Counting Points on Shimura Varieties
Lecture 2

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§1 Integral models

(G,X) Shimura datum, $E(=\mathbb{Q})$ reflex field.

$K \subseteq G(\mathbb{A}_f)$ open compact subgp.

Prime p s.t. (i) $K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$, $K_p \subset G(\mathbb{Q}_p)$

(ii) K_p is hyperspecial, i.e. $\exists$ conn. red. gp. sch $\mathcal{G}/\mathbb{Z}_p$ with $\mathcal{G}_{\mathbb{Q}_p} \cong G_{\mathbb{Q}_p}$ s.t. $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

<u>Expectation</u> $\exists$ a "canonical" smooth integral model $\mathcal{S}_K/\mathbb{Z}_p$ of Sh_K.

<u>Theorem</u> (Vasiu, Kisin, Madapusi Pera–Kim)

This is true if (G,X) is of abelian type.

more general than Hodge type.

<u>"Canonical"</u>: If Sh_K is not projective, the idea is we want to forbid arbitrary deleting points from special fiber.

$\forall g \in G(\mathbb{A}_f^p)$, cpt open $U^p, K^p \subset G(\mathbb{A}_f^p)$, s.t. $g^{-1} U^p g \subseteq K^p$.

$\rightsquigarrow [g] : \boxed{\mathrm{Sh}_{U^pK_p} \longrightarrow \mathrm{Sh}_{K^pK_p}}$ which on $\mathbb{C}$-pts is given by

def'd / E.

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/U^pK_p \longrightarrow G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^pK_p$$
$$(x,y) \longmapsto (x, yg)$$

finite étale

transition maps: $[1]$

with $\varprojlim_{K^p} \mathrm{Sh}_{K^pK_p} = \mathrm{Sh}_{K_p}$ E-scheme.

Implicity: the integral models for fixed $K_p = \mathcal{G}(\mathbb{Z}_p)$
different K^p should satisfy
$\forall [g] : Sh_{U^pK_p} \longrightarrow Sh_{K^pK_p}$,
$\rightsquigarrow$ to extend uniquely to a finite étale $\mathcal{S}_{U^pK_p} \longrightarrow \mathcal{S}_{K^pK_p}$
$\rightsquigarrow$ can form $\mathcal{S}_{K_p} := \varprojlim_{K^p} \mathcal{S}_{K^pK_p}$

$\mathbb{Z}_{(p)}$-scheme.

action also extends to $\mathcal{S}_{K_p}$.

$$\mathcal{S}_{K_p} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = Sh_{K_p} \circlearrowleft G(\mathbb{A}_f^p)$$

<u>Note</u> In order to characterize $\mathcal{S}_{K^pK_p}$,
we first need to characterize
$\mathcal{S}_{K_p}$ with $G(\mathbb{A}_f^p)$-action

b/c $\boxed{\mathcal{S}_{K^pK_p} = \mathcal{S}_{K_p}/K^p}$

Need to characterize the $\mathbb{Z}_{(p)}$-scheme $\mathcal{S}_{K_p}$.

<u>Characterizing condition</u>

$\forall \mathbb{Z}_{(p)}$-scheme T which is regular $\mathbb{Q}$ formally smooth$/\mathbb{Z}_{(p)}$,
every $\mathbb{Q}$-map $T_{\mathbb{Q}} \longrightarrow Sh_{K_p}$
extends (uniquely) to $T \longrightarrow \mathcal{S}_{K_p}$.

* This uniquely characterizes $\mathcal{S}_{K_p}$.

<u>Rmk</u> For us, we need

(Madapusi Pera) In the Hodge-type case,
if Sh_K is projective, so is $\mathcal{S}_K$.

(Lan-Stroh) In the abelian-type case,
no assumption on Sh_K being proj., we have

$H^i_{et,c}(Sh_{K,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = H^i_c(Sh_K) \cong H^i_c(\mathcal{S}_{K,\bar{\mathbb{F}}_p})$.

Apply Lefschetz trace formula on $H_c^i(Sh_K, \overline{\mathbb{F}}_p)$

$\Rightarrow \sum (-1)^i \mathrm{Tr}(Fr^a \mid H_c^i(Sh_K, \overline{\mathbb{F}}_p)) = \# Sh(\mathbb{F}_{p^a})$.

(Also proved for intersection cohomologies).

§2 Conjecture for $\# Sh_K(\mathbb{F}_{p^a})$

With $K = K^p K_p$, K_p hyperspecial.

Conjecture (Kottwitz)

Assume (i) G^{der} is simply connected

(ii) Z_G its max'l $\mathbb{R}$-split subtorus is $\mathbb{Q}$-split. (Z_G is cuspidal)

Then $\# Sh_K(\mathbb{F}_{p^n}) = \sum_{(\gamma_0, \gamma, \delta)} c_1(\gamma_0, \gamma, \delta)\, c_2(\gamma_0, \gamma, \delta)\, O_\gamma(1_{K^p}) \cdot TO_\delta(f_n)$.

Here $(\gamma_0, \gamma, \delta)$ runs through a certain subset of

$$G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^n})$$

deg n unram. ext'n of $\mathbb{Q}_p$

G with p-Frob.

Modulo $\sim$:

$$(\gamma_0, \gamma, \delta) \sim (\gamma_0', \gamma', \delta')$$

if
- γ_0 & γ_0' are conjugate in $G(\overline{\mathbb{Q}})$
- γ & γ' are conjugate in $G(\mathbb{A}_f^p)$
- δ & δ' are G-conjugate in $G(\mathbb{Q}_{p^n})$

i.e. $\exists\, c \in G(\mathbb{Q}_{p^n})$ s.t. $\delta = c \cdot \delta' \cdot \sigma(c)^{-1}$.

where $O_\gamma(1_{K^p})$ is the integral of $1_{K^p} : G(\mathbb{A}_f^p) \longrightarrow \{0,1\}$

on the conj. class of γ in $G(\mathbb{A}_f^p)$.

$TO_\delta(f_n)$ is the integral of f_n

on the G-conj. class of δ in $G(\mathbb{Q}_{p^n})$

$f_n : G(\mathbb{Q}_{p^n}) \longrightarrow \{0,1\}$ is the characteristic func'n of a certain $\mathcal{G}(\mathbb{Z}_{p^n})$-double coset in $G(\mathbb{Q}_{p^n})$ determined by (G,X).

§3 Case $(GL_2, \mathfrak{H}^{\pm}) = (G,X)$

$K = K^p K_p$, $K_p = GL_2(\mathbb{Z}_p)$.

How to arrange this?

is cpt open subgp of $GL_2(\mathbb{A}_f)$.

$\forall K$, $\exists N \gg 0$ s.t. $K \supset \boxed{\widehat{\Gamma(N)}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 \bmod N \right\}$.

then $\forall p \nmid N$, $K \supset K^p \cdot GL_2(\mathbb{Z}_p)$

replace K by $K^p \cdot GL_2(\mathbb{Z}_p)$. okay ☺.

* $S_K = S_{K^p K_p}$: $\forall \mathbb{Z}_{(p)}$-scheme R.

$S_K(R) = \{ (\mathcal{E}, \eta) \mid \mathcal{E}$ elliptic curve $/R$, η K^p-level str.$\}$

i.e. on each conn. comp R_i of R, pick $\bar{r}$.

η is a $\pi_1(R_i, \bar{r})$-stable elt, K^p-orbit of isoms

$$(\hat{\mathbb{Z}}^p)^{\oplus 2} \xrightarrow{\sim} T^p(\mathcal{E}_{\bar{r}}).$$

<u>Recall</u> F is a field. Two semi-simple elts of $GL_n(F)$ are conjugate in $GL_n(F)$ iff they are conjugate in $GL_n(\bar{F})$.

<u>Formula</u> $\# S_K(\mathbb{F}_{p^n}) = \sum_{(\gamma_0, \delta)} c(\gamma_0, \delta) \cdot O_{\gamma_0}(1_{K^p}) \cdot TO_\delta(f_n)$

* γ_0 is an elt of $G(\mathbb{Q})$ (up to conjugacy) & $\mathbb{R}$-elliptic i.e. γ_0 is either central ($\gamma_0 = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$, $\lambda \in \mathbb{Q}^*$) or its char poly is irred. $/\mathbb{R}$ (it has two distict imaginary eigenvalues).

$\gamma_0 \in T(\mathbb{R})$. T is a max'l torus in $G_{\mathbb{R}}$ s.t. $T(\mathbb{R})$ is cpt mod Z_G.

* $\delta \in G(\mathbb{Q}_{p^n})$ s.t. $\delta\cdot\sigma(\delta)\cdot\sigma^2(\delta)\cdots\sigma^{n-1}(\delta) \in G(\mathbb{Q}_{p^n})$

"naive norm" is conj. to γ_0.

δ is taken up to σ-conjugacy in $G(\mathbb{Q}_{p^n})$.

* $O_{\gamma_0}(1_{K^p}) = \int_{G_{\gamma_0}(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)} 1_{K^p}(x^{-1}\cdot\gamma_0\cdot x)\,\boxed{dx}$

quotient Haar measure on $G_{\gamma_0}(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)$

by the choice of a Haar meas on $G(\mathbb{A}_f^p)$

& a Haar meas on $G_{\gamma_0}(\mathbb{A}_f^p)$.

* $TO_\delta(f_n) = \int_{G(\mathbb{Q}_{p^n})_{\delta\sigma}\backslash G(\mathbb{Q}_{p^n})} f_n(x^{-1}\delta\sigma(x))\,dx$

Here $\underline{G(\mathbb{Q}_{p^n})_{\delta\sigma}}$ = σ-centralizer of δ

$= \{g\in G(\mathbb{Q}_{p^n}) \mid g\cdot\delta\cdot\sigma(g)^{-1} = \delta\}$.

actually $\mathbb{Q}_p$-pts of a reductive gp/$\mathbb{Q}_p$

$J_{n,\delta}(R) = \{g\in G(\mathbb{Q}_{p^n}\otimes_{\mathbb{Q}_p} R) \mid g\delta\sigma(g)^{-1} = \delta\}$.

Alternatively, $G_n := \mathrm{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} G$ is a reductive gp/$\mathbb{Q}_p$.

$\theta \in \mathrm{Aut}_{\mathbb{Q}_p}(G_n)$ $\quad(\theta \longleftrightarrow \sigma\in \mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p))$

with $J_{n,\delta} = \{g\in G_n \mid g\,\delta\,\theta(g)^{-1} = \delta\}$.

* $c_1(\gamma_0,\delta)$: Given (γ_0,δ),

we can define a unique inner form I of G_{γ_0}

s.t. $I_{\mathbb{R}}$ is cpt mod Z_G.

$I_{\mathbb{Q}_\ell} \cong \boxed{G_{\gamma_0}}$, $\forall \ell\neq p$,

$I_{\mathbb{Q}_p} \cong \boxed{J_{n,\delta}}$ isom as inner form of G_{γ_0}.

$c_1(\gamma_0,\delta) = \mathrm{vol}(\underline{I(\mathbb{Q})\backslash I(\mathbb{A}_f)})$

is nice space b/c $I_{\mathbb{R}}$ is cpt mod Z_G

& Z_G is cuspidal

Fix Haar measures on $I(\mathbb{A}_f^p) \cong G_{\gamma_0}(\mathbb{A}_f^p) \times J_{n,\delta}(\mathbb{Q}_p)$

compatibly as the Haar measures on

$G_{\gamma_0}(\mathbb{A}_f^p)$ & $J_{n,\delta}(\mathbb{Q}_p)$ in the def'n of O_γ & TO_δ.

$I(\mathbb{Q}) \subset I(\mathbb{A}_f)$ discrete counting meas on $I(\mathbb{Q})$.

Finally: Haar meas on $G(\mathbb{A}_f^p)$ is normalized s.t. $\mathrm{vol}(K^p) = 1$.

Haar meas on $G(\mathbb{Q}_{p^n})$ is normalized s.t. $\mathrm{vol}(GL_2(\mathbb{Z}_{p^n})) = 1$.

$f_n : GL_2(\mathbb{Q}_{p^n}) \longrightarrow \{0,1\}$ characteristic func'n of

$$GL_2(\mathbb{Z}_{p^n}) \boxed{\begin{pmatrix} p & \\ & 1 \end{pmatrix}} GL_2(\mathbb{Z}_{p^n}) \subset GL_2(\mathbb{Q}_{p^n})$$

$\mu(p)$. μ Hodge character of X.

Recall Cartan decomposition:

Exercise ☺.

$$GL_2(\mathbb{Q}_{p^n}) = \coprod_{\substack{a,b \in \mathbb{Z} \\ a \geq b}} GL_2(\mathbb{Z}_{p^n}) \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} GL_2(\mathbb{Z}_{p^n}).$$

Observation If γ_0 shows up, then $\det \gamma_0 = p^n$

$\exists \delta \in G(\mathbb{Q}_{p^n})$ s.t. $\gamma_0 \sim \delta\sigma(\delta) \cdots \sigma^{n-1}(\delta)$

$\det \gamma_0 = N_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} \det \delta$.

$TO_\delta(f_n) \neq 0 \Rightarrow \exists c \in G(\mathbb{Q}_{p^n})$ s.t. $c \cdot \delta \cdot \sigma(c)^{-1} \in GL_2(\mathbb{Z}_p) \begin{pmatrix} p & \\ & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)$

$\Rightarrow \det c \cdot \det \delta \cdot \sigma(\det c)^{-1}$ has p-adic value 1.

$\Rightarrow \det \delta$ has p-adic value 1.

$\Rightarrow \det \gamma_0 \in \mathbb{Q}^\times$ has p-adic value n.

$O_{\gamma_0}(1_{K^p}) \neq 0$ (Always assume K^p is small enough

say $K^p K_p \subset \hat{\Gamma}(N)$ for some $N \geq 3$) "neat".

$\Rightarrow K_p \subset GL_2(\hat{\mathbb{Z}}^p)$

& γ_0 is $\mathbb{A}_f^p$-conj. to some elt $\in GL_2(\hat{\mathbb{Z}}^p)$

$\Rightarrow$ $\det \gamma_0$ has ℓ-adic value 0, $\forall \ell \neq p$.

$\det \gamma_0 > 0$ b/c char poly is irred./$\mathbb{R}$.

$\Rightarrow$ $\det \gamma_0 = p^n$. $\square$

<u>Classify γ_0's</u>:

1) supersingular case (only appears if n is even.)

$$\gamma_0 = \begin{pmatrix} p^{n/2} & \\ & p^{n/2} \end{pmatrix}, \quad G_{\gamma_0} = G = GL_2.$$

$G_{\gamma_0} = G = GL_2$, $I = D^\times$, D = quaternion alg./$\mathbb{Q}$ ram'd at p & ∞.

2) ordinary case: γ_0 is non-central.

$F := \mathbb{Q}(\text{eigenvalues of } \gamma_0) = \mathbb{Q}(\gamma_0)$

is imaginary quadratic field (γ_0 is $\mathbb{R}$-elliptic).

$G_{\gamma_0} \cong \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m = \text{"}F^\times\text{"} \hookrightarrow GL_2$, $I = G_{\gamma_0}$.

$F = \mathbb{Q} \oplus \mathbb{Q}$.

<u>Exercise 1</u> Show that only finitely many γ_0's show up.

<u>Exercise 2</u> $\# S_K(\mathbb{F}_5)$, $K = \widehat{\Gamma(4)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 \bmod 4 \right\}$.

may be very hard.