Lecture 9 Automorphic bundles on Shimura varieties

§1. Algebraic geometric background

<u>Definition</u>. Let G be a flat group scheme over S. A <u>G-torsor</u> or a <u>G-bundle</u> for the Zariski topology is a scheme $\mathcal{E} \xrightarrow{\pi} S$ s.t.

(1) G acts on $\mathcal{E}$ in the sense that

$\exists$ a morphism $\mathrm{act}: G \times_S \mathcal{E} \to \mathcal{E}$ satisfying the "obvious" axioms

(2) $\mathcal{E}$ is locally trivial in the sense that, there's a Zariski covering $\{U_i\}$ of S

$\exists$ an isom. $\mathcal{E} \times_S U \overset{\phi}{\simeq} G \times_S U$ s.t.

$$\begin{tikzcd} G \times_S \mathcal{E} \times_S U \arrow[r, "\mathrm{act}"] \arrow[d, "\wr\, 1\times\phi"'] & \mathcal{E} \times_S U \arrow[d, "\wr\, \phi"] \\ G \times_S G \times_S U \arrow[r, "m"] & G \times_S U. \end{tikzcd}$$

Easy to see $\{\text{isom. classes of } G\text{-torsors on } S\} \leftrightarrow \check{H}^1(S, G) \leftarrow$ Čech cohomology

$$\| $$

$$\varinjlim_{\substack{\{U_i\} \\ \text{affine cover}}} \operatorname{Ker}\Big(\prod_{i<j} G(U_i \cap U_j) \longrightarrow \prod_{i<j<k} G(U_i \cap U_j \cap U_k)\Big)$$

In many ocassions, G is over $\operatorname{Spec}\mathbb{Z}$, then a G-torsor over S means a G_S-torsor over S.

<u>Example</u>: $\{\text{vector bundles of rank } n \text{ over } S\} \overset{\text{bij}}{\longleftrightarrow} \{GL_n\text{-torsor over } S\}$

$\leftrightarrow \check{H}^1(S, GL_n(\mathcal{O}_S))$

* Conversely, if G is defined over $\operatorname{Spec}\mathbb{Z}$ and $G \to GL(V)$ is a representation then there is a natural functor (preserving natural tensor & dual)

$$\{\text{Alg. Rep'ns of } G\} \longrightarrow \{\text{Vector bundles on } S\}$$

$$V \longmapsto \mathcal{G} \overset{G}{\times} V := (\mathcal{G} \times V) \big/ \text{diagonal } G\text{-action}$$

Locally $U \subseteq S$, we make the quotient $(G \times U \times V)/_G \cong X \times U$

but there's a global twist.

Or equivalently, $\check{H}^1(X,G) \to \check{H}^1(X, GL(V))$
$[\mathcal{G}] \longmapsto [\mathcal{G} \overset{G}{\times} V]$.

Back to the case of unitary Shimura variety $\mathcal{M}$ for $G = GU(V)$ of signature (a,b)

$$\begin{array}{ccccc} A & \rightsquigarrow & \omega_{A/\mathcal{M},1} \text{ has rank } b & \text{and} & \omega_{A/\mathcal{M},2} \text{ has rank } a \\ \downarrow & & \wr & & \wr \\ & & \mathcal{G}_1 \ \ GL_b\text{-torsor} & & \mathcal{G}_2 \ \ GL_a\text{-torsor.} \\ \mathcal{M} & & & & \end{array}$$

So any irreducible rep'n $V_b \otimes V_a$ of $GL_b \times GL_a \longleftrightarrow$ some highest wt of $GL_b \times GL_a$
$\rightsquigarrow \left(\mathcal{G}_1 \overset{GL_b}{\times} V_b\right) \otimes \left(\mathcal{G}_2 \overset{GL_a}{\times} V_a\right)$ an automorphic vector bundle on $\mathcal{M}$. $\quad GL_b \times GL_a \subseteq GL_n$

Sections are automorphic forms.

§2. Automorphic vector bundles and local systems

* Let (G,X) be a Shimura datum $K \subseteq G(\mathbb{A}_f)$ open compact subgroup.

$$\rightsquigarrow \quad Sh_K(G,X)(\mathbb{C}) = {}_{G(\mathbb{Q})}\backslash^{X \times G(\mathbb{A}_f)}/_K.$$

<u>Assumption</u> (known as SV5) The max'l $\mathbb{R}$-split subtorus of Z_G is also $\mathbb{Q}$-split

This is equivalent to $Z_G(\mathbb{Q})$ being discrete in $Z_G(\mathbb{A}_f)$

$\Rightarrow X \times G(\mathbb{A}_f)/K \longrightarrow G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is a nice cover.

① <u>Canonical P-torsor</u>:

$h: \mathbb{S} \longrightarrow G_{\mathbb{R}}$ homomorphism, $\mathfrak{g} = \operatorname{Lie} G$

$\theta = \operatorname{Ad}_{h(i)}: \begin{array}{c} G \longrightarrow G \\ \mathfrak{g} \longrightarrow \mathfrak{g} \end{array}$ is a Cartan involution

$K = G^\theta$ = compact mod center

$\mathfrak{g} = \mathfrak{g}^{\theta=1} \oplus \mathfrak{g}^{\theta=-1} = \mathfrak{k} \oplus \mathfrak{p}$

$h: \mathbb{S} \longrightarrow GL_{2,\mathbb{R}} \qquad \mathfrak{g} = \mathfrak{gl}_2$

$x+iy \longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$

$\theta: X \longmapsto \operatorname{Ad}_{\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)}(X)$

$K = G^\theta = SO_2 \cdot \mathbb{R}^\times$

char poly $= x^2+1$
eigenvalues $\pm i$

$\mathfrak{gl}_2 = \mathfrak{gl}_2^{\theta=1} \oplus \mathfrak{gl}_2^{\theta=-1}$

Fact: $G \overset{\text{homeo.}}{\simeq} K \times \exp(\mathfrak{p})$

By (SV1), $\mathfrak{g}_{\mathbb{C}} \cong \underset{\mathfrak{p}^+}{\underset{\|}{\mathfrak{g}^{-1,1}}} \oplus \underset{\mathfrak{k}_{\mathbb{C}}}{\underset{\wr\|}{\mathfrak{g}^{0,0}}} \oplus \underset{\mathfrak{p}^-}{\underset{\|}{\widetilde{\mathfrak{g}^{1,-1}}}}$

- $\mu(\mathbb{G}_m)$ acts on $\mathfrak{p}^+$ by z & on $\mathfrak{p}^-$ by z^{-1}

$$\mathfrak{p}_{\mathbb{C}} \cong \mathfrak{p}^- \oplus \mathfrak{p}^+$$

Then $\mathfrak{p}^\pm$ are stable under K-action

& $[\mathfrak{p}^+,\mathfrak{p}^+] = [\mathfrak{p}^-,\mathfrak{p}^-] = 0$, $[\mathfrak{p}^+,\mathfrak{p}^-] \subseteq \mathfrak{k}_{\mathbb{C}}$

$\rightsquigarrow$ Lie algebra $\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- \subseteq \mathfrak{g}_{\mathbb{C}}$

$\rightsquigarrow$ parabolic subgroup Q

$\rightsquigarrow D = G/K \to G_{\mathbb{C}}/K_{\mathbb{C}} \twoheadrightarrow G_{\mathbb{C}}/Q =: \check{D}$

is an open immersion $\qquad \check{D} \supseteq \exp(\mathfrak{p}^+) \cong \exp(\mathfrak{p}^+)\cdot Q/Q$

$\mathfrak{g}/\mathfrak{k} \xrightarrow{\sim} \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}^-$ is an isomorphism

Fact:

$$\begin{tikzcd} \tilde{\mathcal{P}} := i^*G_{\mathbb{C}} \arrow[d, "Q\text{-torsor}"', "\text{left } G\text{-action}"] & G_{\mathbb{C}} \arrow[d, "Q\text{-torsor}"] \\ D \arrow[r, hook, "i"] & \check{D} = G_{\mathbb{C}}/Q \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} \mathcal{P} \arrow[d, "Q\text{-torsor}"'] \\ Sh_K(G)(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \end{tikzcd}$$

Fact: $Sh_K(G)$ admits a canonical model $/E$

$\mathfrak{k} = \begin{pmatrix} z & c \\ -c & z \end{pmatrix} \qquad \mathfrak{p} = \left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right\}$

$h(x+iy)$ has eigen vectors $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ w/ eigenval $x+iy$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$ w/ eigenval $x-iy$

$$\mathfrak{gl}_{2,\mathbb{C}} = \overset{\mathfrak{g}^{0,0}}{\overset{\|}{\mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} ic & \\ & -ic \end{pmatrix}}} \oplus \overset{\mathfrak{p}^+ = \mathfrak{g}^{-1,1}}{\overset{\|}{\underbrace{\mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}}_{\text{eigenval. } \frac{x+iy}{x-iy}}}} \oplus \overset{\mathfrak{p}^- = \mathfrak{g}^{1,-1}}{\overset{\|}{\underbrace{\mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}}_{\text{eigenval } \frac{x-iy}{x+iy}}}}$$

$\mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} z & \\ & z \end{pmatrix} \oplus$

$\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- = \mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

$\rightsquigarrow Q = \mathrm{Ad}_{\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)}\begin{pmatrix} * & \\ * & * \end{pmatrix}$

$$\begin{tikzcd} D = GL_2(\mathbb{R})/SO_2\mathbb{R}^\times \arrow[r] \arrow[d, equal] & GL_2(\mathbb{C})/K_{\mathbb{C}} \arrow[r, "\cdot\left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)"] & GL_2(\mathbb{C})/\left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right) \arrow[d, equal] \\ \mathfrak{h}^\pm \arrow[rr] & & \check{D} = \mathbb{P}^1 \end{tikzcd}$$

$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto [b:d]$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} * & ai+b \\ * & ci+d \end{pmatrix} \mapsto \frac{ai+b}{ci+d}$

$$\begin{tikzcd} \text{natural } B_{\mathbb{C}}\text{-torsor} \arrow[d] & GL_2(\mathbb{C}) \arrow[d] \\ \mathfrak{h}^\pm \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$

$$\begin{tikzcd} \omega_{E^\vee/M_K} \subseteq H_1^{dR}(E^{univ}/M_K) \arrow[d, "Q\text{-torsor}"'] \\ M_K(\mathbb{C}) = GL_2(\mathbb{Q})\backslash \mathfrak{h}^\pm/K \end{tikzcd}$$

so is the Q-torsor $\mathcal{P}$. reflex field

Given a rep'n of Q: $Q \to GL(W)$ W fin. dim$/E$

$\rightsquigarrow$ vector bundle $\underline{W} := \mathcal{P} \times^{Q} W \,/ Sh_K(G)$

- $T_{\check{D}} \cong \underline{(\mathfrak{g}/\mathfrak{q})} \cong \mathfrak{p}^+$ (for adjoint Q-action)

$\Omega^1_{\check{D}} \cong \Omega^1_{G_{\mathbb{C}}/Q} = \underline{(\mathfrak{p}^+)^*}$ (for the adjoint Q-action)

$$Q = \mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \cong \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \xrightarrow{(m,n)} \mathbb{C}^\times$$

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto a^m d^n$$

$\rightsquigarrow$ a line bundle on $\mathbb{P}^1$ & on M_K

It is $\omega_E^{n-m} \otimes \left(\Lambda^2 \mathcal{H}^1_{dR}(E/M_K)\right)^{\otimes -n}$

(There's a "duality issue, $V \leftrightarrow$ homology theory)

$\mathfrak{p}^-$ corresponds to $m=-1$, $n=1$

$\leftrightarrow \Omega^1_{M_K} \cong \omega^{\otimes 2} \otimes \underbrace{\left(\Lambda^2 \mathcal{H}^1_{dR}(E/M_K)\right)^{\otimes(-1)}}$ on M_K

trivial bundle so people usually ignore.

② Betti local system

$$\begin{array}{c} D \times G(\mathbb{A}_f)/K =: \widetilde{Sh}_K(G) \\ \Big\downarrow \\ Sh_K(G)(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K \end{array}$$

locally const $G(\mathbb{Q})$-sheaf

For every rep'n $G \to GL(V)\,/\mathbb{Q}$,

$$\rightsquigarrow \mathcal{L}^B_V := \widetilde{Sh}_K(G) \times^{G(\mathbb{Q})} V$$

$$\Big\downarrow$$

$$Sh_K(G)(\mathbb{C})$$

Betti local system on $Sh_K(G)(\mathbb{C})$

$$\begin{array}{c} E \\ \downarrow f \\ M_K \end{array}$$

$GL_2(\mathbb{Q})$-torsor is the monodromy of $R^1 f_* \mathbb{Q}_E =: \mathcal{L}_{std}$

For rep'n $\mathrm{Sym}^{k-2} \otimes \det^m : G \to GL(V)$

$$\begin{array}{c} E \\ \downarrow f \\ M_K \end{array} \quad \mathcal{L}^B_V = \mathrm{Sym}^{k-2}\left(R^1 f_* \mathbb{Q}_E\right)^\vee \otimes \det\left(R^1 f_* \mathbb{Q}_E\right)^{-m}$$

③ Automorphic vector bundle w/ integrable connection.

If V is a rep'n of G,

$\rightsquigarrow$ a rep'n of P

$\rightsquigarrow \underline{V}$ coherent sheaf on $Sh_K(G)$

$\exists$ a Gauss-Manin connection:

$$\nabla : \underline{V} \longrightarrow \underline{V} \otimes_{\mathcal{O}_{Sh_K(G)}} \Omega^1_{Sh_K(G)}$$

Gauss-Manin connection

$$\begin{array}{ccc} \nabla_{GM} : \mathcal{H}^1_{dR}(E/M_K) & \longrightarrow & \mathcal{H}^1_{dR}(E/M_K) \otimes \Omega^1_{M_K} \\ \| & & \| \\ \underline{std^\vee} & & std^\vee \end{array}$$

Taking symmetric power of this gives general ∇_{GM}.

(*Alternatively*: we have an integrable connection on principal bundle $\mathcal{P} \overset{Q}{\times} G =: \mathcal{G}$)

④ Étale local system on $Sh_K(G)$

$$\begin{array}{c} G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K^\ell \\ \downarrow \\ Sh_K(G)(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \end{array}$$

K_ℓ-local system $\mathcal{G}^{et,\ell}$

For each rep'n V of G,

$$\rightsquigarrow \mathcal{L}_V^{et,\ell} := \mathcal{G}^{et,\ell} \overset{K_\ell}{\times} V_{\mathbb{Q}_\ell}$$

$$\begin{array}{ccc} E & \rightsquigarrow & R^1 f_* \mathbb{Q}_{\ell,E} \text{ rank } 2 \\ \downarrow f & & \| \quad K_\ell\text{-local system} \\ M_K & & \mathcal{L}_{std^\vee}^{et,\ell} \end{array}$$

From this, we get $\mathcal{L}^{et,\ell}_{Sym^{k-2}\otimes \det^m}$

Comparison of local systems & automorphic line bundles

Given a rep'n $G \to GL(V) \, / \mathbb{Q}$

* Betti $\longleftrightarrow$ de Rham local system

Riemann-Hilbert correspondence

$$\left(\mathcal{L}_V^B \otimes_{\underline{\mathbb{Q}}_{Sh_K(G)^{an}}} \mathcal{O}^{an}_{Sh_K(G)}, 1\otimes d\right) \cong \left(\underline{V}, \nabla_{GM}\right)$$

E.g. GL_2, $std^\vee$.

$$\begin{array}{c} E \\ \downarrow f \\ M_K \end{array} \quad \left(\left(R^1 f_* \underline{\mathbb{Q}}\right) \otimes_{\underline{\mathbb{Q}}_{M_K}} \mathcal{O}^{an}_{M_K}, 1\otimes d\right) \cong \left(\mathcal{H}^1_{dR}(E/M_K), \nabla_{GM}\right)$$

* Betti $\longleftrightarrow$ Étale comparison

$$\mathcal{L}_V^B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longleftrightarrow \mathcal{L}_V^{et,\ell}$$

E.g. $R^1 f_*^{an} \underline{\mathbb{Q}}^{Betti}_{\ell,E} \longleftrightarrow R^1 f_{et,*} \underline{\mathbb{Q}}^{et}_{\ell,E}$

Proof of the comparison:

$$\begin{tikzcd} D \times G(\mathbb{A}_f)/K^\ell \arrow[d, "\text{quot by } G(\mathbb{Q}) \times K_\ell"] \\ Sh_K(G) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \end{tikzcd}$$

— $\mathcal{L}_V^B \otimes \mathbb{Q}_\ell$ vs. $\mathcal{L}_V^{et,\ell}$

The local system $\mathcal{L}_V^B$ is obtained by quotienting

$$D \times G(\mathbb{A}_f)/K^\ell \times V \quad \text{by}$$

$$(x,v) \sim (\gamma, g_\ell)\cdot(x,v) = (\gamma x g_\ell, \gamma v).$$

The local system $\mathcal{L}_V^{et,\ell}$ is obtained by quotienting

$$D \times G(\mathbb{A}_f)/K^\ell \times V_{\mathbb{Q}_\ell} \text{ by}$$

$$(x',v') \sim (\gamma, g_\ell)\cdot(x',v') = (\gamma x' g_\ell, g_\ell^{-1} v')$$

Key:

$$\begin{tikzcd} D \times G(\mathbb{A}_f)/K^\ell \times V \otimes \mathbb{Q}_\ell \arrow[r, leftrightarrow] & D \times G(\mathbb{A}_f)/K^\ell \times V \\ (x, v) \arrow[r, leftrightarrow] \arrow[d, squiggly, "(\gamma, g_\ell)_{Betti}"] & (x, x_\ell^{-1} v) \arrow[d, squiggly, "(\gamma, g_\ell)_{et,\ell}"] \\ (\gamma x g_\ell, \gamma v) \arrow[r, leftrightarrow] & (\gamma x g_\ell, g_\ell x_\ell^{-1} v) \end{tikzcd}$$