Counting Points on Shimura Varieties
Lecture 1

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Ref [Kot 92] Points on Shimura varieties. JAMS
[Kot 90] Shimura varieties and λ-adic rep'ns. (conference proceeding). Ann Arbor, vol I.
[Kis 10] Kisin, Integral models. JAMS.
[– 17] Mod p points. JAMS.
[KSZ] Kisin-Shin-Zhu to appear.

§1 Hasse-Weil zeta functions

X smooth proj. $/\mathbb{Q}$.

$\forall\!\!\!\forall p$, $\exists$ "good" integral model" $\mathfrak{X}_p/\mathbb{Z}_p$.

almost all — sm. proj. scheme$/\mathbb{Z}_p$, whose generic fiber is $X_{\mathbb{Q}_p}$.

Local zeta factor: $\zeta_p(X,s) = \exp\left(\sum_{n=1}^{\infty} \#\mathfrak{X}_p(\mathbb{F}_{p^n}) \cdot \frac{p^{-ns}}{n}\right)$

LTF proper smooth base change

$H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$, $\ell \neq p$ or $H^i_{et}(\mathfrak{X}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$

$$\prod_{i=0}^{2\dim X} \det(1 - \underset{\text{geom.}}{\mathrm{Frob}_p} \cdot T \mid H^i_{et})^{(-1)^{i+1}} \Big|_{T = p^{-s}}$$

everything is well-defined (esp. $\zeta_p(X,s)$) as long as $\mathfrak{X}_p$ exists.

$$\zeta(X,s) = \prod_{a.a.\,p} \zeta_p(X,s) \quad (\mathrm{Re}\, s \gg 0).$$

Ultimate conj: $\zeta(X,s)$ has a meromorphic continuation to $\mathbb{C}$.

E.g. $X = \operatorname{Spec} \mathbb{Q} \rightsquigarrow \zeta(X,s) =$ Riemann's zeta.

§2 Hasse-Weil zeta functions for Shimura varieties

Theorem (Eichler-Shimura, cf. Xiao's course)

$$X = X_0(N),\quad \zeta(X,s) = \underbrace{\zeta(s)}_{H^0}\,\underbrace{\zeta(s-1)}_{H^2}\,\underbrace{\prod_{i=1}^{g} L(f_i,s)^{-1}}_{H^1}$$

where $\{f_1,\cdots,f_g\}$ is an eigenbasis of $S_2(\Gamma_0(N))$

$L(f_i,s)$ = L-func. of f_i built from the Hecke-eigenvalues of f_i.

Hecke: has mero. cont. to $\mathbb{C}$.

Rmk If we replace $H^i_{et}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)$ by $H^i_{et}(X_{\bar{\mathbb{Q}}},\mathcal{L})$

suitable local system on X

(built from rep'ns of $G = GL_2$, cf. LX)

then we see higher-weight modular forms in the analogue of $\zeta(X,s)$.

§3 Generalized SV

Shimura datum (G,X).

G: reductive group$/\mathbb{Q}$, e.g. GL_2

X: $G(\mathbb{R})$-conjugacy class of an $\mathbb{R}$-homo. cplx str. on X.

$$\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}.$$

$K \subset G(\mathbb{A}_f)$ congruent open subgp.

$$\rightsquigarrow Sh_K(G,X) = Sh_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K = \coprod_{i=1}^{m} X_i/\Gamma_i.$$

X_i is a connected component of X.

Γ_i is an arithmetic subgp of $G(\mathbb{Q}) \curvearrowright X_i$.

complex manifold $\xrightarrow{\text{Baily-Borel}}$ quasi-proj. variety $/\mathbb{C}$

Shimura - Deligne - Borovoi - Milne $\rightsquigarrow$ Sh_K has a canonical model $/E$
1970 - 1990 a number field $/\mathbb{C}$

In a lot of cases (PEL), the canonical model of Sh_K/E
can be directly defined as a moduli space of AV's
+ polarization + endomorphism str. + level str.
$\rightsquigarrow$ integral models
e.g. modular curve / Siegel modular varieties / some unitary SV.

More recently (Rasin, Kisin, Madapusi Pera - Kim, Kisin - Pappas),
hyperspecial level at $p>2$. $p=2$ some parabolic level at p.
have constructed integral models beyond PEL case.

Expectation the set of $\mathbb{F}_p$ points of a suitable integral model
also has a group theoretic description similar to
$Sh_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

For simplicity, $E=\mathbb{Q}$ below.

Conj Hasse-Weil ζ of SV $\longleftrightarrow$ explicit autom. L-func.

Langlands' idea $\zeta(Sh_K, s) \xleftarrow{\ \forall p\ } \{\#\mathcal{S}_K(\mathbb{F}_{p^n}) \mid n\}$

$\mathcal{S}_K$: "good" int. model $/\mathbb{Z}_p$.

Langlands: need to count
AV + str's $/\mathbb{F}_{p^n}$

autom. rep'ns of G $\xleftrightarrow[\text{Selberg, Arthur}]{\text{Tr Formula}}$ orbital integrals

i.e. $G(\mathbb{A}) \curvearrowright L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ i.e. integrals of some func'ns on $G(\mathbb{A})$ over a conjugacy class in $G(\mathbb{A})$.

Rmk 1) When G/Z_G contains a $\mathbb{Q}$-split torus,

(e.g. $G = GL_2$, $G/Z_G = PGL_2 \supset \mathbb{G}_m$)

Sh_K is NOT projective $/E$.

Related problem $G(\mathbb{Q})\backslash G(\mathbb{A})$ is non-compact

$\rightsquigarrow$ $f \in C_c^\infty(G(\mathbb{A}))$

$Tr(f | L^2(G(\mathbb{Q})\backslash G(\mathbb{A})))$ doesn't make sense.

* Trace formula becomes an identity between two quantities whose def'ns are really complicated.

2) For applications, we are not just satisfied w/ understanding

$Gal(\bar{E}/E) \curvearrowright H^i_{et}(Sh_{K,\bar{E}}, \mathbb{Q}_\ell)$.

We want to also understand

$\mathbb{Z}[Gal(\bar{E}/E)] \times \mathcal{H}(G(\mathbb{A}_f)/\!/K) \curvearrowright H^i_{et}(Sh_{K,\bar{E}}, \mathbb{Q}_\ell)$

For this, we need to understand

for a fixed $f \in \mathcal{H}(G(\mathbb{A}_f)/\!/K)$

$\{Tr(f \times Frob_p^a \mid H^i_{et}) \mid a\}$ $\forall\forall p$. (depending on f)

For the fixed f, for almost all p, we have

$K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$, $K_p \subset G(\mathbb{Q}_p)$

$f = f^p f_p$, $f^p \in \mathcal{H}(G(\mathbb{A}_f^p)/\!/K^p)$

$f_p = 1_{K_p} : G(\mathbb{Q}_p) \longrightarrow \{0,1\}$.

WMA by linearity, $f^p = 1_{K^p g K^p}$, $g \in G(\mathbb{A}_f^p)$.

$$\sum_i (-1)^i \mathrm{Tr}(f \times \mathrm{Frob}_p^a \mid H^i_{et}) = \#\text{ fixed points of the correspondence}$$

$$\begin{tikzcd} & \mathcal{S}_{(K^p \cap g K^p g^{-1})\cdot K_p} \arrow[r, "\mathrm{Frob}_p^a"] \arrow[dl, "g"'] & \mathcal{S}_{(K^p \cap g K^p g^{-1})\cdot K_p} \arrow[d] \\ \mathcal{S}_K & & \mathcal{S}_K \end{tikzcd}$$

$K = K^p K_p$ cf. $\# \mathcal{S}_K(\mathbb{F}_{p^n})$

3) Instead of $H^i(Sh_{K,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$

can look at a local system $\mathcal{L}$ assoc. w/ a rep'n of G.

§4 More precise conjectures

(G, X) Shimura datum. Reflex field E $(=\mathbb{Q})$.

Assume $\begin{cases} G_{der} \text{ is simply connected} \\ \text{the max'l } \mathbb{R}\text{-split } \mathbb{G}_m \text{ in } Z_G \text{ is } \mathbb{Q}\text{-split.} \end{cases}$

e.g. $G = GL_2$ or GSp_{2g}.

$K \subseteq G(\mathbb{A}_f)$. p prime s.t. $K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$, $K_p \subset G(\mathbb{Q}_p)$

i.e. $\exists$ connected reductive gp scheme $\mathcal{G}/\mathbb{Z}_p$,

w/ generic fiber $G_{\mathbb{Q}_p}$. s.t. $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Z}_p)$.

These assumption on p are satisfied $\forall\forall p$.

e.g. $G = GL_2$,

$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 \bmod N \right\}$

assumptions on p are satisfied if $p \nmid N$

$\mathcal{O}_{E,(\mathfrak{p})}$, $\mathfrak{p} \mid p$

Conj. For such p, $\exists$ canonical integral model $\mathcal{S}_K / \mathbb{Z}_{(p)}$ of $Sh_K / \mathbb{Q}$ $\leftarrow E$

which is smooth over $\mathbb{Z}_{(p)}$.

Moreover, the $G(\mathbb{A}_f^p)$-action on $\varprojlim_{K^p} Sh_{K_p K^p}$

should extend to $G(\mathbb{A}_f^p) \circlearrowright \varprojlim_{K^p} \mathcal{S}_{K_p K^p}$

Also, if Sh_K is proj,

$\rightsquigarrow$ expect: $\mathcal{S}_K$ is proj.

if Sh_K is non-cpt:

$\rightsquigarrow$ expect: the Baily-Borel compactification of Sh_K extends to a similar comp. of $\mathcal{S}h_K$.

No worrying :)

Theorem (Vasiu, Kisin, Madapusi Pera-Kim)

The above conjecture for the existence of integral models is true if (G, X) is of abelian type,

(closely related to Hodge-type).