'd over $\mathbb{R}$.

$$\begin{tikzcd}
T_hX = \mathfrak{g}/\mathfrak{g}^{00} \arrow[r] \arrow[d, "\cong"'] \arrow[dr, "d\varphi"] & \mathrm{End}(V)/\mathrm{End}(V)^{00} \arrow[d, phantom, "\wr"] \\
\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \arrow[r] & \mathrm{End}(V)/F^0\mathrm{End}(V)
\end{tikzcd}$$

injective when $G \to GL(V)$ is faithful.

This defines a natural complex structure on $X$ so that $X \to Gr$ is holomorphic.

Note: When choosing $V = (\mathfrak{g}, \mathrm{Ad})$, $d\varphi$ is an isomorphism

$X \hookrightarrow Gr$ is an open embedding.

(2) The Griffiths transversality translate to that the image of $d\varphi$ lies in $F^{-1}\mathrm{End}(V)/F^0\mathrm{End}(V)$

$\Longleftrightarrow$ Hodge types on $(\mathfrak{g}, \mathrm{Ad}\circ h)$ can only be $(-1,1), (1,-1), (0,0)$.

(3) This is some results from Lie theory. Omit here.

## §2. Classification of Shimura data

- We say that $(G,X)$ is of <u>Hodge type</u> if
  $\exists$ embedding $G \hookrightarrow GSp_{2g}$ s.t. $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to GSp_{2g,\mathbb{R}}$
  is conjugate to $z = x+iy \longmapsto \begin{pmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{pmatrix}$

  i.e. $Sh_G$ can be viewed as a moduli space of abelian varieties (with Hodge tensors)

- We say that $(G', X')$ is of <u>abelian type</u> if $\exists$ $(G,X)$ of Hodge type s.t.
  * $\exists$ $G_{der} \to G'_{der}$ isog. inducing $G_{ad}(\mathbb{R})\cdot X \xrightarrow{\sim} G'_{ad}(\mathbb{R})\cdot X'$

  Basically, it means Hodge type up to center, except for one particular case.

Roughly, $\{$All loc. symm. spaces$\} \supset \{$Shimura varieties$\} \supset \{$abelian type$\} \supset \{$Hodge type$\} \supset \{$PEL type$\}$

quite a gap. | difference not that arithmetically important | usually not a problem | technical issue, after Kisin's breakthrough.

<u>Classification</u>: By positivity, $G_{ad}$ is the product of $\mathbb{Q}$-simple factors, which must take the forms of $\mathrm{Res}_{F/\mathbb{Q}} G'$ for an absolutely simple group $G'$ over a totally real field $F$

<u>Type A</u>: $\hat{G} = GL_{n/\mathbb{C}}$, $\mu = (\overbrace{1,\cdots,1}^{a},0,\cdots,0)$, $\hat{G} \xrightarrow{\wedge^a} GL_{\binom{n}{a}}(\mathbb{C})$ PEL type

$G = U(a, n-a)$ $\quad h: \mathbb{S} \to G_{\mathbb{R}}$

$z \longmapsto \mathrm{diag}\{\overbrace{z/\bar{z}, \cdots, z/\bar{z}}^{a}, 1, \cdots, 1\}$

<u>Type B</u> $\hat{G} = GSp_{2n}$, $\mu \leftrightarrow$ vector rep'n $\quad$ spin rep'n. $\quad$ Hodge type

$G = GSpin(2, 2n-1) \hookrightarrow GSp_{2^{2n}}$

<u>Type C, D</u> $\hat{G} = GSpin_{n/\mathbb{C}}$, $\mu \leftrightarrow$ spin rep'n of $\hat{G}$

$G = \begin{cases} GSp(n-1) & n \text{ odd} \leftarrow \text{Siegel case. PEL type} \\ GSpin(2, n-2) & n \text{ even} \leftarrow \text{Hodge type} \end{cases}$

$GSpin(2,n-2) \hookrightarrow GSp_{2^{n-2}}$

<u>Type $D^H$</u> $\hat{G} = SO_{2n}$ vector rep'n.

$G = GSO_{2n} \hookrightarrow GSp_{4n}$ $\quad$ Hodge type

* For Hodge type: can't mixed up $D$ & $D^H$ ...

Type $E_6, E_7$, $\exists$ minuscule rep'ns but not of abelian type.

## §3 Shimura reciprocity law and canonical model

* Reflex field : $(G,X) \rightsquigarrow$ a $G(\mathbb{C})$-conjugacy class of cocharacters $\mu: \mathbb{G}_m \to G_{\mathbb{C}}$

  $\{\mu: \mathbb{G}_m \to G_{\mathbb{C}}\}$ is a natural variety, can be defined over a number field $E \subseteq \mathbb{C}$

  E is called the reflex field.

  Explicitly, $\underset{\mu}{\overset{\cup\!\!\!\!|}{X_*(T_{\mathbb{C}})^{dom}}} \circlearrowleft \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$ where $\mathbb{Q}^{alg}$ = alg. closure of $\mathbb{Q}$ inside $\mathbb{C}$

  Then $E = E(G,X)$ = subfield of $\mathbb{Q}^{alg}$ fixed by $\mathrm{Stab}_\mu(\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}))$

  Note: The reflex field is always a subfield of $\mathbb{C}$ ! i.e. a number field with a specific cplx embedding.

Theorem. The tower of Shimura variety $Sh(G,X) = (Sh_K(G,X))_{K \leq G(\mathbb{A}_f)}$

$$\left( Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K \right)$$

admits a canonical model over the reflex field $E = E(G,X)$

↳ now, explain this.

* If $G = T$ is a torus (i.e. $T_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}}^n$), $h: \mathbb{S} \to T_{\mathbb{R}}$ is invariant under conjugation

  $\Rightarrow X = \{h\}$ is a singleton

  & $\mu: \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ is def'd over the reflex field $E \subseteq \mathbb{C}$.

  $\rightsquigarrow$ For $K \leq T(\mathbb{A}_f)$ open compact, $Sh_K(T)(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / K$ is a finite set.

  To define a model of $Sh_K(T)$ over $E$, it's enough to specify $\mathrm{Gal}(\mathbb{Q}^{alg}/E)$-action.

Shimura reciprocity map: $E$ = reflex field

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}^{alg}/E) \arrow[r, two heads] \arrow[rrrd, bend right, "\mathrm{Rec}_\mu"'] & \mathrm{Gal}(E^{ab}/E) & E^\times \backslash \mathbb{A}_E^\times / E_{\mathbb{R}}^{\times,0} \arrow[l, "\mathrm{Art}"', "\cong"] \arrow[r] & \mathbb{G}_{m,E}(\mathbb{Q}) \backslash \mathbb{G}_{m,E}(\mathbb{A}_f) \arrow[d, "\mu_E"] \\
& & T_E(\mathbb{Q}) \backslash T_E(\mathbb{A}_f) \arrow[r, "\mathrm{Nm}_{E/\mathbb{Q}}"] & T(\mathbb{Q}) \backslash T(\mathbb{A}_f)
\end{tikzcd}$$

↑ sending uniformizer to geom. Frob.

The canonical model of $Sh_K(T)$ over $\mathrm{Spec}\, E$ is the $E$-scheme structure s.t.
the induced $\tau \in \mathrm{Gal}(\mathbb{Q}^{alg}/E)$ on $Sh_K(T)(\mathbb{C})$ is given by right translate by $\mathrm{Res}_\mu(\tau)$.

- For general $(G,X)$, a canonical model of the Shimura variety is an $E$-scheme $Sh_K(G,X)$ s.t. for every morphism $(T,\{h\}) \to (G,X)$ of Shimura data, the natural morphism

$$T(\mathbb{Q})\backslash \{h\} \times T(\mathbb{A}_f)/K\cap T(\mathbb{A}_f) \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is induced by a morphism $Sh_{K\cap T(\mathbb{A}_f)}(T,\{h\}) \to Sh_K(G,X) \times_{\mathrm{Spec} E} \mathrm{Spec}\, E(T,\{h\})$

(Basically, there are "enough" such $(T,\{h\}) \to (G,X)$ to rigidify the scheme structure of $Sh_K(G,X)$)

<u>Example</u> for modular curves / Siegel modular varieties

$V \simeq \mathbb{Q}^{2g}$, $\{\,,\,\}$ non-deg. alternating form

$\rightsquigarrow GSp(V) = \{(g,c) \in GL(V) \times \mathbb{G}_m,\ \{gx,gy\} = c\cdot\{x,y\}\ \forall x,y \in V\}$

$$1 \to Sp(V) \to GSp(V) \xrightarrow{c} \mathbb{G}_m \to 1$$

$$\rightsquigarrow \quad h: \mathbb{S} \longrightarrow GSp(V_\mathbb{R}) \longrightarrow \mathbb{G}_m \qquad \text{over } \mathbb{R},\ \{\,,\,\} \leftrightarrow \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$$

$$x+iy \longmapsto \begin{pmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{pmatrix} \longrightarrow x^2+y^2$$

$$\mu: \mathbb{G}_{m,\mathbb{C}} \hookrightarrow \mathbb{S}_\mathbb{C} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_\mathbb{C}} GSp(V_\mathbb{C}) \longrightarrow \mathbb{G}_m$$

$$z \longmapsto (z,1) \longmapsto \begin{pmatrix} zI_g & \\ & I_g \end{pmatrix} \longmapsto z.$$

For $K \subseteq GSp(V)(\mathbb{A}_f)$ open compact subgroup

Consider $Sh_K(GSp(V), \mathfrak{h}_g^\pm) \longrightarrow Sh_{c(K)}(\mathbb{G}_m, \{pt\})$

$$\begin{array}{ccc} \| & & \\ GSp(V)(\mathbb{Q})\backslash \mathfrak{h}_g^\pm \times GSp(V)(\mathbb{A}_f)/K & \xrightarrow{c} & \mathbb{Q}^\times\backslash \mathbb{A}_f^\times/c(K) \end{array}$$

Reciprocity map: $\mathrm{Gal}_\mathbb{Q} \to \mathrm{Gal}_\mathbb{Q}^{ab} \simeq \mathbb{Q}^\times\backslash \mathbb{A}_f^\times = \mathbb{G}_m(\mathbb{Q})\backslash \mathbb{G}_m(\mathbb{A}_f) \xrightarrow{\mu} \mathbb{G}_m(\mathbb{Q})\backslash \mathbb{G}_m(\mathbb{A}_f) \to \mathbb{Q}^\times\backslash \mathbb{A}_f^\times/c(K)$.

So when $K = \widehat{\Gamma(N)}$, the Galois group acts on the $\pi_0(Sh_K(GSp(V), \mathfrak{h}_g^\pm))$ via the cyclotomic character.