Counting Points on Shimura Varieties
Lecture 6

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Last time Arthur's invariant TF.

G red. gp/$\mathbb{Q}$, G^{der} simply connected.

$I: C_c^\infty(G(\mathbb{A})) \longrightarrow \mathbb{C}$ invariant distribution

If $G(\mathbb{Q})\backslash G(\mathbb{A})$ is cpt. $I(f) = \mathrm{Tr}(R(f) \mid L^2(G(\mathbb{Q})\backslash G(\mathbb{A})))$

In general: more complicated.

Geometric exp

$$I(\cdot) = \sum_{\substack{\gamma_0 \in G(\mathbb{Q})/\mathrm{conj} \\ \gamma_0:\ \mathbb{Q}\text{-elliptic}}} \boxed{I(G_{\gamma_0})} \cdot O_{\gamma_0}(\cdot) + \text{more complicated terms}$$

$= \mathrm{vol}(G_{\gamma_0}(\mathbb{Q})\backslash G_{\gamma_0}(\mathbb{A}))$ w.r.t. a canonical Haar meas.

Spectral exp

multiplicity π apps in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_{disc}$

$$I(\cdot) = \sum_{\substack{\pi \\ \text{unitary irrep'ns of } G(\mathbb{A})}} \boxed{m_\pi^{disc}}\, \mathrm{tr}(\cdot \mid \pi) + \text{more complicated terms}$$

Problem These distributions $I(\cdot)$, $O_{\gamma_0}(\cdot)$, $\mathrm{tr}(\cdot\mid\pi)$ are NOT stable.

i.e. invariant under stable conjugacy $= G(\mathbb{A}\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$-conjugacy

$\boxed{\text{Want}}$ a stable distribution $S: C_c^\infty(G(\mathbb{A})) \longrightarrow \mathbb{C}$

s.t. $S \overset{\text{geom. exp}}{=} \sum$ stable distr'ns of a geom nature.

$\overset{\text{spec. exp}}{=} \sum$ stable distr'ns of a spectral nature.

§1 An overview of stabilization

Example How to make $I_{ell} = \displaystyle\sum_{\substack{\gamma_0 \in G(\mathbb{Q})/\sim \\ \gamma_0 \text{ elliptic}}} \tau(G_{\gamma_0}) \cdot O_{\gamma_0}(\cdot)$ stable?

Tamagawa number

Step 1 $$I_{ell} = \sum_{\substack{\gamma_0 \in G(\mathbb{Q})/\text{stab}\sim \\ \gamma_0 \text{ elliptic}}} \tau(G_{\gamma_0}) \cdot \sum_{\substack{\gamma \in G(\mathbb{A})/\sim \\ \gamma \overset{\text{stab}}{\sim} \gamma_0 \\ \boxed{inv(\gamma,\gamma_0)=0}}} O_\gamma(\cdot)$$

$\sim$: conj. in $G(\mathbb{A})$

$\tau(G_{\gamma_0})$ is well-defin'd for γ_0 up to stab conj. $\tau(I)=\tau(I')$ if I is an inner form of I' (Kottwitz)

Here $inv(\gamma,\gamma_0) \in K(\gamma_0)$ (finite ab. gp).

Point: $inv(\gamma,\gamma_0)=0 \iff \gamma$ is $G(\mathbb{A})$-conj. to some $G(\mathbb{Q})$

Rmk The above formula is very similar to PCE !

$$\sum_{(\gamma_0,\gamma,\delta)} c_1 \cdot c_2 \cdot O_\gamma(\cdot) \cdot TO_\delta(\cdot)$$

with $\gamma_0 \in G(\mathbb{Q})$/stab conj., γ, δ adelic up to conj./G-conj.

& $\alpha(\gamma_0,\gamma,\delta)=0$.

Step 2 Apply Fourier inversion to the finite ab. gp $K(\gamma_0)$

$$I_{ell} = \boxed{\sum} \tau(G_0) \cdot O_\gamma = \boxed{\sum} \frac{\tau(G_{\gamma_0})}{|K(\gamma_0)^D|} \sum_{\kappa \in K(\gamma_0)^D} \underbrace{\langle \kappa, inv(\gamma,\gamma_0)\rangle \cdot O_\gamma(\cdot)}_{= O^\kappa_{\gamma_0}(\cdot)}$$

First $\sum$: γ_0 up to stab, γ up to conj. s.t. $\gamma_0 \overset{\text{stab}}{\sim} \gamma$, $inv(\gamma,\gamma_0)$ dies

Second $\sum$: γ_0 up to stab, γ up to conj. s.t. $\gamma_0 \overset{\text{stab}}{\sim} \gamma$ but No condition on $inv(\gamma,\gamma_0)$

$$= \sum_{\gamma_0 \text{ up to stab}} \frac{\tau(G_{\gamma_0})}{|K(\gamma_0)^D|} \sum_{\kappa \in K(\gamma_0)^D} \underbrace{\sum_{\substack{\gamma \in G(\mathbb{A})/\sim \\ \gamma \overset{\text{stab}}{\sim} \gamma_0}} \langle \kappa, inv(\gamma,\gamma_0)\rangle\, O_\gamma(\cdot)}_{= O^\kappa_{\gamma_0}(\cdot)}$$

$O^\kappa_{\gamma_0}(\cdot)$ κ-orbital integral

e.g. $\kappa=1$. $O^\kappa_{\gamma_0}(\cdot) = \sum_{\substack{\gamma \in G(\mathbb{A})/\sim \\ \gamma \overset{\text{stab}}{\sim} \gamma_0}} O_\gamma(\cdot) =: SO_{\gamma_0}(\cdot)$ "stable orbital integral".

This is a stable distribution on $G(\mathbb{A})$!

Idea For a non-trivial $O^\kappa_{\gamma_0}(\cdot)$ "comes from" a stable distribution differential group.

More precisely: from (γ_0,κ), construct a new red. gp

$\boxed{H_\kappa}/\mathbb{Q}$ called an endoscopic gp (with additional data relating H_κ & G)

& $\boxed{\gamma_\kappa} \in H_\kappa(\mathbb{Q})$ up to stab. conj.

Want to relate $O^\kappa_{\gamma_0}(\cdot)$ w/ $SO_{\gamma_\kappa}(\cdot)$ ← a stab distri'n on $H_\kappa(\mathbb{A})$.

Step 3 (Hard!) $\forall f \in C_c^\infty(G(\mathbb{A}))$,

want to find $f^{H_\kappa} \in C_c^\infty(H_\kappa(\mathbb{A}))$

called "Langlands-Shelstad transfer" s.t. $O^\kappa_{\gamma_0}(f) = SO_{\gamma_\kappa}(f^{H_\kappa})$.

Here f^{H_κ} should depend only on H_κ (+ additional data), not on γ_κ.

Involves hard work by L-S, Waldspurger, Laumon, Ngo.

(FL proved by Ngo).

Step 4 Put everything together:

Thm (Kottwitz, assuming Step 3).

$$\boxed{I_{ell}(f) = \sum_{\substack{H \\ \text{endoscopic gps of } G}} \iota(G,H)\, ST^H_{ell,*}(f^H)}$$ ← is a stabilization of $I_{ell}(\cdot)$.

- f^H: LS-transfer
- $ST^H_{ell,*}(\cdot)$ = "ell & (G,H)-regular part of stable TF"

$$= \sum_{\gamma_H \in H(\mathbb{Q})/\text{stab.}\sim} \underline{\tau(H)\cdot SO_{\gamma_H}(\cdot)}$$

↑ is a stable distribution on H.

Rmk Arthur later stabilizes all the terms in geom & spectral exp's of I

$$\Rightarrow I(f) = \sum_H \iota(G,H)\, ST^H(f^H)$$

↑ full stable TF for H.

ST^H is stable distri'n on $H(\mathbb{A})$

which has a geom exp & spectral exp

into smaller stable distri'ns.

§2 Back to PcF

$$\sum(-1)^i \mathrm{Tr}(f^p \times \mathrm{Frob}_p^r \mid H^i_{et,c})$$

Conj. $\cancel{\# S_K(\mathbb{F}_{p^n})} = \sum_{\substack{(\gamma_0,\gamma,\delta) \\ \alpha(\gamma_0,\gamma,\delta)=0}} c_1(\cdots)\, c_2(\cdots)\cdot \cancel{O_\gamma(1_{K^p})}\, TO_\delta(f_n)$

$O_\gamma(f^p)$

Theorem (Kottwitz)

RHS can be stabilized in a similar way as I_{ell}

i.e. $\text{RHS} = \sum_{H} \iota(G,H)\cdot \boxed{ST^H_{ell,*}}(\boxed{f^H_{sh}})$ ← both same as before.

H endoscopic gp of G

$f^H_{sh} = f^H_\infty \cdot f^H_p \cdot f^{H,p,\infty}$ $\in \mathcal{C}^\infty_c(H(\mathbb{A}_f^p))$

$f^H_\infty \in \mathcal{C}^\infty(H(\mathbb{R}))$, $f^H_p \in \mathcal{C}^\infty_c(H(\mathbb{Q}_p))$

$f^{H,p,\infty}$: LS-tranfer of $f^p \in \mathcal{C}^\infty_c(G(\mathbb{A}_f^p))$.

Expectation 1 When $Sh_K(G,X)$ is proj,

$\forall H,\ ST^H_{ell,*}(f^H_{sh}) = ST^H(f^H_{sh})$

$$\Rightarrow \sum(-1)^i \mathrm{Tr}(f^p \times \mathrm{Frob}_p^r \mid H^i_{et}) = \sum_H \iota(G,H)\, ST^H(f^H_{sh}).$$

Expectation 2 Non-proj.

$\sum(-1)^i \mathrm{Tr}(f^p \times \mathrm{Frob}_p^r \mid IH^i_{et}(\overline{Sh_K})) = \boxed{\sum(-1)^i \mathrm{Tr}(- \mid H^i_c)} +$ more terms

intersection cohom. of BB cpt'n

$\boxed{\sum(-1)^i \mathrm{Tr}(- \mid H^i_c)} = \sum \iota(G,H)\, ST^H_{ell,*}(f^H_{sh})$.

$= \sum \iota(G,H)\, ST^H(f^H_{sh})$ (*)

Morel, Zhu's thesis.

Rmk In general, from (*), we expect to be able to relate LHS of (*) to automorphic L-func'ns.

(need some more ingredients: Arthur's multiplicity conj.)

Point need to relate ST^H back to autom L-func'ns for autom rep'ns of G.

For some classical groups, some unitary similitude gps, everything can be made to work ☺.

§3 Proof of PCF for (G,X) of Hodge-type

$(G,X) \xhookrightarrow{i} (GSp(V), \mathcal{H}^{\pm})$.

Stage 1 Canonical integral models:

Fix $K = K^p K_p \subset G(\mathbb{A}_f)$. K_p hyperspecial, K^p small.

up to shrinking K^p, replacing i by different choice.

Can assume:

$$Sh_K(G,X) \xrightarrow{\text{closed embed.}} Sh_U(GSp(V) \times_{\mathbb{Q}} E).$$

$U = U^p U_p \subset GSp(V)(\mathbb{A}_f)$.

U_p hypersp. $U_p = GSp(V_{\mathbb{Z}_p})$

a self-dual $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$.

Sh_U has integral model $\mathcal{S}_U / \mathbb{Z}_{(p)}$

which is a moduli of polarized ab. schs

$$Sh_K \hookrightarrow Sh_U \times E \hookrightarrow \mathcal{S}_U \times_{\mathbb{Z}_{(p)}} O_{E,(\wp)}.$$

Def'n $\mathcal{S}_K :=$ normalization of Zariski closure of Sh_K in $\mathcal{S}_U \times_{\mathbb{Z}_{(p)}} O_{E,(\wp)}$.

Hard part to prove $\mathcal{S}_K$ is smooth $/ O_{E,(\wp)}$.

Step 1 $G \hookrightarrow GSp(V)$ is the stabilizer of certain tensors s_α on V $(\otimes, \oplus, Sym^k, Alt^k)$

We can arrange each s_α extends to a $\mathbb{Z}_p$-linear tensor on $V_{\mathbb{Z}_p}$.

Moreover, $\mathcal{G} \hookrightarrow GSp(V_{\mathbb{Z}_p})$ is precisely the stabilizer of s_α.

red model $/\mathbb{Z}_p$ of $G_{\mathbb{Q}_p}$ s.t. $\mathcal{G}(\mathbb{Z}_p) = K_p$.

Step2 $K/\mathbb{Q}_p$ finite ext'n, res. field $\mathbb{Q}$.

$x \in \mathcal{S}_K(\mathcal{O}_K)$, $\mathcal{S}_K \longrightarrow \mathcal{S}_U \times \mathcal{O}_{E,(p)}$.

$\rightsquigarrow A_x$ on $\mathcal{O}_K$.

$\rightsquigarrow$ p-adic rep'n $\mathrm{Gal}(\bar{K}/K) \circlearrowright T_p(A_{x,\bar{K}})$

This can be identified with $V_{\mathbb{Q}_p}$

In particular can view s_α as a tensor on this.

Fact s_α is $\mathrm{Gal}(\bar{K}/K)$-invariant.

By p-adic comparison

s_α "transforms" to tensor $s_{\alpha,0}$ on $M_0(A_{x,k})[\frac{1}{p}]$

Integral p-adic Hodge theory

(Breuil-Kisin modules & the relationship with p-adic gps)

$\Rightarrow s_{\alpha,0}$ is a tensor on $M_0(A_{x,k})$.

Step3 Use these integral tensors $s_{\alpha,0}$ to write down

a deformation space of $A_{x,k}[p^\infty]$ + $s_{\alpha,0}$

(Faltings) This space is formally smooth $/W(k)$.

Step4 Relate the space in Step 3 w/ local structure of $\mathcal{S}_K$.

$\Rightarrow \mathcal{S}_K$ is smooth.

Stage2 Classifying "isogeny classes"

Work with $\mathcal{S}_{K_p}(\bar{\mathbb{F}}_p) = \varprojlim_{K^p} \boxed{\mathcal{S}_{K^pK_p}}(\bar{\mathbb{F}}_p)$

If we understand this set + $G(\mathbb{A}_f^p)$-action + Frob.-action

$\Rightarrow$ PcF !

Def'n $x, x' \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ are called isogenous if

$\exists$ quasi-isog. $f: A_x \longrightarrow A_{x'}$ st.

f takes each $s_{\alpha,0}$ on $M_0(A_x)[\frac{1}{p}]$
to $s_{\alpha,0}$ on $M_0(A_{x'})[\frac{1}{p}]$
and f preserves similar tensors on ℓ-adic Tate modules $\ell \neq p$

Kisin classified isog. classes in a group theoretic manner
(essentially similar to Honda-Tate theory).
in the language of Tate's special lifting theorem.
"Every A.V./$\mathbb{F}_p$ is isog. to red'n of a CM A.V."

Theorem (Kisin)

Every isog class in $S_K(\overline{\mathbb{F}}_p)$ contains a point
which is the reduction of a special point
i.e. a point on Sh_K coming from $Sh(T,h)$

$$(T,h) \longrightarrow (G,X).$$

torus

Stage 3 Parametrize points in a fixed isog. class

Fix $x_0 \in S_{K_p}(\overline{\mathbb{F}}_p)$. Want to parametrize the isog. class of x_0.

$x_0 \rightsquigarrow A_{x_0}$ + tensors on $T^p(A_{x_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$

tensors on $M_0(A_{x_0}) \circlearrowleft F$.

$$X^p = \{ \text{isoms } \underset{s_\alpha}{V_{\mathbb{A}_f^p}} \xrightarrow{\sim} \underset{s_\alpha}{\boxed{T^p(A_{x_0}) \otimes_{\mathbb{Z}} \mathbb{Q}}} \text{ preserving tensors} \}$$

$$X_p = \left\{ \text{lattices } \Lambda \subset M_0(A_{x_0})[\tfrac{1}{p}] \;\middle|\; \begin{array}{l} (\Lambda, F) \text{ is a Dieudonné module} \\ \text{of } \boxed{\text{dimension}} = \dim A_{x_0} \\ + \text{ compatibility w/ } s_{\alpha,0} \end{array} \right\}$$

Rmk X_p is an affine Deligne-Lustzig set.

It has a purely group theoretic description.

$\Rightarrow$ isog. class of $x_0 \longleftrightarrow I_x(\mathbb{Q})\backslash(X^p \times X_p)$.

$I_x(\mathbb{Q})$ = gp of self-quasi-isogenies of $\mathcal{A}_x$ preserving ℓ-adic & crystalline tensors.

I_x : red. gp $/\mathbb{Q}$ (like I_{E_0} in GL_2 case).

In GL_2 case:
$I_{E_0}(\mathbb{Q})\backslash Y^p \times Y_p$
$\parallel$
$I(\mathbb{Q})\backslash Y^p \times Y_p$

Stage 4 After rewriting X^p & X_p in a more gp-theoretic way, we obtain a red. gp$/\mathbb{Q}$, I

in GL_2, I is like the gp attached to (γ_0, δ)

s.t. $I(\mathbb{A}_f^p)$ naturally acts on $X^p \times X_p$.

Also, $I_x(\mathbb{Q})$ acts on $X^p \times X_p$

$\rightsquigarrow$ get an embedding $I_x(\mathbb{Q}) \xhookrightarrow{z} I(\mathbb{A}_f^p) \circlearrowright X^p \times X_p$.

Problem $z(I_x(\mathbb{Q})) \neq I(\mathbb{Q})$

Rather, they are only conjugate by $I^{ad}(\mathbb{A}_f^p)$.

* If $I^{ad}(\mathbb{A}_f^p)$-conj. is the same as $I(\mathbb{A}_f^p)$-conj., ok!

E.g. GL_2-case ☺.

Upshot In reality, we have $I_x(\mathbb{Q})\backslash X^p \times X_p$

Ideally, we want $I(\mathbb{Q})\backslash X^p \times X_p$. But they're not the same!

Discrepency is measured by $\tau_x \in I^{ad}(\mathbb{A}_f^p)$

s.t. $z(I_x(\mathbb{Q})) = \mathrm{Int}(\tau_x)(I(\mathbb{Q}))$

Need extra new ideas to "control" the τ_x's for different isog classes and to show that with the suitable control, they don't affect the desired PCF.