Exercise for lecture 1: Adelic interpretation of modular forms and automorphic representations

Problem 1.1 (Dirichlet characters and Hecke characters)**.** (1) Let N be an integer, and let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Show that

$$\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0} \cong \prod_p \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

is a Hecke character (whose restriction to $\mathbb{R}^\times_{>0}$ is trivial). Especially, explain well the middle isomorphism.

(2) Let χ and ω be as above. The grossen character ω induces a character of $\mathbb{A}^\times$, which must take the form of $\prod_v \omega_v$ over all places v, where each $\omega_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ is a character of $\mathbb{Q}_v^\times$. If $v = p$ is coprime to N, what does ω_v look like, especially what is $\omega_p(p)$? Can you also describe other ω_v?

(3) Conversely, given a grossen character ω of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ that is trivial on $\mathbb{R}^\times_{>0}$. How to determine the minimal N such that ω comes from a Dirichlet character of level N?

(4) Let F be a number field and let $\chi : \mathrm{Cl}(F) \to \mathbb{C}^\times$ be a character of the ideal class group; show that χ induces a Hecke character of F, that is, a character of $F^\times\backslash\mathbb{A}_F^\times$.

Problem 1.2 (Adelic interpretation of Γ_1-level structure)**.** Let N be a positive integer. Let χ be a Dirichlet character of $(\mathbb{Z}/N\mathbb{Z})^\times$ and let ω be defined as in Problem 1.1(1). Imitate the argument in the lecture to show that, there is a natural embedding:

$$\begin{aligned} S_k(\Gamma_1(N);\chi) &\hookrightarrow \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2(\mathbb{Q});\omega|\cdot|^{k-2}) \\ f &\longmapsto F_f(\gamma g_\infty u) = \det(g_\infty)^{k-1} j(g_\infty, i)^{-k} f(g_\infty \cdot i)\chi(u), \end{aligned}$$

for every $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})$, $u \in \widehat{\Gamma}_0(N)$. Here $S_k(\Gamma_1(N);\chi)$ is the space of cusp forms such that

$$f\Big(\frac{az+b}{cz+d}\Big) = \chi(d)(cz+d)^k f(z), \qquad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

Problem 1.3 (Classical Hecke operators vs. adelic Hecke operators)**.** Suppose that $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ is an open compact subgroup such that $\det(K) = \widehat{\mathbb{Z}}^\times$. Let $\Gamma := K \cap \mathrm{GL}_2(\mathbb{Q})$. Let $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ be an element.

(1) Show that there exists $g_i \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\Gamma g \Gamma = \coprod_i g_i \Gamma \quad \text{and} \quad KgK = \coprod_i g_i K.$$

(Hint: first pretend that $\mathrm{GL}_2(\mathbb{Q})$ is dense in $\mathrm{GL}_2(\mathbb{A}_f)$ to prove the statement, and then show that the condition $\det(K) = \widehat{\mathbb{Z}}^\times$ plus the strong approximation theorem can remedy the situation.)

(2) Assume that $K = \widehat{\Gamma}_0(N)$ and $\Gamma = \Gamma_0(N)$. Show that the Hecke algebra action T_ℓ on the space of modular forms is compatible with the action of $\mathbf{1}_{K\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell^{-1} \end{smallmatrix}\right)K}$ on the space of automorphic forms. (Caveat: T_ℓ corresponds to the cosets $\Gamma_0(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)\Gamma_0(N)$. The inversion $\ell \rightsquigarrow \ell^{-1}$ comes from: the adelic Hecke operator comes at the place at ℓ, but the Hecke operator for modular forms is at ∞. The transportation is through the diagonally embedded $\mathrm{GL}_2(\mathbb{Q})$.)

(3) Moreover, if $K = \prod_p K_p$ for $K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$, we may rewrite

$$KgK = \prod_p (K_p g_p K_p) = \prod_p \left(\coprod_i g_{i,p} K_p \right)$$

for elements $g_{i,p} \in \mathrm{GL}_2(\mathbb{Q}_p)$.

Problem 1.4 (adelic Hecke operators computation)**.** Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$. For $i \in \mathbb{Z}_{\geq 0}$, write $T_i := \mathbf{1}_{K\left(\begin{smallmatrix} p^i & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ and $S = \mathbf{1}_{pK}$. Show that the Hecke algebra $\mathcal{H} \cong \mathbb{C}[T_1, S^{\pm 1}]$ and express each T_i in terms of T_1 and S explicitly.

Problem 1.5 (More general Hecke algebra)**.** Let F_v be a local field, let G an algebraic group, and set $G_v := G(F_v)$. Let K_1 and K_2 be open compact subgroups of G_v.

(1) Show that the space $\mathbb{C}_c[K_1 \backslash G_v / K_2]$ is an $\big(\mathcal{H}(G_v; K_1),\ \mathcal{H}(G_v; K_2)\big)$-bimodule.

(2) Let π_v be a smooth representation of V. Show that there is an explicit map

$$\mathbb{C}_c\big[K_1 \backslash G_v / K_2\big] \times \pi_v^{K_2} \longrightarrow \pi_v^{K_1}$$

that is $\mathcal{H}(G_v, K_1)$-equivariant, and is compatible with the $\mathcal{H}(G_v, K_2)$-action on the two factors on the left.

Exercise for lecture 2: Representations over nonarchimedean local fields

Problem 2.1 (Steinberg representations)**.** Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and B the upper triangular matrices in G. Let $|\cdot| : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the character given by p-adic absolute values.

(1) Use Frobenius reciprocity to show that there is a natural map $\mathbf{1} \to \mathrm{Ind}_B^G \mathbf{1}$. Give explicitly the vector in $\mathrm{Ind}_B^G \mathbf{1}$ that is the image of $\mathbf{1}$.

(2) Accept that there is a natural map $\mathrm{Ind}_B^G \delta_B \to \mathbf{1}$, where δ_B is the modulus character. Show that the extension

$$0 \to \mathrm{St}_G \to \mathrm{Ind}_B^G \delta_B \to \mathbf{1} \to 0$$

does not split. (Hint: Use Frobenius reciprocity to compute $\mathrm{Hom}_G(\mathbf{1}, \mathrm{Ind}_B^G \delta_B)$.)

(3) Can you write down an explicit map $\mathrm{Ind}_B^G \delta_B \to \mathbf{1}$? (This has something to do with integration.)

Problem 2.2 (Universal principal series)**.** Write $G = \mathrm{GL}_2(\mathbb{Q}_p)$, B the upper triangular matrices in G, and $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Consider the trivial representation $\mathbf{1}$ of K, and its compactly supported induction

$$\text{c-Ind}_K^G \mathbf{1} = \{f : G \to \mathbb{C} \text{ compactly supported; } f(kg) = f(g),\ \forall k \in K, g \in G\}.$$

(1) Show that $\mathrm{End}_G(\text{c-Ind}_K^G \mathbf{1}) \cong \mathbb{C}_c[K\backslash G/K]$ as algebra. (Hint: first give a map from the RHS to LHS by using its action on $\text{c-Ind}_K^G \mathbf{1}$, and then use Frobenius reciprocity to show that this is an isomorphism as vector spaces)

(2) Let $\chi = \chi_1 \times \chi_2 : B(\mathbb{Q}_p) \to \mathbb{C}^\times$ be a character, such that both χ_i are unramified and $\chi_i(p) = \alpha_i \in \mathbb{C}^\times$. Then $\text{n-Ind}_B^G \chi$ admits a K-invariant vector. Show that there is a natural map

$$\text{c-Ind}_K^G \mathbf{1} \to \text{n-Ind}_B^G \chi.$$

which factors through

$$\text{c-Ind}_K^G \mathbf{1} / (T_1 - p^{1/2}(\alpha_1 + \alpha_2), T_2 - \alpha_1\alpha_2) \cdot \text{c-Ind}_K^G \mathbf{1},$$

where $T_1 = \mathbf{1}_{K\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ and $T_2 = \mathbf{1}_{pK}$.

(3) When $\alpha_1/\alpha_2 \notin \{p, p^{-1}\}$, show that

$$\text{c-Ind}_K^G \mathbf{1} / \big(T_1 - p^{1/2}(\alpha_1 + \alpha_2), T_2 - \alpha_1\alpha_2\big) \to \text{n-Ind}_B^G \chi$$

is surjective. (It is in fact an isomorphism; can you prove that?)

When $\alpha_1/\alpha_2 = p$ or p^{-1}. Discuss the image of the corresponding map. (This uses Problem 2.1.)

Problem 2.3 (Explicit computation for Satake isomorphism)**.** For $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$, compute explicitly the image of $\mathbf{1}_{K\left(\begin{smallmatrix} p^i & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ under the Satake isomorphism

$$\mathrm{Sat} : \mathbb{C}_c[K\backslash G/K] \to \mathbb{C}_c[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W$$

where T denote the diagonal matrices and $W \cong S_2$ is the Weyl group, in which the nontrivial element swaps the factors in T.

Can you generalize your computation to $G = \mathrm{GL}_n$ and for the Hecke operators associated to the cosets

$$\mathrm{GL}_n(\mathbb{Z}_p)\,\mathrm{Diag}\,\{\underbrace{p,\dots,p}_{i},1,\dots,1\}\,\mathrm{GL}_n(\mathbb{Z}_p)?$$

Problem 2.4 (p-stabilization)**.** Write $G = \mathrm{GL}_2(\mathbb{Q}_p)$, B the upper triangular matrices in G, and $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Set

$$\mathrm{Iw}_p = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

Let $\chi = \chi_1 \times \chi_2 : B(\mathbb{Q}_p) \to \mathbb{C}^\times$ be a character, such that both χ_i are unramified and $\chi_i(p) = \alpha_i \in \mathbb{C}^\times$. Consider $\pi = \text{n-Ind}_B^G\chi$.

(1) Show that $\dim \pi^{\mathrm{Iw}_p} = 2$ and write out a set of basis vector explicitly.

(2) Show that the natural map

$$\begin{aligned} (\pi^K)^{\oplus 2} &\longrightarrow \pi^{\mathrm{Iw}_p} \\ (x,y) &\longmapsto x - \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) y \end{aligned}$$

is an isomorphism (except possibly for particular values of α_1 and α_2). Compute the corresponding matrices (with respect to the two bases).

(3) Consider the operator $U_p := \mathbf{1}_{\mathrm{Iw}_p \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \mathrm{Iw}_p} \in \mathbb{C}_c[\mathrm{Iw}_p\backslash G/\mathrm{Iw}_p]$. Then U_p acts on π^{Iw_p}; find the eigenvalues (in terms of α_1 and α_2).

(4) Consider the operator $\mathrm{AL}_p := \mathbf{1}_{\mathrm{Iw}_p \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \mathrm{Iw}_p} \in \mathbb{C}_c[\mathrm{Iw}_p\backslash G/\mathrm{Iw}_p]$. Then AL_p acts on π^{Iw_p}; find out how AL_p acts on the two eigenspaces of U_p (at least when α_1 and α_2 avoid some particular values).

(5) Explore the structure of $\mathbb{C}_c[\mathrm{Iw}_p\backslash G/\mathrm{Iw}_p]$; what are the generators? This algebra acts on π^{Iw_p} and gives the known structure theory related to the so-called p-stabilization process.

Exercise for lecture 3: $(\mathfrak{g}, K)$-modules and Matsushima formula

Problem 3.1 (Casimir operator)**.** Consider the three operators in $\mathfrak{sl}_2$:

$$F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

We explain a general way to construct Casimir operator (for semisimple Lie algebras).

(1) Consider the Killing form (which is symmetric bilinear) defined on $\mathfrak{sl}_2$:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \qquad \mathfrak{sl}_2 \times \mathfrak{sl}_2 &\longrightarrow \mathbb{C} \\ (X, Y) &\longmapsto \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \in \mathbb{C}. \end{aligned}$$

Show that, with respect to the basis $\{F, H, E\}$, the matrix for the symmetric bilinear Killing form is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

From this, we see that the dual basis are $\{\frac{1}{4}E, \frac{1}{8}H, \frac{1}{4}F\}$ in order.

(2) Prove abstractly that the Killing form is G-equivariant, i.e. $\langle \mathrm{ad}_g(X), \mathrm{ad}_g(Y) \rangle = \langle X, Y \rangle$, for $X, Y \in \mathfrak{sl}_2$ and $g \in \mathfrak{sl}_2$. From this, deduce purely abstractly that

$$C := E \cdot E^* + F \cdot F^* + H \cdot H^* = \frac{1}{4}(EF + FE + \frac{1}{2}H^2)$$

commutes with $\mathfrak{sl}_2$ in $U(\mathfrak{sl}_2)$, namely C belongs to the center $Z(U(\mathfrak{sl}_2))$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$. (Note that: this abstract construction works for every semisimple Lie algebra $\mathfrak{g}$, producing a *Casimir* operator of degree 2 in the center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$. In the case of $\mathfrak{sl}_2$, one can show that $Z(U(\mathfrak{sl}_2)) = \mathbb{C}[C]$ is the polynomial algebra generated by this degree 2 Casimir operator. For general semisimple algebra $\mathfrak{g}$, the generators of $Z(U(\mathfrak{g}))$ may of higher degree.) Remark on notation: In different literature, the definition Casimir operator may be differed by a scalar, but this is not important.

Problem 3.2 (Computation in classification of $(\mathfrak{g}, K)$-modules for $\mathfrak{sl}_2$)**.** Let $\mathfrak{g} = \mathfrak{sl}_2$ and $K = \mathrm{SO}_2$. Set

$$\kappa = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad R = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right), \quad L = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right).$$

Then the Casimir operator is $\Omega = -\frac{1}{4}\kappa^2 - \frac{i}{2}\kappa + LR$. Consider the following construction of a $(\mathfrak{g}, K)$-module: starting with v_1 on which $\pi(\kappa)v_1 = iv_1$, define

$$v_{2k+1} = \pi(R^k)v_1, \quad v_{1-2k} = \pi(L^k)v_1.$$

so that $\pi(\kappa)v_\ell = i\ell v_\ell$ for ℓ odd. Suppose that the Casimir operator acts by γ. Determine whether this constructs an irreducible $(\mathfrak{p}, K)$-module, and when it is not irreducible, find the the subquotients.

Also, discuss the special case of limit of discrete series.

Exercise for lecture 4: Moduli of elliptic curves and geometric modular forms

Problem 4.1 (Quasi-isogeny of abelian varieties versus lattices)**.** Let A_0 be an abelian variety over $\mathbb{C}$ with principal polarization $\lambda_0 : A_0 \xrightarrow{\cong} A_0^\vee$. Show that there is an equivalence of categories:

$$\left\{\begin{matrix}\text{Abelian varieties } A \text{ with a quasi-isogeny } \alpha : A \to A_0 \\ \text{together with a principal polarization } \lambda : A \to A^\vee \\ \text{such that } \lambda = \alpha^\vee \circ \lambda_0 \circ \alpha\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\widehat{\mathbb{Z}}\text{-lattices } \Lambda \text{ in } \widehat{V}(A_0) \text{ which is} \\ \text{self-dual under the symplectic pairing}\end{matrix}\right\}$$

$$A \longmapsto \widehat{T}(A)$$

Now, suppose that A_0 is defined over $\mathbb{Q}$, show that under the above correspondence, the $\widehat{\mathbb{Z}}$-lattice Λ of $\widehat{V}(A_0)$ is stable under the $\mathrm{Gal}_{\mathbb{Q}}$-action if and only if it comes from an abelian variety over $\mathbb{Q}$.

Problem 4.2 (Γ_0-level structure)**.** We give a moduli interpretation of modular curve with $\Gamma_0(p)$-level structure, when p is a prime number.

(1) Show that the following two functors are equivalent.

$$\mathcal{M}, \mathcal{M}' : \ \mathbf{Sch}_{/\mathbb{Z}_{(p)}} \longrightarrow \mathbf{Sets}$$

$$S \longmapsto \mathcal{M}(S) = \left\{\begin{matrix}\text{isomorphism classes of isogenies } \beta : E \to E' \\ \text{of degree } p \text{ between two elliptic curves over } S\end{matrix}\right\}.$$

$$S \longmapsto \mathcal{M}'(S) = \left\{\begin{matrix}\text{isomorphism classes of } (E, C) : \\ E \text{ is an elliptic curve over } S \\ C \text{ is a subgroup of } E[p] \text{ of degree } p\end{matrix}\right\}.$$

They are represented by a stack[1] $Y_0(p)$ over $\mathbb{Z}_{(p)}$ (but not smooth over the fiber at p). This will not give a scheme, as we will see in Problem 4.4; however we can “pretend" that it is a scheme for most purpose. We will come to study its geometry later.

(2) Using either moduli problem, explain what the Hecke correspondence at p looks like.

Problem 4.3 (sheaf for modular forms using rationalized moduli problem)**.** If one uses moduli problem of elliptic curves up to isogeny, the sheaf ω is not immediately defined. Let us recall the moduli problem first (or rather its integral version): let p be a prime number, and let K^p be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$, we define

$$\mathcal{M}'_K : \mathbf{Sch}^{\text{loc. noe.}}_{/\mathbb{Z}_{(p)}} \longrightarrow \mathbf{Sets}$$

$$S \longmapsto \mathcal{M}'_K(S) = \left\{\begin{matrix}\textit{equiv.} \text{ classes of } (E', \eta'); \quad E' \text{ is an elliptic curve over } S; \\ \text{choosing a geom. point } \bar{s} \text{ on each conn. component of } S \\ \eta' : \mathbb{A}_f^{(p),\oplus 2} \xrightarrow{\simeq} \widehat{V}^{(p)}(E') \text{ is a } \pi_1(S, \bar{s})\text{-stable } K^p\text{-orbit of isoms.}\end{matrix}\right\}.$$

[1]It is a stack but not a scheme because the moduli problem is supposed to be the quotient $Y_0(p)/\{1, \left(\begin{smallmatrix}-1 & 0 \\ 0 & -1\end{smallmatrix}\right)\}$, so every point has nontrivial automorphism. Or in the language of moduli problem, $[-1] : E \to E$ is an automorphism of a pair (E, C) but it induces trivial map on $\mathcal{M}$ if this were represented by a scheme.

Here, $\widehat{V}^{(p)}(E')$ is the rationalized Tate modules of E' away from p. We say (E', η') and (E'', η'') are equivalent if there is a *prime-to-p* quasi-isogeny $\alpha : E' \dashrightarrow E''$ such that $\alpha \circ \eta' = \eta''$ (as K^p-orbit). This $\mathcal{M}'_K$ is represented by a smooth curve M'_K over $\mathbb{Z}_{(p)}$.

Task 1: Show that this defines the same moduli problem as the usual moduli problem for $K^p \operatorname{GL}_2(\mathbb{Z}_p)$ (over $\mathbb{Q}$).

Normally, we define the automorphic line bundle to be: take the universal elliptic curve $E^{\mathrm{univ}} \to M'_K$ (with the zero section s), and then define $\omega := s^* \Omega^1_{E^{\mathrm{univ}}/M'_K}$. But the problem here is that we don't have an isomorphism class of universal elliptic curves but only an equivalent class of elliptic curves.

There are two possible solutions:

(1) Fix a $\widehat{\mathbb{Z}}^{(p)}$-lattice $\Lambda^{(p)}$ of $\mathbb{A}_f^{(p),\oplus 2}$ that is invariant under K^p. And in the equivalence class, choose the one where $\eta' : \mathbb{A}_f^{(p),\oplus 2} \xrightarrow{\simeq} \widehat{V}^{(p)}(E')$ matches $\Lambda^{(p)}$ with $\widehat{T}^{(p)}(E')$. Then define ω using that E'.

(2) Just define ω using any E' in the equivalent class and show that for any two equivalent (E', η') the corresponding sheaf has a canonical isomorphism.

Problem 4.4 (Quadratic twists of elliptic curves)**.** We discuss the question of quadratic twist of elliptic curves.

Classical definition: For an elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$, a *quadratic twist* is the elliptic curve $E_D : Dy^2 = x^3 + ax + b$ for some $D \in \mathbb{Q}$ typically square-free. The two curves E and E_D are not isomorphic over $\mathbb{Q}$ but are isomorphic over $\mathbb{Q}(\sqrt{D})$. A key feature is that there is a j-invariant attached to D as follows: the modular function $j : \operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} \xrightarrow{\cong} \mathbb{C}$ gives a bijection. (Here I used double slash to indicate "coarse moduli problem"; we may temporarily ignore this now.) The statement above amounts to say $j(E) = j(E_D)$.

Moreover, via the isomorphism $j : \operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} \xrightarrow{\cong} \mathbb{C}$, we can endow $\operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$ a natural $\mathbb{Q}$-structure (namely, a rational point on it means a point with j-invariant in $\mathbb{Q}$.) But we still write $\operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$ for it to mean the corresponding $\mathbb{Q}$-scheme.

Galois cohmology explanation: Elliptic over $\mathbb{C}$ (or over $\overline{\mathbb{Q}}$) up to isomorphism are determined by the j-invariant.

(1) Prove that, given an elliptic curve E over $\mathbb{Q}$, any other elliptic curves that are isomorphic to E over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$, called *forms of* E, are classified by $H^1(\mathbb{Q}, \operatorname{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}}))$.

(2) Find $\operatorname{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}})$ for all $E_{\overline{\mathbb{Q}}}$. Show that unless $j(E) = 0$ or 1728, $\operatorname{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}}) = \{\pm 1\}$. Deduce that in this case, all forms of E are quadratic twists.

Explanation using moduli stack: (Let use try if this explanation makes sense.) If we consider the moduli problem of elliptic curves, call it $\mathcal{M}$, it is represented by a stack. On an open subset, it looks like $U/\{\pm I_2\}$ where U is an open subset of $\operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} - \operatorname{SL}_2(\mathbb{Z})\{i, e^{2\pi i/3}\}$. Here $\pm I_2$ acts trivially on U. But as a stack, it is natural to keep this quotient. In other words, we have a natural morphism $\mathcal{M} \to \operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$. Again, this can be defined over $\mathbb{Q}$.

Giving a j-invariant (say over $\mathbb{Q}$ but not at 0 or 1728) amounts to a morphism $x : \operatorname{Spec}\mathbb{Q} \to \operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$, we can take the fiber product:

$$\begin{array}{ccc} \big[\operatorname{Spec}\mathbb{Q}/\{\pm 1\}\big] & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Q} & \xrightarrow{x} & \operatorname{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}. \end{array}$$

Again, $\left[\operatorname{Spec}\mathbb{Q}/\{\pm 1\}\right]$ is the stack given by "quotienting" $\operatorname{Spec}\mathbb{Q}$ by the trivial $\{\pm 1\}$-action. In the fancier language, this is the classifying space for $\{\pm 1\}$. So a $\operatorname{Spec}\mathbb{Q}$-point of $\left[\operatorname{Spec}\mathbb{Q}/\{\pm 1\}\right]$ corresponds to a $\{\pm 1\}$-torsor over $\operatorname{Spec}\mathbb{Q}$, that is a quadratic extension of $\mathbb{Q}$ (including $\mathbb{Q}\times\mathbb{Q}$).

Explicitly, for a quadratic extension $\mathbb{Q}(\sqrt{D})$, we have $\iota_D : \operatorname{Spec}\mathbb{Q}(\sqrt{D}) \to \operatorname{Spec}\mathbb{Q}$, equivariant for the $\{\pm 1\}$-action, where -1 acts by natural Galois action on $\mathbb{Q}(\sqrt{D})$ and trivially on $\mathbb{Q}$. Taking the quotient of ι_D by the $\{\pm 1\}$-action gives $\iota_D : \operatorname{Spec}\mathbb{Q} \to \left[\operatorname{Spec}\mathbb{Q}/\{\pm 1\}\right]$.

Exercise for lecture 5: Tate curves and Gauss–Manin connections

Problem 5.1 (q-expansion of U_p-operator)**.** Let $N \geq 4$ be an integer, and let p be a prime number that divides N, say $p^r \| N$ for some $r \geq 1$. In this case, we usually write U_p for the Hecke operator at p.

Recall that the modular curve $Y_1(N)$ classifies, for a $\mathbb{Q}$-scheme S, a pair (E, i) where E is an elliptic curve over S, and $i : \mu_{N,S} \to E[N]$ an embedding.

Let f be a Katz modular form of weight k. Then $U_p(f)$ is the Katz modular form, whose evaluation on a test object (E, i, ω) over a $\mathbb{Q}$-algebra R (such that $\operatorname{Spec} R$ is connected) is

$$U_p(f)(E, i, \omega) = p^{k-1} \sum_{\substack{C \subset E[p] \\ C \not\subseteq \mathrm{Im}(i)}} f(E/C, i_C, \omega_C),$$

where the sum is taken over all subgroups of $E[p]$ of order p that is different from the one in $\Im(i)$, i_C is the embedding $\mu_{N,S} \xrightarrow{i} E[N] \to E/C$ (as $C \not\subseteq \mathrm{Im}(i)$, this is an inclusion), and $\omega_C = \check{\pi}^*(\omega)$ with $\check{\pi}$ the map defined by the factorization $\mathrm{mult}_p : E \to E/C \xrightarrow{\check{\pi}} E$.

Give the q-expansion expression of $U_p(f)$ in terms of that of f.

Problem 5.2 (Coherent sheaf with intergrable connection is locally free)**.** Let X be a smooth variety over a field k of characteristic zero. Let M be a coherent sheaf on X with an integrable connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_X$. The goal is to prove that M is locally free as an $\mathcal{O}_X$-module.

To see this, it is enough to work locally in a formal neighborhood of a point x, and hence we may practically replace X with $\operatorname{Spec} k[\![x_1, \dots, x_n]\!]$, and then M_x is a finite $k[\![x_1, \dots, x_n]\!]$-module.

(1) Show that M admitting an integrable connection implies that M_x carries *commuting* differential operators $\partial_{x_1}, \dots, \partial_{x_n}$.

(2) Given any $e \in M_x$, show that the following expression

$$\sum_{a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}} \frac{(-x_1)^{a_1} \cdots (-x_n)^{a_n}}{a_1! \cdots a_n!} \partial_{x_1}^{a_1} \cdots \partial_{x_n}^{a_n}(e)$$

is a (or rather unique) horizontal section of M_x (namely killed by all ∂_i), with the same reduction as e modulo $(x_1, \dots, x_n)$.

(3) Prove that M_x is a finite free $k[\![x_1, \dots, x_n]\!]$-module. (A maybe a direct way to prove this is to show that taking horizontal lifts of elements in a basis of $M_x/(x_1, \dots, x_n)$ to M, there is no relation among these lift.)

Remark: This explains why $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ is locally free as a coherent sheaf (because it carries a Gauss–Manin connection).

Problem 5.3 (Gauss–Manin connection for elliptic curves)**.** The goal of this problem is to compute explicitly the Gauss–Manin connection on family of elliptic curves. Let S be an affine scheme.

(1) We start with a general elliptic curve E/S, and let ∞ denote the zero section of the elliptic curve. Set $U := E \backslash \infty$ and $j : U \to S$ the natural inclusion. Show that the following natural morphisms

$$\big[\mathcal{O}_E \to \Omega^1_{E/R}\big] \longrightarrow \big[\mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)\big] \longrightarrow \big[j_*\mathcal{O}_U \to j_*\Omega^1_{U/R}\big]$$

induce isomorphisms on $\mathbb{H}^1(E, -)$, namely the 1st hypercohomology of the complex (not necessarily on other degrees).

(2) Prove that

$$H^1\big(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)\big) \cong H^0\big(E, \Omega^1_{E/R}(2\infty)\big).$$

and show that if we write $y^2 = x^3 + ax + b$ for $a, b \in \Gamma(S, \mathcal{O}_S)$, this cohomology has two basis $\frac{dx}{y}$ and $\frac{xdx}{y}$.

Using the last isomorphism of (1), show that $\frac{dx}{y}$ and $\frac{xdx}{y}$ give a basis of the cokernel of $H^0(U, \mathcal{O}_U) \xrightarrow{d} H^0(U, \Omega^1_{U/R})$.

(3) On the affine part U of E, show that there exists $A(x), B(x) \in \Gamma(S, \mathcal{O}_S)[x]$ such that

$$A(x)(x^3 + ax + b) + B(x)(3x^2 + a) = 1.$$

(Explicitly, if $\Delta := 4a^3 + 27b^2$, then $A(x) = \frac{-18ax+27b}{\Delta}$ and $B(x) = \frac{6ax^2-9bx+4a^2}{\Delta}$)

Using this, deduce that

$$\frac{dx}{y} = A(x)ydx + 2B(x)dy,$$

as differentials in $\Omega^1_{U/R}$ (but not in $\Omega^1_{U/k}$ when $S = \operatorname{Spec} k[t]$) (It may simplify the notation if we write $P(x) = x^3 + ax + b$.)

(4) Going through the definition of Gauss–Manin connection (and use its compatibility with its restriction to U) to give a recipe to compute, for a family of elliptic curve $y^2 = x^3 + a(t)x + b(t)$ with $a(t), b(t) \in k[t]$, the Gauss–Manin connection, in terms of $A(x)$ and $B(x)$ above.

Remark: the computation will be very formidable to implement in practice; we are just talking about a way to compute Gauss–Manin connection in principle.

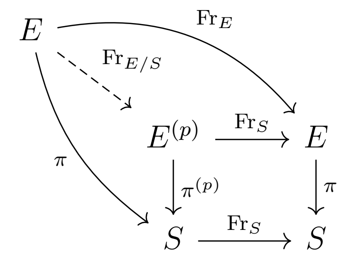
Exercise for lecture 6: Galois representations associated to modular forms

Problem 6.1. Let $f : X \to S$ be a proper smooth morphism over smooth schemes over $\mathbb{C}$. Write out the proof of Griffith transversality in general, namely the Gauss–Manin connection sends

$$\nabla : \mathrm{Fil}^i(\mathcal{H}^n_{\mathrm{dR}}(X/S)) \to \mathrm{Fil}^{i-1}(\mathcal{H}^n_{\mathrm{dR}}(X/S)) \otimes \Omega^1_{S/\mathbb{C}}.$$

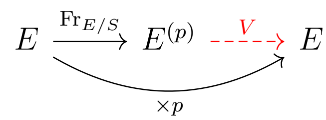
Show that ∇ induces a $\mathcal{O}_S$-coherent map $\mathrm{gr}^i(\mathcal{H}^n_{\mathrm{dR}}(X/S)) \to \mathrm{gr}^{i-1}(\mathcal{H}^n_{\mathrm{dR}}(X/S)) \otimes \Omega^1_{S/\mathbb{C}}$.

Problem 6.2 (Hasse invariants)**.** We first recall the general setup of relative Frobenius (focusing on elliptic curves): Let $\pi : E \to S$ be a morphism of $\mathbb{F}_p$-varieties, then on each of E and S there is a Frobenius morphism Fr_E and Fr_S, given by raising the coordinate functions to p-th power



Here the square is the Cartesian pullback. Show that there is a natural map $\mathrm{Fr}_{E/S}$ that makes the diagram commute.

When E is an elliptic curve, the relative Frobenius $\mathrm{Fr}_{E/S}$ facts as



This S-morphism V is called a Verschiebung morphism. It induces a morphism

$$\mathcal{H}^1_{\mathrm{dR}}(E/S) \xrightarrow{V^*} \mathcal{H}^1_{\mathrm{dR}}(E^{(p)}/S) \cong \mathcal{H}^1_{\mathrm{dR}}(E/S)_{\mathcal{O}_S,\mathrm{Fr}_S}\mathcal{O}_S.$$

It is a general fact that the image of V^* is precisely $\omega_{E^{(p)}/S} \cong \omega_{E/S} \otimes_{\mathcal{O}_S,\mathrm{Fr}_S} \mathcal{O}_S$.

Explain why $\omega_{E^{(p)}/S} \cong \omega_{E/S}^{\otimes p}$. Applying this discussion to the universal case, gives a morphism

$$V^* : \omega \to \omega^p.$$

Show that this defines a canonical section $h \in H^0(Y_K, \omega^{p-1})$, called the *Hasse invariant*. Its zeros are precisely the supersingular points.

Problem 6.3 (q-expansion of Hasse invariants)**.** (1) From the expression of Tate curve $\mathrm{Tate}_q \cong \mathbb{C}_p^\times/q^{\mathbb{Z}}$ viewed as rigid analytic elliptic curve, deduce that

$$1 \to \mu_p \to \mathrm{Tate}_q[p] \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

(2) From this, deduce that, viewing Tate curve over $\mathbb{Z}_p((q))$, the natural map

$$\mathrm{Tate}_q \xrightarrow{\text{mult. by } p} \mathrm{Tate}_{q^p}$$

lifts the Frobenius morphism modulo p.

(3) Show that the Hasse invariant h has q-expansion equal to 1.

Problem 6.4 (Counting supersingular elliptic curves)**.** Assume that the prime $p \geq 7$ for simplicity.[2] Recall that supersingular elliptic curves over $\overline{\mathbb{F}}_p$ are in one-to-one correspondence with their j-invariants. Classically, determine the number of j-invariants uses an explicit form of Hasse invariant, but this can be done in a much more abstractly.

The moduli stack of elliptic curve is $X(\mathrm{SL}_2(\mathbb{Z}))$; its coarse moduli space is given by taking j-invariants $j : X(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{P}^1$.

We make the computation over a cover. Consider the modular curve $X(\Gamma(5))$.[3] It is a Galois cover of $X(\mathrm{SL}_2(\mathbb{Z}))$ with Galois group $\mathrm{SL}_2(\mathbb{F}_5)$[4] (in the sense of function field extension, as there are ramifications at cusps).[5] When compositing with the j-invariant map, $X(\Gamma(5))$ becomes a Galois cover of $\mathbb{P}^1$ with Galois group $\mathrm{PSL}_2(\mathbb{F}_5) \simeq A_5$.

(1) The ramification degree of the cover $X(\Gamma(5)) \to \mathbb{P}^1$ at $\tau = i$ is 2, at $\tau = e^{2\pi i/3}$ is 3, and at $\tau = \infty$ is 5. Check using Riemann-Hurwtiz formula that the genus of $X(\Gamma(5))$ is zero.

(2) Now, assume all the computation we did in (1) works over $\overline{\mathbb{F}}_p$, and write $\overline{X}(\Gamma(5))$ for the mod p fiber. Over $\overline{X}(\Gamma(5))$, the Kodaira–Spencer isomorphism gives an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{\overline{X}(\Gamma(5))}(\log \overline{C})$, where $\overline{C}$ is the cusps, namely the (reduced subscheme of) the preimage of $\infty \in \mathbb{P}^1_{\overline{\mathbb{F}}_p}$. Show that the degree of ω on $\overline{X}(\Gamma(5))$ is 5, and compute the number of supersingular points over $\overline{X}(\Gamma(5))$.

(3) Prove the following statements.

- The j-invariant 0 ($\tau = e^{2\pi i/3}$) corresponds to a supersingular curve over $\overline{\mathbb{F}}_p$, if and only if $p \equiv 2 \bmod 3$.
- The j-invariant 1728 ($\tau = i$) corresponds to a supersingular curve over $\overline{\mathbb{F}}_p$, if and only if $p \equiv 3 \bmod 4$.
- The number of supersingular j-invariants that are not 0 or 1728 is $\lfloor \frac{p}{12} \rfloor$.

[2]One may use similar argument with $\Gamma(3)$ and $\Gamma(4)$ to get the result for prime $p = 5$.

[3]Here we lied a little. The genuine $X(\Gamma(5))$ by definition is over $\mathbb{Q}(\zeta_5)$, but as we consider everything over $\mathbb{C}$, we make base change $X(\Gamma(5)) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, this will split $X(\Gamma(5))$ into 4 connected component. What we use below is one of the component.

[4]This is not the same as S_5, as S_5 has no center, but $\mathrm{SL}_2(\mathbb{F}_5)$ sits in an exact sequence $0 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{F}_5) \to A_5 \to 1$.

[5]This is easy to see on the moduli problem or over $\mathbb{C}$.

Exercise for lecture 7: Siegel modular varieties, Shimura varieties of PEL type

Problem 7.1 (Siegel half space versus Hodge filtration)**.** Complete the proof of description of $\mathbb{C}$-points of Siegel space. In particular, explain the following two points:

(1) Why is providing a Hodge filtration for abelian varieties equivalent to giving a complex structure on $\Lambda \otimes \mathbb{R}$?

(2) Deduce that, if $J = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ gives the complex structure on $\Lambda \otimes \mathbb{R}$, then $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(iI_g)$ belongs to $\mathfrak{H}_g^{\pm}$.

Problem 7.2 (Siegel space as homogeneous)**.** (1) Consider $\mathrm{Sp}_{2g}(\mathbb{R})$ acting on $\mathfrak{H}_g$ given by $Z \mapsto (AZ+B)(CZ+D)^{-1}$. Show that this action is well-defined. What is the centralizer?

(2) Similarly consider the $\mathrm{GSp}_{2g}(\mathbb{R})$-action on $\mathfrak{H}_g^{\pm}$. What is the centralizer? Observe that this action factors through $\mathrm{PSp}_{2g}(\mathbb{R})$. Give an example of an element which turns $\mathfrak{H}_g$ into $\mathfrak{H}_g^{-}$.

What we are getting at here is a small subtlety for Shimura varieties, which we will encounter later. Let G be a reductive group over $\mathbb{R}$; the locally Hermitian space X we consider is technically $G_{\mathrm{ad}}(\mathbb{R})/K_{\mathrm{ad}}$, where G_{ad} is the quotient of G by its center, and K_{ad} is the maximal compact subgroup of G_{ad}. So $G(\mathbb{R})$ naturally acts on X and the action factors through $G_{\mathrm{ad}}(\mathbb{R})$. But the image $G(\mathbb{R})$ in $G_{\mathrm{ad}}(\mathbb{R})$ is typically only a connected component. (3) Explain the case when $G = \mathrm{Sp}_{2g,\mathbb{R}}$ using the exact sequence $1 \to Z(G) \to G \to G_{\mathrm{ad}} \to 1$.

Problem 7.3 (Fake moduli problem for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$)**.** We explain the moduli problem for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, using a variant of the moduli problem for $G' := (\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2)^{\det\in\mathbb{G}_m}$. Fix a totally real field F. Choose and fix a set of representative $\{\mathfrak{c}_1, \dots, \mathfrak{c}_{h^+}\}$ of the strict ideal class group of F, i.e. the quotient of fractional ideals of F by principal ideals generated by totally positive elements.

(1) For each abelian variety A over some scheme S equipped with a faithful $\mathcal{O}_F$-action, and for an ideal $\mathfrak{c} \subset \mathcal{O}_F$, the following definition of abelian variety $A \otimes_{\mathcal{O}_F} \mathfrak{c}$ makes sense: choose an element $\delta \in \mathcal{O}_F$ such that $\delta\mathcal{O}_F \subseteq \mathfrak{c}$, so that $\delta\mathfrak{c}^{-1}$ is a genuine ideal of $\mathcal{O}_F$. Let

$$H := A[\delta\mathfrak{c}^{-1}] = \{x \in A \mid \text{for every } a \in \delta\mathfrak{c}^{-1},\ a \cdot x = 0_A\}$$

be the subgroup of A killed by elements in $\delta\mathfrak{c}^{-1}$. We define $A \otimes_{\mathcal{O}_F} \mathfrak{c} := A/H$. Show that this $A \otimes_{\mathcal{O}_F} \mathfrak{c}$ is independent of the choice of δ, and carry a natural action of $\mathcal{O}_F$.

More canonically, we view A as a group functor on all S-schemes: for an S-scheme $A(T) := \mathrm{Hom}_S(T, A)$, then $(A \otimes_{\mathcal{O}_F} \mathfrak{c})(T) := \mathrm{Hom}_S(T, A) \otimes_{\mathcal{O}_F} \mathfrak{c}$ is a group functor represented by an abelian variety (as constructed above).

Let D denote the discriminant of F, and $\mathfrak{d}_F$ the different ideal of F. For each i, $\mathcal{M}_{\mathfrak{c}_i}$ is the moduli space over $\mathbb{Z}[\frac{1}{DN}]$, such that for every $\mathbb{Z}[\frac{1}{DN}]$-scheme S, $\mathcal{M}_{\mathfrak{c}_i}(S)$ is the isomorphism classes of triples (A, λ, i) such that

- A is an abelian scheme over S of dimension $[F : \mathbb{Q}]$, equipped with an action of $\mathcal{O}_F$,
- $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c}_i \xrightarrow{\simeq} A^\vee$ is an $\mathcal{O}_F$-equivariant polarization ($\mathfrak{c}_i$-polarization),[6] and

[6] Rigorously speaking, a $\mathfrak{c}_i$-polarization is an isomorphism $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c}_i \simeq A^\vee$ such that the natural morphism $\mathfrak{c}_i \to \mathrm{Hom}_{\mathcal{O}_F}(A, A \otimes \mathfrak{c}_i) \xrightarrow{\lambda} \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ induces an isomorphism between $\mathfrak{c}_i$ with "symmetric" elements in $\mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ and totally positive elements $\mathfrak{c}_i^+$ in $\mathfrak{c}_i$ with polarizations in $\mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$. Here symmetric morphism $\alpha : A \to A^\vee$ means that the dual morphism $A \cong A^{\vee\vee} \xrightarrow{\alpha^\vee} A^\vee$ is the same as α.

- $i : \mathfrak{d}_F^{-1} \otimes_{\mathbb{Z}} \mu_N \to A[N]$ is an embedding of group scheme over S. (Twisting by $\mathfrak{d}_F^{-1}$ will not affect this definition, but it will benefit our later discussion of compactifications.)

Define $\mathcal{M} := \coprod_i \mathcal{M}_{\mathfrak{c}_i}$; it is a smooth scheme over $\mathbb{Z}[\frac{1}{DN}]$ of dimension $[F : \mathbb{Q}]$.

(2) Prove that if $\mathfrak{c}$ and $\mathfrak{c}'$ are two ideals in the same strict ideal class. Show that there is an (not quite canonical) isomorphism $\mathcal{M}_{\mathfrak{c}} \simeq \mathcal{M}_{\mathfrak{c}'}$.

(3) Show that $\mathcal{O}_F^{\times,>0}$ (totally positive units) acts on each $\mathcal{M}_{\mathfrak{c}}$ by sending

$$(A, \lambda, i) \mapsto (A, u\lambda, i) \quad u \in \mathcal{O}_F^{\times,>0}.$$

Let $\mathcal{O}_{F,N}^{\times}$ denote the subgroup of $\mathcal{O}_F^{\times}$ consisting of elements that are congruent to 1 modulo N. Show that the action of the subgroup $(\mathcal{O}_{F,N}^{\times})^2$ is trivial on each $\mathcal{M}_{\mathfrak{c}}$.

The Shimura variety associated to $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ with $\Gamma_1(N)$-level structure is isomorphic to

$$Y_1(N) := \mathcal{M} \Big/ \Big(\mathcal{O}_F^{\times,>0} / \big(\mathcal{O}_{F,N}^{\times}\big)^2\Big)$$

A reference for more general level structure and for the complex points of this moduli problem is section 2.3 of Yichao Tian and Liang Xiao, p-adic cohomology and classicality of overconvergent Hilbert modular forms, in Astérisque 382 (2016), 73–162.

(4) The polarization $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c}_i \xrightarrow{\simeq} A^\vee$ induces an $\mathcal{O}_M$-linear perfect pairing

$$H^1_{\mathrm{dR}}(A/\mathcal{M}) \times \big(H^1_{\mathrm{dR}}(A/\mathcal{M}) \otimes_{\mathcal{O}_F} \mathfrak{c}_i^{-1}\big) \to \mathcal{O}_{\mathcal{M}},$$

which in turn defines a natural $\mathcal{O}_M \otimes_{\mathbb{Z}} \mathcal{O}_F$-linear isomorphism

$$\wedge^2_{\mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathcal{O}_F} H^1_{\mathrm{dR}}(A/\mathcal{M}) \cong \mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathfrak{c}_i \mathfrak{d}_F^{-1}$$

Explain where the factor $\mathfrak{d}_F$ comes from.

(5) Let L denote the Galois closure of $F(\sqrt{u};\ u \in \mathcal{O}_F^{\times,>0})$ inside $\mathbb{C}$, and let $\mathcal{O}_L$ denote the ring of integers of L. We base change $\mathcal{M}$ to $\mathcal{O}_L$ to define line bundles ω_τ and $\epsilon_\tau := \wedge^2_{\mathcal{O}_{\mathcal{M}}}(\mathcal{H}^1_{\mathrm{dR}}(A)_\tau)$, for embeddings $\tau : F \to L$. Recall that for a paritious weight $\kappa = ((k_\tau)_{\tau \in \Sigma}, w) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$, we can define a line bundle

$$\omega^\kappa := \bigotimes_{\tau \in \Sigma} \Big(\omega_\tau^{k_\tau} \otimes_{\mathcal{O}_{\mathcal{M}}} \epsilon_\tau^{(w-k_\tau)/2}\Big).$$

In a natural way, we let $\mathcal{O}_F^{\times,>0}$ to act on ω_τ and ϵ_τ by, $u \in \mathcal{O}_F^{\times}$

- sending a section s of ω_τ to $u^{-1/2} \cdot \langle u \rangle^*(s)$, and
- sending a section s of ϵ_τ to $u^{-1} \cdot \langle u \rangle^*(s)$,

where $\langle u \rangle$ is the action of $\mathcal{O}_F^{\times,>0}$ on $\mathcal{M}_{\mathfrak{c}}$ mentioned above.

Show that the induced action of $\mathcal{O}_F^{\times,>0}$ on ω^κ is compatible with the action on $\mathcal{M}$ and hence we may descent ω^κ to $Y_1(N)$ (but not each individual ω_τ and ϵ_τ).

Exercise for lecture 8: General theory of Shimura varieties

Problem 8.1 (h versus μ)**.** Let T be a torus over $\mathbb{R}$. Show that there is a one-to-one correspondence between

$$\begin{array}{ccc}\big\{\text{homomorphisms } h:\mathbb{S}\to T\big\} & \longleftrightarrow & \big\{\text{homomorphisms } \mu:\mathbb{G}_{m,\mathbb{C}}\to T_{\mathbb{C}}\big\}\\ h & \longmapsto & \mu_h\end{array}$$

where $\mu_h : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z\mapsto(z,1)} \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} T_{\mathbb{C}}$.

Problem 8.2 (Shimura set associated to CM types)**.** Let E be a CM field with F its maximal totally real subfield. Recall that a CM type is a set of embeddings $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C})$ such that $\mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C}) = \Phi \sqcup \Phi^c$, where $\Phi^c := \{c\circ\phi;\ \phi\in\Phi\}$ and c denotes the complex conjugation. Consider the torus $T := \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$. It comes equipped with a cocharacter

$$\begin{array}{rcl}\mu_\Phi : \mathbb{G}_{m,\mathbb{C}} & \longrightarrow & T_{\mathbb{C}} \cong \prod_{\phi\in\Phi} \mathbb{G}_{m,E}\times_{E,\phi}\mathbb{C}\\ z & \longmapsto & z \text{ at each } \phi\in\Phi.\end{array}$$

The group T admits a subgroup $T^{\mathbb{Q}}$ whose R-points for a $\mathbb{Q}$-algebra R is

$$T^{\mathbb{Q}}(R) = \big\{x \in T(R) = (R\otimes_{\mathbb{Q}} E)^\times;\ \mathrm{Nm}_{E/F}(x)\in R^\times\big\}.$$

(1) Observe that μ_Φ has image in $T^{\mathbb{Q}}$.

(2) By the previous problem, μ_Φ corresponds to $h_\Phi : \mathbb{S}\to T_{\mathbb{R}}$ (or even $h_\Phi^{\mathbb{Q}} : \mathbb{S}\to T_{\mathbb{R}}^{\mathbb{Q}}$)

(3) Show that the reflex field E_Φ of $(T,\{h_\Phi\})$ or $(T^{\mathbb{Q}},\{h_\Phi^{\mathbb{Q}}\})$ can be described as follows: let $\mathbb{Q}^{\mathrm{alg}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let H denote the subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ that stabilizers the CM type Φ, that is for any $h\in H$, $\{h\circ\phi;\ \phi\in\Phi\} = \Phi$. Then E_Φ is the subfield of $\mathbb{Q}^{\mathrm{alg}}$ fixed by H.

(4) Take a special case: $E = E_0F$ for E_0 an imaginary quadratic field and F a totally real field. Fix one embedding $\tau : E_0 \to \mathbb{C}$. Show that this induces a CM type $\Phi_\tau := \{\phi\in\mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C});\ \phi|_{E_0} = \tau\}$. Show that the reflex field of this Φ_τ is just E_0. What's the corresponding Shimura reciprocity map?

Problem 8.3 (Computation of the reflex field of a special type of Shimura curve)**.** This type of Shimura curve appears in the study of generalizations of Heegner points to the totally real case.

Let F be a totally real field, and let B be a quaternion algebra over F such that there is a unique $\tau_0 : F\to\mathbb{R}$:

$$B\otimes_{F,\tau}\mathbb{R} \cong \begin{cases}\mathrm{M}_2(\mathbb{R}) & \tau=\tau_0\\ \mathbb{H} & \tau\neq\tau_0\end{cases}$$

Let $G = \mathrm{Res}_{F/\mathbb{Q}}\,B^\times$. Then we can define a Shimura datum for G, by taking h to be the $G(\mathbb{R})$-conjugacy class of

$$\begin{array}{rcl}h : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times & \longrightarrow & G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})\times\prod_{\tau\neq\tau_0}\mathbb{H}^\times\\ z = x+iy & \longmapsto & \Big(\big(\begin{smallmatrix}x & -y\\ y & x\end{smallmatrix}\big), 1, \dots, 1\Big).\end{array}$$

Show that the reflex field of this Shimura datum is F embedded in $\mathbb{C}$ via τ_0, precisely the one that we used above.

(The upshot is that the Shimura curve is then defined over F embedded in $\mathbb{C}$ via τ_0. Somehow, one should intrinsically think of this Shimura curve defined over F canonically, and associated to B intrinsically. Namely, if we change how F embeds into $\mathbb{C}$, it will affect accordingly how the Shimura curve over F is embedded in $\mathbb{C}$.)

Problem 8.4 (Geometric connected components of Shimura varieties)**.** Let (G, X) denote a Shimura datum and let G_{ab} denote the maximal abelian quotient of G and $\nu : G \to G_{\mathrm{ab}}$ the natural map. Then each $h : \mathbb{S} \to G_{\mathbb{R}}$ in X induces the same homomorphism $h_{\mathrm{ab}} : \mathbb{S} \to G_{\mathbb{R}} \to G_{\mathrm{ab},\mathbb{R}}$. So we have a natural morphism of Shimura data

$$(G, X) \to (G_{\mathrm{ab}}, \{h_{\mathrm{ab}}\}).$$

If $K \subseteq G(\mathbb{A}_f)$ is an open compact subgroup then $\nu(K)$ is an open compact subgroup of $G_{\mathrm{ab}}(\mathbb{A}_f)$.

(This problem is taken from Milne's Introduction to Shimura varieties, [Mi05, Theorem 5.17].) **Assume that the derived subgroup G^{der} is simply-connected.** Then we will prove below that the natural map $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{\nu(K)}(G_{\mathrm{ab}}, \{h_{\mathrm{ab}}\})$ "almost" induces an isomorphism on the set of geometric connected components. More precisely, let Z denote the center of G and set

$$G_{\mathrm{ab}}(\mathbb{R})^\dagger := \mathrm{Im}(Z(\mathbb{R}) \to G_{\mathrm{ab}}(\mathbb{R})) \quad \text{and} \quad G_{\mathrm{ab}}(\mathbb{Q})^\dagger := G_{\mathrm{ab}}(\mathbb{Q}) \cap G_{\mathrm{ab}}(\mathbb{R})^\dagger.$$

Then the natural map

$$\mathrm{Sh}_K(G, X) \to G_{\mathrm{ab}}(\mathbb{Q})^\dagger \backslash G_{\mathrm{ab}}(\mathbb{A}_f)/\nu(K) \tag{8.4.1}$$

induces a bijection on the geometric connected components.

(1) First look at what this statement entails in some examples: consider $G = \mathrm{GL}_{2,\mathbb{Q}}$, that is the case of modular curves. In this case, the maximal abelian quotient is given by $\nu = \det : \mathrm{GL}_{2,\mathbb{Q}} \to G_{\mathrm{ab}} = \mathbb{G}_{m,\mathbb{Q}}$. So $G_{\mathrm{ab}}(\mathbb{R})^\dagger = \mathbb{R}^{>0}$ and $G_{\mathrm{ab}}(\mathbb{Q})^\dagger = \mathbb{Q}^{\times,>0}$. If we take $\Gamma_1(N)$-level structure, it corresponds to $\widehat{\Gamma}_1(N) = \big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}});\ c \equiv 1, d \equiv 0 \pmod N \big\}$. The determinant is the entire $\widehat{\mathbb{Z}}^\times$. So

$$\pi_0^{\mathrm{geom}}(\mathrm{Sh}_{\widehat{\Gamma}_1(N)}(\mathrm{GL}_{2,\mathbb{Q}})) = \mathbb{Q}^{\times,>0} \backslash \mathbb{A}_f^\times / \widehat{\mathbb{Z}}^\times = \{1\}.$$

In this case, the modular curve is always connected.

On the other hand, when the level structure is $\Gamma(N)$, corresponding to $\widehat{\Gamma}(N) := \big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}});\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \big\}$, whose determinant is $(1 + N\widehat{\mathbb{Z}})^\times$. In this case

$$\pi_0^{\mathrm{geom}}(\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}})) = \mathbb{Q}^{\times,>0} \backslash \mathbb{A}_f^\times / (1 + N\widehat{\mathbb{Z}})^\times = (\mathbb{Z}/N\mathbb{Z})^\times.$$

We can further discuss the Galois action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ on the set of geometric connected component (which comes from the Shimura reciprocity map for G_{ab} and $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}_{2,\mathbb{C}} \xrightarrow{\nu} \mathbb{G}_{m,\mathbb{C}}$ sending $z \to z$)

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{Art} \mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}^\times_{>0} = \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times.$$

From this, we see that the Galois action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ on $(\mathbb{Z}/N\mathbb{Z})^\times$ is factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. There is another way to explain this: $\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}})$ is an irreducible curve over $\mathbb{Q}(\zeta_N)$, but when we view it naturally over $\mathbb{Q}$ instead, and make base change, we see that $\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}}) \times_{\mathbb{Q}} \mathbb{C}$ has $(\mathbb{Z}/N\mathbb{Z})^\times$-geometric connected components.

(2) Now we indicate the proof of (8.4.1). For this, we need to accept a few blackbox theorems from [PR94, Theorem 6.4, 6.6]: (these are very useful statements)

- (vanishing of nonarchimedean cohomology for simply-connected groups) If G is simply-connected semisimple group over $\mathbb{Q}_\ell$, then $H^1(\mathbb{Q}_\ell, G) = \{1\}$.
- (Hasse principle for simply-connected group and adjoint group) For an algebraic group G over $\mathbb{Q}$, we define
$$Ш^1_f(\mathbb{Q}, G) := \mathrm{Ker}\left(H^1(\mathbb{Q}, G) \to \prod_{\ell\neq\infty} H^1(\mathbb{Q}_\ell, G)\right).$$
Then if G is simply-connected and semisimple, then
$$Ш^1_f(\mathbb{Q}, G) \to H^1(\mathbb{R}, G)$$
is an isomorphism. (If G is semisimple and adjoint, this is injective.)
- If G is a simply-connected real reductive group (or a compact real reductive group), then $G(\mathbb{R})$ is connected.
- (Strong approximation for simply-connected groups) If G is a simply-connected group over a number field F; suppose that v is a place of F such that $G(F_v)$ is non-compact at each F-simple factor of G, then $G(F)$ is dense in $G(\mathbb{A}_F^{(v)})$.

Applying these statements, we prove the following in turns.

- Let X^+ denote the connected component of X, then the stabilizer of X^+ under the $G(\mathbb{R})$ action is $G(\mathbb{R})_+ :=$ preimage of the connected component of $G_{\mathrm{ad}}(\mathbb{R})$ in $G(\mathbb{R})$. Set $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Then
$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K.$$
- If G^{der} is simply-connected, then $G(\mathbb{R})_+ = G^{\mathrm{der}}(\mathbb{R}) \cdot Z(\mathbb{R})$.
- If G^{der} is simply-connected, then $G(\mathbb{A}_f) \to T_{\mathrm{ab}}(\mathbb{A}_f)$ is surjective and sends open compact subgroups to open compact subgroups.

Concludes eventually that (8.4.1) induces an isomorphism between geometric connected components.

Remark: the geometric connected component of more general Shimura varieties is somewhat subtle, see the discussion in Deligne's article in Corvallis.

References

[Mi05] J. Milne, Introduction to Shimura varieties, Harmonic Analysis, the Trace Formula and Shimura Varieties, Clay Mathematics Proceedings, Volume 4 (2005), 265-378.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, Vol. **139**, Academic Press Inc. Boston, MA, 1994, xii+614 pp.

Lost energy to provide more exercises for Lecture 9–10.