Springer Fibers and Quiver Varieties
Lecture 1: Lie Algebras

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§1 Overview

Nakajima quiver var = { certain [quiver rep'n] } e.g.
$$\left[\begin{tikzcd} & \mathbb{C} \arrow[d] & \\ \mathbb{C} \arrow[r, shift left] & \mathbb{C}^2 \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C} \arrow[l, shift left] \end{tikzcd}\right]$$

[which are geometric realizations of

(1) $U(\mathfrak{g})$, $\mathfrak{g}$ Kac-Moody Lie alg

(2) Integrable rep'n of $\mathfrak{g}$.

(3) Cotangent bundle of partial flag var.]

Springer fiber $\mathcal{B}_\lambda$ = { certain flags }

e.g. $0 \subset \langle e_1 + e_2 \rangle$

<u>Punchline</u> (1) Springer correspondence

$$\left\{\begin{matrix}\text{nilpotent orbit}\\ \text{of } \mathfrak{gl}_n\end{matrix}\right\} \xleftrightarrow{1-1} \mathrm{Irr}\,\underbrace{\boxed{\Sigma_n}}_{}$$
symmetric grps

$H^{top}(\mathcal{B}_\lambda)$ "Springer rep'n".

$\mathcal{O} \longleftrightarrow$ Jordan type λ

$(n_1, n_2, \cdots, n_k)$ sizes of Jordan blocks

(2) Irred. of $\mathcal{B}_\lambda$ are "useful" in rep'n theory.

What's the rep'n theory?

(= the study of rep'ns of associated algs).

<u>Def'n</u> A rep'n of alg A is an alg homomorphism
$$\rho : A \longrightarrow \mathrm{End}(V) \text{ for some vect space } V$$
i.e. $\rho(a)\rho(b) = \rho(ab)$, $\forall a, b \in A$.

<u>Def'n</u> (equivalently) An A-module is a v.s. V with an A-action s.t. $a.(b.v) = (ab).v$, $\forall a,b \in A$, $v \in V$.

A submodule of V is a subspace $W \subseteq V$ s.t. $A.W \subseteq W$.

$\rightsquigarrow$ V is called simple / irred if V has no proper subs.

V is called indecomposable if $V \neq W_1 \oplus W_2$.

<u>Typical Problems in Rep(A):</u>

(1) Classify and describe irred. & indecomposable modules.

(2) Do this for finite dim'l case.

<u>Examples</u> (1) G = fin grp $\rightsquigarrow$ grp alg $\underset{\text{f.g. dim'l alg}}{\mathbb{C}[G]} = \mathrm{Span}_{\mathbb{C}}\{a_g \mid g \in G\}$

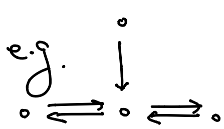
s.t. $a_g \cdot a_h = a_{gh}$, $\forall g,h \in G$

(2) $\mathfrak{g}$ = Lie alg $\rightsquigarrow$ universal enveloping alg

$A = U(\mathfrak{g}) = \mathrm{Span}_{\mathbb{C}}\{\text{PBW basis}\} \leftarrow$ w.r.t. Lie brackets

(∞-dim'l in general)

(3) Q = quiver (= finite directed graph) e.g.



$\rightsquigarrow$ path alg

$= \mathrm{Span}_{\mathbb{C}}\{a_x \mid x \text{ is a path in } \Gamma\}$

$a_x \circ a_y = 0$ unless $x \to y$ nose to tail

<u>§2 Lie Algebras</u>

<u>Def'n</u> A Lie alg is a v.s. equip'd with a Lie bracket

$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ s.t.

(L1) $[\cdot,\cdot]$ bilinear

(L2) $[x,x]=0, \ \forall x \in \mathfrak{g}$

(L3) (Jacobi) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$.

exercise: (L1)+(L2) $\Rightarrow [x,y]=-[y,x]$.

<u>Rmk</u> Adjoint operator: $\mathrm{ad}_x : y \longmapsto [x,y]$

$(L3) \Leftrightarrow \mathrm{ad}_x[y,z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z]$ c.f. Leibniz rule

<u>Examples</u> (1) $\mathfrak{gl}_n(\mathbb{C}) = \mathrm{Mat}_n(\mathbb{C})$ with $[A,B]=AB-BA$

Classical Lie algs:

(2) $\mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid tr(A)=0\}$ type A_{n-1}

(3) $\mathfrak{sp}_{2n}(\mathbb{C}) = \{A \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid MA + A^TM = 0\}$ symplectic

where $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ type C_n

(4) $\mathfrak{so}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid MA + A^TM = 0\}$

where $M = \begin{cases} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} & 2 \mid n \quad \text{type } D_n \\ \begin{pmatrix} 1 & & \\ & 0 & I_n \\ & I_n & 0 \end{pmatrix} & 2 \nmid n \quad \text{type } B_n \end{cases}$

<u>Example</u> (Type A_1)

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} = \mathbb{C}\underbrace{\begin{pmatrix} & 1 \\ & \end{pmatrix}}_{e} \oplus \mathbb{C}\underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{h} \oplus \mathbb{C}\underbrace{\begin{pmatrix} & \\ 1 & \end{pmatrix}}_{f}.$$

$\rightsquigarrow$ $[e,f] = ef - fe = h$

$[h,e] = 2e, \ [h,f] = -2f$

(i.e. ad_h has eigenvectors e, f & eigenvalues $2, -2$).

<u>Def'n</u> An ideal of $\mathfrak{g}$ is a subspace I s.t. $[\mathfrak{g}, I] \subseteq I$

A Lie alg is simple if it has no proper ideals.

Thm (Cartan decomp) If $\mathfrak{g}$ is simple then

$\exists$ Cartan subalg $\mathfrak{h} \in \mathfrak{g}$ s.t. $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right)$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid ad_h x = \alpha(h) x, \ \forall h \in \mathfrak{h}\}$ $\leftarrow$ root space if $\mathfrak{g}_\alpha \neq 0$

for fixed $\alpha : \mathfrak{g} \to \mathbb{C}$ $\leftarrow$ root if $\alpha \neq 0$.

$\rightsquigarrow \Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\}\}$ set of roots

Moreover, $\dim \mathfrak{g}_\alpha = 1$, $\forall \alpha \in \Phi$

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in \Phi$

$\Phi = -\Phi$, etc.

Example (cont.) $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$,

$\mathfrak{h} = \mathbb{C}h$, $\mathfrak{g}_\alpha = \mathbb{C}e$, $\mathfrak{g}_{-\alpha} = \mathbb{C}f$, $\alpha : \mathfrak{h} \to \mathbb{C}$, $h \mapsto 2$

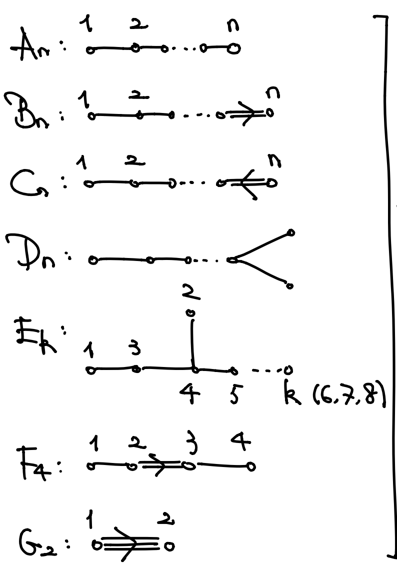
Thm (classification)

$\{$(simple) Lie alg$\} \xleftrightarrow{1-1} \{$(irred.) root systems$\}$

$\updownarrow$ 1-1

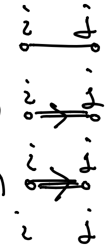
$\{$(indecomposable) Cartan matrices$\}$

$\updownarrow$ 1-1



$\leftarrow$ $\{$(connected) Dynkin diagrams$\}$

<u>Recipe 1</u>: Dynkin $\rightsquigarrow$ Cartan matrix



(i) $a_{ii} = 2$, $\forall i$

(ii) i — j $\quad a_{ij} = a_{ji} = -1$

(iii) $i \Rightarrow j$ $\quad a_{ij} = -1$, $a_{ji} = -2$

(iv) $i \Rrightarrow j$ $\quad a_{ij} = -1$, $a_{ji} = -3$

(v) i $\quad j$ $\quad a_{ij} = a_{ji} = 0$.

Recipe 2 Cartan matrix $\rightsquigarrow$ Lie alg

$$\mathfrak{g}(A) = \langle e_i, f_i, h_i \rangle_{i=1,\dots,n} / \sim$$

Relation $\sim$: $[h_i, h_j] = 0, \ \forall i, j,$

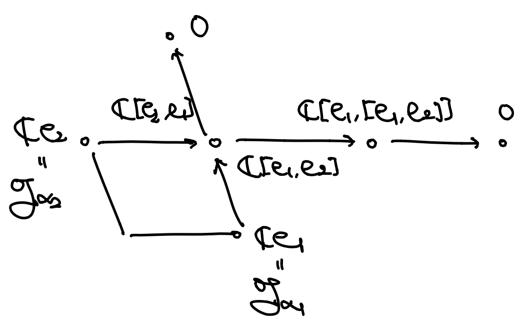
$$\left.\begin{array}{l} [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \\ [e_i, f_j] = \delta_{ij} h_i \end{array}\right\} \text{Chevalley relation}$$

$\mathrm{ad}_{e_i}^{1-a_{ji}}(e_j) = 0 = \mathrm{ad}_{f_i}^{1-a_{ji}}(f_j), \ \forall \underline{\underline{i \neq j}}$ Serre relation

(to describe root systems)

Example (Type B_2)

$$\overset{1}{\circ} \Rightarrow \overset{2}{\circ} \longmapsto A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \longmapsto \mathfrak{g}(A) = \langle e_1, e_2, f_1, f_2, h_1, h_2 \rangle.$$



Serre relation:

$$\mathrm{ad}_{e_1}^{1-a_{21}}(e_2) = \mathrm{ad}_{e_1}^{3}(e_2) = 0$$

$$\mathrm{ad}_{e_2}^{1-a_{12}}(e_1) = \mathrm{ad}_{e_2}^{2}(e_1) = 0$$

Interlude: Generalized Cartan matrices

Def'n $a_{ij} \leq 0$ if $i \neq j$, and $a_{ii} = 2$.
Also, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
$\}$ $\rightsquigarrow$ can define Kac-Moody Lie algs

Classes (i) (finite) $\exists u > 0$ s.t. $Au > 0$ $\rightsquigarrow$ $\dim \mathfrak{g}_\alpha = 1$

(ii) (affine) $\exists u > 0$ s.t. $Au = 0$ $\rightsquigarrow$ $\dim \mathfrak{g}_\alpha = \begin{cases} 1, & \text{"real"} \\ e, & \text{"imag"} \end{cases}$

(iii) (indefinite) $\exists u < 0$ s.t. $Au < 0$ $\rightsquigarrow$ still unknown.

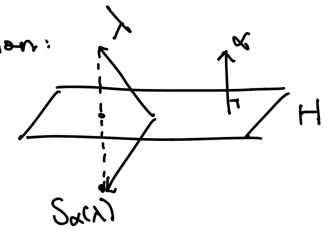
The set Φ is a root system $\Phi \subseteq E := \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$ euclidean space.
equipped w/ inner product (Killing form)

(R1) $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$, $\forall \alpha \in \Phi$

(R2) $S_\alpha(\Phi) = \Phi$ where $\forall \alpha \in \Phi$,

$$S_\alpha(\lambda) := \lambda - (\lambda, \alpha^\vee)\Phi, \quad \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.$$

reflexion:



(R3) $(\beta, \alpha^\vee) \in \mathbb{Z}$, $\forall \alpha, \beta \in \Phi$.

$\rightsquigarrow$ The Weyl group of $\mathfrak{g}(A)$ is

$$W = \langle S_\alpha \mid \alpha \in \Phi \rangle \leq GL(E) \hookrightarrow \Sigma_{|\Phi|}.$$

} some kind of coxeter grp

$\rightsquigarrow$ the length function $l : W \to \mathbb{Z}_{\geq 0}$

$\exists$ Bruhat order on W.

§3 Representation Theory of Lie Algebras

Goal (1) Construct & classify irreducibles as quotients of Verma modules. } needs univ enveloping algs (UEA)

(2) Understand fin dim'l irreducibles by Weyl's character formula

(3) The ∞-dim'l case: Kazhdan-Lusztig theory

For Lie alg $\mathfrak{g}$, define an associated alg (called UEA)

$$U(\mathfrak{g}) := \Big(\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}\Big) / \underbrace{\langle x\otimes y - y \otimes x - [x,y]\rangle}_{J}$$

abbrev: $x_1 \otimes \cdots \otimes x_k + J = x_1 \cdots x_k$.

Thm (Poincaré-Birkhoff-Witt) (assume $\dim \mathfrak{g} < \infty$)

If $\{x_i\}_{i \in I}$ is a basis of $\mathfrak{g}$, $(I, \leq)$ is totally ordered.

Then $\{x_{i_1}^{r_1} \cdots x_{i_n}^{r_n} : r_i \geq 0,\ i_1 < \cdots < i_n\}$ is a basis of $U(\mathfrak{g})$.

<u>Example</u> (1) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \mathrm{Span}_{\mathbb{C}}\{e, f, h\}$

$e < f < h \Rightarrow U(\mathfrak{g})$ has a basis $\{e^a f^b h^c : a, b, c \in \mathbb{Z}_{\geq 0}\}$

(2) $f < h < e \Rightarrow U(\mathfrak{g})$ has a basis $\{f^a h^b e^c : a, b, c \in \mathbb{Z}_{\geq 0}\}$.

In particular, we can split Φ into $\Phi^+ \sqcup (-\Phi^+) = \Phi^+ \sqcup \Phi^-$.

Fix an ordering $\Phi^+ = \{\beta_1 < \cdots < \beta_m\}$ $\rightsquigarrow$ nonzero $e_i \in \mathfrak{g}_{\beta_i}$, $f_i \in \mathfrak{g}_{-\beta_i}$

$\Phi^- = \{h_1 < \cdots < h_m\}$ $\rightsquigarrow$ similarly.

$\Rightarrow \{f_1^{a_1} \cdots f_m^{a_m} h_1^{b_1} \cdots h_\ell^{b_\ell} e_1^{c_1} \cdots e_m^{c_m},\ a_i, b_i, c_i \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U(\mathfrak{g})$.

For each $\lambda \in \mathfrak{h}^*$, define the Verma module $M(\lambda) := U(\mathfrak{g}).V_\lambda^+$

s.t. $\begin{cases} e_i \cdot V_\lambda^+ = 0, & \forall i = 1, \dots, m \\ h \cdot V_\lambda^+ = \lambda(h) V_\lambda^+, & \forall h \in \mathfrak{h} \end{cases}$

<u>Philosophy</u> We don't wanna define rep'n theories for each assoc alg rather than construct them over known theory of Lie alg (so there comes lots of relations to suit Lie types).

<u>Eventually</u> $\boxed{\mathrm{Rep}(\mathfrak{g}) \simeq \mathrm{Rep}(U(\mathfrak{g}))}$.

$\Rightarrow M(\lambda)$ has a basis $\{f_1^{a_1} \cdots f_m^{a_m} V_\lambda^+\}$.

<u>Fact</u> (1) $M(\lambda)$ has a unique max'l submodule $N(\lambda)$

$\rightsquigarrow M(\lambda)/N(\lambda) =: L(\lambda)$ unique irred quot

<u>E.g.</u> $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $(\lambda : \mathfrak{h} \to \mathbb{C}) \equiv \lambda(h) \in \mathbb{C}$.

$M(\lambda)$ with basis $\{V_\lambda^+, fV_\lambda^+, f^2 V_\lambda^+, \dots\}$

$\rightsquigarrow M(0)$ with basis $\{V_0^+, fV_0^+, f^2 V_0^+, \dots\}$.

$N(0)$ with basis $\{fV_0^+, f^2 V_0^+, \dots\}$.

$\Rightarrow L(0) = $ 1-dim'l irred $= \{\overline{V_0^+}\}$.

(2) If L is irred. then $L \simeq L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

$\rightsquigarrow$ $M = M(\lambda)$ decomposes into

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \quad M_\mu = \{m \in M \mid h.m = \lambda(h).m,\ \forall h \in \mathfrak{h}\}.$$

$$\Rightarrow \operatorname{char} M = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e(\mu).$$

<u>Thm</u> (Weyl) $\dim L(\lambda) < \infty$

$$\Rightarrow \operatorname{char} L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{l(w)} e(w \cdot 0)}$$

dot action.

<u>Thm</u> (Kazhdan-Lusztig) $\operatorname{char} L(\lambda) = \sum_{w \in W} \mathbb{Z} \cdot \operatorname{char} M(w \cdot \lambda)$ (∞-dim'l. vague)

coeff given by Hecke alg

has sth to do with Weyl grp