Counting Points on Shimura Varieties
Lecture 5

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Last time Rmk. In general, suppose $E/\mathbb{F}_q \rightsquigarrow \gamma_0 \in GL_2(\mathbb{Q})$

E is supersingular $\Leftrightarrow \exists k,\ \gamma_0^k$ is central.

More precisely, γ_0^k is central

$\Leftrightarrow (\mathrm{End}_{\mathbb{F}_q} E) \otimes \mathbb{Q}$ is a quaternion algebra.

There exist examples of ss elliptic curves

s.t. γ_0 is <u>NOT</u> central,

but some power of γ_0 is central.

e.g. $q=p=3$, $\pi = \sqrt{-3}$ Weil 3-number.

$\rightsquigarrow E/\mathbb{F}_3$, E is ss but $\gamma_0 \sim \begin{pmatrix} \sqrt{-3} & \\ & \sqrt{-3} \end{pmatrix}$ not central,

$\gamma_0^2 = \begin{pmatrix} 3 & \\ & 3 \end{pmatrix}$ central.

But we proved:

$\forall (E,\eta) \in S_K(\mathbb{F}_q)$, then E is s.s. $\Leftrightarrow \gamma_0$ is central.

i.e. examples as above do not extend to a pair (E,η) over $\mathbb{F}_q$.

§1 General formula

(G,X) Shimura datum. $\begin{cases} G^{der} \text{ simply connected} \\ Z_G \text{ is cuspidal} \end{cases}$

$K = K^p K_p$, K^p "small enough" & K_p hyperspecial.

E reflex field. Fix a prime $\mathfrak{p}$ of E over p.

$\rightsquigarrow$ conjectural canonical integral model $S_K / \mathcal{O}_{E,(\mathfrak{p})}$.

Take some $q = p^n$ s.t. $\mathbb{F}_q \supset$ residue field of $\mathcal{O}_{E,(\mathfrak{p})}$.

$\# \mathfrak{Z}_K(\mathbb{F}_q) \overset{\text{conj.}}{=} \sum_{(\gamma_0,\gamma,\delta)} c_1(\gamma_0,\gamma,\delta)\, c_2(\gamma_0)\, O_\gamma(1_{K^p})\, TO_\delta(f_n)$

Here: (γ_0,γ,δ) runs through $G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^n})$

with • γ_0 is $\mathbb{R}$-elliptic

i.e. $\exists$ max'l torus $T \subset G_{\mathbb{R}}$ $/\mathbb{R}$ s.t.

$\gamma_0 \in T(\mathbb{R})$ & $(T/Z_G)(\mathbb{R})$ is cpt.

• γ is "strictly conjugate" to γ_0

i.e. $\gamma \sim \gamma_0$. $G(\overline{\mathbb{A}}_f^p)$, $\overline{\mathbb{A}}_f^p := \mathbb{A}_f^p \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$

Think $\gamma = (\gamma_\ell)_{\ell \neq p}$, $\gamma_\ell \in G(\mathbb{Q}_\ell)$, $\gamma_\ell \overset{G(\overline{\mathbb{Q}}_\ell)}{\sim} \gamma_0$.

• $\delta \in G(\mathbb{Q}_{p^n})$, $\delta \cdot \sigma(\delta) \cdots \sigma^{n-1}(\delta) \overset{G(\overline{\mathbb{Q}}_p)}{\sim} \gamma_0$

i.e. the stable conjugacy class of γ_0

is the deg n norm of δ.

Now, given (γ_0,γ,δ) as above. + some hypothesis

One defines a <u>Cartan Galois cohomological invariant</u>

"Kottwitz invariant".

$\alpha(\gamma_0,\gamma,\delta)$ lying in some finite abelian gp

depending only on γ_0.

In the summantion, only those (γ_0,γ,δ) w/ $\alpha(\gamma_0,\gamma,\delta)=0$ should appear.

* Summation is over (γ_0,γ,δ) satisfying $\alpha(\gamma_0,\gamma,\delta)=0$

up to an equivalence relation

$(\gamma_0,\gamma,\delta) \sim (\gamma_0',\gamma',\delta')$

if $\gamma_0 \overset{G(\overline{\mathbb{Q}})}{\sim} \gamma_0'$, $\gamma \overset{G(\mathbb{A}_f^p)}{\sim} \gamma'$, and

δ: σ-conjugate to δ' in $G(\mathbb{Q}_{p^n})$

<u>Key Point</u> if $(\gamma_0,\gamma,\delta) \sim (\gamma_0',\gamma',\delta')$,

then $\alpha(\gamma_0,\gamma,\delta)=0 \Leftrightarrow \alpha(\gamma_0',\gamma',\delta')=0$.

Summand $\quad C_1(\gamma_0,\gamma,\delta)\, C_2(\gamma_0)\, O_\gamma(1_{K^p})\, TO_\delta(f_n)$

- $C_1(\gamma_0,\gamma,\delta)$: given (γ_0,γ,δ),

(i) write $\gamma = (\gamma_\ell)_{\ell\neq p}$. $\gamma_\ell \in G(\mathbb{Q}_\ell)$.

G_{γ_ℓ} is an inner form of $\boxed{(G_{\gamma_0})_{\mathbb{Q}_\ell}}$

(ii) $J_{n,\delta}$ is an inner form of $(G_{\gamma_0})_{\mathbb{Q}_p}$.

Want global inner form I of $G_{\gamma_0}/\mathbb{Q}$.

s.t. (i) $I_{\mathbb{R}}/Z_{G,\mathbb{R}}$ is cpt.

(ii) $I_{\mathbb{Q}_\ell} \cong G_{\gamma_\ell}$ as inner forms of $(G_{\gamma_0})_{\mathbb{Q}_\ell}$.

(iii) $I_{\mathbb{Q}_p} \cong J_{n,\delta}$ as inner forms of $(G_{\gamma_0})_{\mathbb{Q}_p}$.

$\rightsquigarrow$ Then we define

$$\boxed{C_1(\gamma_1,\gamma,\delta) = \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))}$$

Note For a general (γ_0,γ,δ), no reason why global I should exist.

But if $\alpha(\gamma_0,\gamma,\delta)=0$, then I exists!

Actually : $\alpha(\gamma_1,\gamma,\delta)=0$ is stronger than I exists.

$C_2(\gamma_0) := \#\ker(Ш(G_{\gamma_0}) \longrightarrow H^1(\mathbb{Q},G))$

where $Ш(G_{\gamma_0}) = \ker(H^1(\mathbb{Q},G_{\gamma_0}) \longrightarrow H^1(\mathbb{A},G_{\gamma_0}))$

$= \bigoplus_v H^1(\mathbb{Q}_v,G_{\gamma_0})$ "direct sum" of pointed sets

$O_\gamma(1_{K^p}) = \int_{G(\mathbb{A}_f^p)_\gamma\backslash G(\mathbb{A}_f^p)} 1_{K^p}(x^{-1}\gamma x)\,dx$

$TO_\delta(f_n)$ = same as in GL_2.

Here $f_n : G(\mathbb{Q}_{p^n}) \longrightarrow \{0,1\}$ is as follows

- $\forall h\in X$ = a $G(\mathbb{R})$-conjugacy class of homo's $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$.

- $h_{\mathbb{C}} : \underset{\substack{\uparrow\\ id:\mathbb{C}\to\mathbb{C}}}{\mathbb{G}_m} \times \underset{\substack{\uparrow\\ (\bar{\cdot}):\mathbb{C}\to\mathbb{C}}}{\mathbb{G}_m} \longrightarrow G_{\mathbb{C}}$

• $\mu_h : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$ "Hodge cocharacter of h"

$z \longmapsto h_{\mathbb{C}}(z,1)$

The $G(\mathbb{C})$-conj. class of μ_h is def'd/E. (by def'n of E).

⇝ <u>In particular</u> if F/E field ext'n,

we can get a (conj. class of cochars of G) /F.

Now if G is quasi-split /F,

this is the same as a G(F)-conj. class of

F-rational cochars: $\mathbb{G}_{m,F} \longrightarrow G_F$.

<u>Now</u> G is quasi-split / $\mathbb{Q}_p$

⇝ $\mathbb{Q}_q \supset E_{\mathfrak{p}} \supset E$

⟶ $G(\mathbb{Q}_q)$-conj. class of cochars of $G_{\mathbb{Q}_q}$.

Take one member μ which extends to a cochar of $\mathcal{G}_{\mathbb{Z}_q}$

Recall $\mathcal{G}$ is a red. gp sch/$\mathbb{Z}_p$ s.t. $K_p = \mathcal{G}(\mathbb{Z}_p)$.

f_n = char func'n of $\mathcal{G}(\mathbb{Z}_q)\mu(p)\cdot\mathcal{G}(\mathbb{Z}_q)$

(indep. of the choice of μ).

E.g. GL_2, f_n = char func'n $GL_2(\mathbb{Z}_q)\begin{pmatrix} p & \\ & 1\end{pmatrix}GL_2(\mathbb{Z}_q)$.

<u>Rmk</u> $TO_\delta(f_n) \neq 0$ then "Kottwitz homomorphism"

$$\kappa : G(\widehat{\mathbb{Q}_p^{ur}}) \longrightarrow \pi_1(G)_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

sends δ to some fixed elt in $\pi_1(G)_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ (def'd by X).

Actually, this condition is needed in order to define $(\gamma_0, \gamma, \delta)$

<u>Rmk</u> Why should we expect for $\boxed{(\gamma_0, \gamma, \delta)}$, global I should exist?

In PEL case $A/\mathbb{F}_q$ + PEL structure

$\rightsquigarrow \gamma \longleftrightarrow \mathrm{Frob} \in \mathrm{End}(A) \to T^{(p)}(A)$ — $\xi(r)$-module is also equipped w/ PE structure.

$\rightsquigarrow \gamma \in G(\mathbb{A}_f^p)$, $\delta \longleftrightarrow$ abs Frob $\circlearrowleft M_0(A)$.

Global I: is isomorphic to the $\mathbb{Q}$-gp:

$\forall \mathbb{Q}$-alg. $R \longmapsto ((\mathrm{End}_{\mathbb{F}_q}(A, \text{PE str.})) \otimes_{\mathbb{Z}} R)^\times$

Rmk $\# Sh_K(\mathbb{F}_q) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \mid H^i_{et}(Sh_{K,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$.

More generally:

$$\boxed{\sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \times f^p \mid H^i_{et}(\cdot \cdot))}$$

with $f^p \in \mathcal{H}(G(\mathbb{A}_f^p) /\!/ K^p)$.

Similar formula, where $O_\gamma(1_{K^p})$ is replaced by $O_\gamma(f^p)$.

§2 Known cases of the conjecture

Kottwitz Around 1990: proved the conj. for
PEL type Shimura varieties of type A, C.

1990 – Now Some sporadic cases beyond PEL type
but closely related to PEL.

Recent (Kisin – Shin – Zhu)
All abelian type cases Remove / G^{der} simply connected \ Z_G is cuspidal.

(See next Lecture).

§3 Informal introduction to Trace Formulas

Langlands "Compare" the formula for $\# Sh_K(\mathbb{F}_q)$ w/ TF from rep'n theory.

i.e. stable Arthur-Selberg TF.

Basics "nice" topological gp H
(Hausdorff, locally cpt, unimodular)
discrete subgp $\Gamma \subseteq H$.
$L^2(\Gamma\backslash H)$ as an H-rep'n.
Here H acts by right translation
$$R: \ R(h): L^2(\Gamma\backslash H) \longrightarrow L^2(\Gamma\backslash H)$$
unitary rep'n.
$$\varphi \longmapsto (x \longmapsto \varphi(xh))$$

Fundamental Question How R "decomposes" into
irreducible unitary rep'ns of H?

E.g. $H=\mathbb{R}$, $\Gamma=\mathbb{Z}$.
Unitary irred. rep'ns of H parametrized by $y \in i\mathbb{R}$
$$\pi_y : H \longrightarrow GL_1(\mathbb{C})$$
$$x \longmapsto e^{yx}$$

$$L^2(\mathbb{Z}\backslash\mathbb{R}) \xrightarrow[\text{isometry}]{\sim} L^2(\mathbb{Z})$$
$$\varphi \longmapsto \boxed{\begin{array}{l}\hat{\varphi}: \mathbb{Z} \longrightarrow \mathbb{C} \\ \quad n \longmapsto \hat{\varphi}(n)\end{array}}$$
$= n$-th Fourier coefficient of φ
$= \int_{\mathbb{Z}\backslash\mathbb{R}} \varphi(x) e^{-2\pi i n x} dx$

H-action on $L^2(\mathbb{Z})$ is given by
$x \in H = \mathbb{R}$, $\psi \in L^2(\mathbb{Z})$.
$$(x\cdot\psi)(n) = e^{2\pi i n x} \cdot \psi(n)$$
$$L^2(\mathbb{Z}) \cong \hat{\bigoplus_n} \mathbb{C} \cong \hat{\bigoplus_n} \pi_{2\pi i n}$$
$$\psi \longleftrightarrow (\psi(n))_n.$$
The nth copy of $\mathbb{C}$ is $\pi_{2\pi i n}$ as an H-rep'n.

Conclusion $\boxed{L^2(\Gamma\backslash H)} \cong \widehat{\bigoplus_n} \pi_{2\pi i n}$ as H-rep'ns

We say $L^2(\Gamma\backslash H)$ decomposes discretely.

In contrast, $H=\mathbb{R}$, $\Gamma=\{0\}$

Fourier transform $L^2(\Gamma\backslash H) \cong$ "$\int_{y\in i\mathbb{R}} \pi_y \, dy$". "continuous spectrum".

Key Whether $\Gamma\backslash H$ is cpt !!

From now on, assume $\Gamma\backslash H$ is cpt.

ring under convolution

TF: We have an associated $\boxed{C_c(H)} \curvearrowright L^2(\Gamma\backslash H)$

$f \in$

$$R(f): L^2(\Gamma\backslash H) \longrightarrow L^2(\Gamma\backslash H)$$

$$R(f) = \int_H R(h) f(h) \, dh.$$

fixed Haar measure on H.

i.e. $[R(f)(\varphi)](x) = \int_H \varphi(xh) f(h) \, dh.$

$L^2(\Gamma\backslash H)$

Now, $\forall f \in C_c(H)$

disc. decomp.

$$\mathrm{Tr}(R(f)|_{L^2(\Gamma\backslash H)}) = \sum_\pi m_\pi \, \mathrm{tr}(f|\pi)$$

"spectral expansion".

"Geometric expansion":

$$\mathrm{Tr}(R(f)|L^2(\Gamma\backslash H)) = \sum_{\gamma\in\Gamma} \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma)\cdot O_\gamma(f).$$

up to conj.

$$O_\gamma(f) = \int_{H_\gamma\backslash H} f(x^{-1}\gamma x) \, dx.$$

Example / Exercise In the case $\mathbb{Z}\backslash\mathbb{R}$:

equality between geom. exp & spectral exp.
amounts to Poisson Summantion Formula.

In the case $\Gamma\backslash H$ non-cpt:

- $\mathrm{Tr}(R(f)\,|\,L^2(\Gamma\backslash H))$ <u>doesn't</u> make sense
- Geom. exp & spectral exp. <u>don't</u> make sense.

Now: want to apply this idea to $\Gamma\backslash H = G(\mathbb{Q})\backslash G(\mathbb{A})$ for some reductive $G/\mathbb{Q}$.

<u>BAD news</u> $G(\mathbb{Q})\backslash G(\mathbb{A})$ often non-cpt even if you replace G by G^{ad} :(

(E.g. $G = GL_2$ or SL_2 non-cpt!)

<u>Arthur</u> Invariant TF:

He defines an $\boxed{\text{invariant}}$ distribution on $G(\mathbb{A})$.

conjugation-invariant.

i.e. $I : \mathcal{C}_c^\infty(G(\mathbb{A})) \longrightarrow \mathbb{C}$ with $f \longmapsto I(f)$

"I is like $f \longmapsto \mathrm{Tr}(R(f)\,|\,L^2(G(\mathbb{Q})\backslash G(\mathbb{A})))$".

I has geometric exp: Assume G_{der} is simply connected.

$$I(\cdot) = \sum_{\substack{\gamma\in G(\mathbb{Q})/\mathrm{conj.} \\ \text{elliptic}}} \underline{\mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))} \cdot \underset{<\infty}{O_\gamma(\cdot)}$$

Tamagawa number $t(G_\gamma)$ of G_γ.

$+$ $\boxed{\text{Some much more complicated terms}}$.

NOT easy to explain.

I has spectral exp:

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{disc} \oplus L^2_{cont}.$$

$$L^2_{disc} = \widehat{\bigoplus_{\pi}} m_\pi^{disc} \cdot \pi.$$

π unitary irrep'ns of G

$$I(\cdot) = \sum_\pi m_\pi^{disc} \cdot \mathrm{tr}(\cdot\,|\,\pi) + \boxed{\text{Some much more complicated terms}}$$

Preview This invariant TF has a problem:

$I(\cdot)$ or $O_\gamma(\cdot)$ or $\mathrm{tr}(\cdot|\pi)$

are NOT invariant under stable conjugacy.