Seminar on Geometric Representation Theory (2022 Spring)

ON LUSZTIG'S MIDDLE EXTENSION OF PERVERSE SHEAVES

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ABSTRACT. This is the note of a seminar talk by Wenhan. We present a gentle and sketchy overview of Lusztig's construction [Lus81] for Springer correspondence via perverse sheaves.

1. OVERVIEW

Readers are assumed to have an acquirement of the geometry of orbits as well as constructible derived categories, perverse sheaves, and intersection homology theory. The standard references are [GM80], [Sho88, Chap III], and [Yun16, Lec I]. As well as Yun's lecture series at Park City Math Institute in 2015, the course [dC15] by de Cataldo on perverse sheaves is particularly recommended.

Group representation theory dictates that all unipotent conjugacy classes of $\mathrm{GL}_n(k)$ are in bijection with all irreducible characters of the symmetric group of degree n over k. It is seen as the first phenomenon of the classical Springer correspondence. When k is a finite field, the conception of Green polynomial arises from representations of $\mathrm{GL}_n(k)$ and has to do with partitions of n.

1.1. **Recap: Springer correspondence.** Let $k = \overline{\mathbb{F}}_q$ be an algebraically closed field of characteristic $p > 0$ and $\mathbf{G}$ be a connected reductive algebraic group defined over k. Let W denote the Weyl group of $\mathbf{G}$. Given the Borel variety $\mathcal{B}$ in $\mathbf{G}$, we would define for any $e \in \mathbf{G}$ that

$$\mathcal{B}_e := \{\mathbf{B} \subset \mathbf{G} \text{ Borel subgroup such that } e \in \mathbf{B}\}.$$

as a subvariety of $\mathcal{B}$. Fix another prime $\ell \neq p$ and consider the ℓ-adic cohomology theory of $\mathcal{B}_e$. It turns out that $H^i(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell)$, the étale cohomology of $\mathcal{B}_e$ with $\overline{\mathbb{Q}}_\ell$-coefficients, carries the Springer representation of W. Recall that all these cohomological groups admit a natural action via the centralizer quotient $A_{\mathbf{G}}(e) = C_{\mathbf{G}}(e)/C^\circ_{\mathbf{G}}(e)$, which morally commutes with the action of W itself. Therefore, we attain the following result (see also [Yun16, Thm 1.5.1]).

Proposition 1.1 (Decomposition of top cohomology)**.** *Let $d_e = \dim_k \mathcal{B}_e$ for each fixed $e \in \mathbf{G}$. The top cohomology of $\mathcal{B}_e$ has order $2d_e$ and admits an $W \times A_{\mathbf{G}}(e)$-action, i.e.,*

$$H^{2d_e}(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\rho \in \mathrm{Irr}(A_{\mathbf{G}}(e))} V(e,\rho) \otimes \rho,$$

where ρ runs through all conjugacy classes of irreducible representations of $A_{\mathbf{G}}(e)$, and $V(e,\rho)$ is the W-module corresponding to ρ.

On Proposition 1.1, we consider the case where $e \in \mathcal{N}$, where $\mathcal{N}$ denotes the collection of all representatives of nilpotent classes in $\mathbf{G}$. In [Spr76, Thm 6.10], Springer has proved that $V(e,\rho)$ is either zero or an irreducible representation of W. Accordingly, the classical Springer correspondence is induced by this construction.

Theorem 1.2 (Springer correspondence)**.** *There exists a bijection*

$$\mathrm{Irr}(W) \longleftrightarrow \{(e,\rho) \in \mathcal{N} \times \mathrm{Irr}(A_{\mathbf{G}}(e)) : V(e,\rho) \neq 0\}.$$

This original result is neat but doesn't work over more general setups because Springer's proof relies on the positive-characteristic assumption. Fortunately, based on [GM80], Lusztig gives a more general approach to reconstruct Theorem 1.2 via perverse sheaves [Lus81]. The punchline lies in that this does work for any $\mathbf{G}$ as well as its Lie algebra $\mathfrak{g}$ over an arbitrary algebraically closed field.

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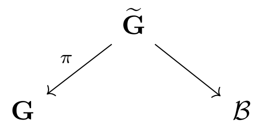
1.2. **Lusztig's construction.** The main task is to describe the W-module $V(e,\rho)$. Let's first consider the derived category $\mathcal{D}(X,\overline{\mathbb{Q}}_\ell)$ for an algebraic variety X over k. It contains all complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X. Then we can associate the cohomology sheaves $\mathcal{H}^*(K)$ to any object $K^\bullet \in \mathcal{D}(X,\overline{\mathbb{Q}}_\ell)$, with the shifting operation given by $\mathcal{H}^m(K[n]) = \mathcal{H}^{m+n}(K)$. From this, one can define a full subcategory $\mathcal{D}_c^b(X,\overline{\mathbb{Q}}_\ell)$ consisting of bounded complexes whose all cohomology sheaves are constructible.

Keeping the notations as before, we define the extended algebraic variety

$$\widetilde{\mathbf{G}} := \{(e,\mathbf{B}) \in \mathbf{G}\times\mathcal{B} \mid e\in\mathbf{B}\}$$

that classifies all conjugate Borel subgroups in $\mathbf{G}$. Let $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$ be the natural projection.



It's not hard to verify that the direct image of the ℓ-adic constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\widetilde{\mathbf{G}}$ along with π lands in the derived category of constructible ℓ-adic sheaves on $\mathbf{G}$. Also, there is a full abelian subcategory $\mathsf{Perv}(\mathbf{G}) \subset \mathcal{D}_c^b(\mathbf{G},\overline{\mathbb{Q}}_\ell)$ consisting of the so-called *perverse sheaves*, which will be discussed later. Here are three key facts by Lusztig:

- the morphism $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$ is semismall, which means that π is proper, and that, in a natural sense, there would never be too many locus of $\mathbf{G}$ whose fibers have high dimensions;
- through some algebro-geometric argument, the previous fact implies that $\pi_*\overline{\mathbb{Q}}_\ell \in \mathcal{D}_c^b(\mathbf{G},\overline{\mathbb{Q}}_\ell)$ shifts to be a semisimple object in $\mathsf{Perv}(\mathbf{G})$, that is,
$$\pi_*\overline{\mathbb{Q}}_\ell[\dim_k \mathbf{G}] \in \mathsf{Perv}(\mathbf{G});$$
moreover, it is a middle extension of its restriction to any open dense subset of $\mathbf{G}$;
- also, the endomorphism algebra of $\pi_*\overline{\mathbb{Q}}_\ell$ can be realized as the group algebra of Weyl group for $\mathbf{G}$, i.e.,
$$\overline{\mathbb{Q}}_\ell[W] \simeq \mathrm{End}_{\mathsf{Perv}(\mathbf{G})}(\pi_*\overline{\mathbb{Q}}_\ell).$$

1.3. **Fiber cohomology as the Weyl representation.** Note that the Springer fiber $\mathcal{B}_e$ is nothing but $\pi^{-1}(e)$ for each $e \in \mathcal{B}$ by definition. The last fact above deduces that W acts on stalks of cohomology sheaves $\mathcal{H}^i(\pi_*\overline{\mathbb{Q}}_\ell) \simeq R^i\pi_*\overline{\mathbb{Q}}_\ell$ at the nilpotent elements. Therefore, the action of W on $H^i(\mathcal{B}_e,\overline{\mathbb{Q}}_\ell)$ is induced from this. In other words, the étale cohomology groups of Springer fibers can be seen as representations of the Weyl group.

On Lusztig's construction, the classical Springer correspondence in Theorem 1.2 can be established as follows (once again). Since $\pi_*\overline{\mathbb{Q}}_\ell$ is a semisimple perverse sheaf, it admits a decomposition

$$\pi_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\chi\in\mathrm{Irr}(W)} (\pi_*\overline{\mathbb{Q}}_\ell)_\chi \otimes \chi$$

where the first factors are simple perverse sheaves up to shifts. Then for each $\chi \in \mathrm{Irr}(W)$, $(\pi_*\overline{\mathbb{Q}}_\ell)_\chi$ is morally isomorphic to an intersection cohomology complex of the Zariski closure of some unipotent class in $\mathbf{G}$. This loosely defines a map sending χ to $e \in \mathcal{N}$. The above is the rough idea to recover Theorem 1.2.

2. Perverse sheaves

Recall that we have used Borel–Moore homology theory $H_*^{\mathrm{BM}}(-)$ to construct the representation of the Weyl group in the previous talks. We concern about the constructible derived category of sheaves, denoted by $\mathcal{D}(-) := \mathcal{D}(-,\overline{\mathbb{Q}}_\ell)$.

2.1. **Verdier duality.** This is a crash course on perverse sheaves. Now let X be an algebraic variety over $k = \overline{\mathbb{F}}_q$ with $q = p^r$. Fix another prime $\ell \neq p$.

- A *local system* on X is a locally constant $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L} \in \mathcal{D}(X)$, which is a twisted form of the constant sheaf. Let $\mathcal{L}^\vee$ denote its dual local system.
- A sheaf is said to be *constructible* if there is a stratification on X such that each cohomology sheaf restricted to each stratum is a locally constant sheaf of finite rank.

Note that $\overline{\mathbb{Q}}_\ell$ can be regarded as a constant sheaf on X. This, together with the dualizing complex ω_X, are two basic and essential objects in the constructible derived category. The relation between them is revealed by Poincaré duality.

Proposition 2.1 (Poincaré duality). *Suppose X is a smooth and oriented manifold of pure dimension d. Then there is a canonical morphism*

$$\overline{\mathbb{Q}}_\ell[d] \xrightarrow{\cong} \omega_X.$$

The relative version of Proposition 2.1 is given by two of the so-called six operations. Recall that in general, given any morphism $f : X \to Y$, we obtain

$$\mathcal{D}(X) \underset{f^!, f^*}{\overset{f_!, f_*}{\rightleftarrows}} \mathcal{D}(Y).$$

Theorem 2.2 (Relative Poincaré duality). *Suppose $f : X \to Y$ is a smooth and oriented morphism of relative dimension d. Then there is a canonical isomorphism*

$$f^*[d] \cong f^!.$$

Note that the Poincaré duality can be understood as a shifting relationship between operations. Moreover, there is a contravariant functor

$$D_X : \mathcal{D}(X) \to \mathcal{D}(X)^{\mathrm{op}}$$

such that $D_X^2 = \mathrm{Id}_{\mathcal{D}(X)}$ to realize this kind of symmetry. Here D_X is called the *Verdier duality* on X. In particular, when X is the space of a single point, D_X is the usual duality for graded vector spaces or flags.

2.2. **Characterizing the intersection complex.** Whenever $d = 2n$ is even in Proposition 2.1, the isomorphism can be rewrite as $\omega_X[-n] \cong \overline{\mathbb{Q}}_\ell[n]$. Namely, the shifted constant sheaf $\overline{\mathbb{Q}}_\ell[n]$ is self-dual. More generally, one may replace $\overline{\mathbb{Q}}_\ell$ by a local system $\mathcal{L}$ on X, and then

$$D_X(\mathcal{L}[n]) \cong \mathcal{L}^\vee[n].$$

Hence $\mathcal{L}[n]$ is self-dual. Yet this fails to be valid when X is not smooth in general. However, the self-dual property of local systems can be recovered on a singular variety in the following sense. There exists an object in $\mathcal{D}(X)$ such that it is self-dual via D_X, and its restriction on any smooth open dense subvariety of X is nothing but the shifted constant sheaf.

Theorem 2.3 (Goresky–MacPherson). *Let X be a singular algebraic variety. Fix an open smooth subvariety $U \subset X$ of pure dimension d and a local system $\mathcal{L}$ on U. Then there exists an object $\mathrm{IC}_X(\mathcal{L})[d] \in \mathcal{D}(X)$ together with an isomorphism*

$$\mathrm{IC}_X(\mathcal{L})[d]|_U \simeq \mathcal{L}[d],$$

such that

1. $\mathcal{H}^k(\mathrm{IC}_X(\mathcal{L})[d]) = 0$ *for all* $k < -d$,
2. $\dim \mathrm{Supp}\, \mathcal{H}^k(\mathrm{IC}_X(\mathcal{L})[d]) < -k$ *for all* $k > -d$, *and*
3. $\dim \mathrm{Supp}\, \mathcal{H}^k(D_X(\mathrm{IC}_X(\mathcal{L})[d])) < -k$ *for all* $k > -d$.

Moreover, this $\mathrm{IC}_X(\mathcal{L})[d]$ is uniquely characterized by all properties above up to a unique isomorphism in $\mathcal{D}(X)$.

It is known that $\mathrm{IC}_X(\mathcal{L})[d] \in \mathcal{D}(X)$, the intersection complex of $\mathcal{L}$ on X, satisfies the desired self-dual property, i.e.,

$$D_X \mathrm{IC}_X(\mathcal{L})[d] \cong \mathrm{IC}_X(\mathcal{L}^\vee)[d].$$

2.3. **Definition of perverse sheaves.** To define perverse sheaves, we restrict Verdier duality to the full subcategory of constructible sheaves, say $D_X : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)^{\mathrm{op}}$. Remember that we are going to realize $\mathrm{IC}_X(\mathcal{L})[d]$ in Theorem 2.3 as a perverse sheaf.

Definition 2.4. A complex $K \in \mathcal{D}_c^b(X)$ is a *perverse sheaf* if for each integer $k \in \mathbb{Z}$,

1. $\dim \mathrm{Supp}\, \mathcal{H}^k(K) \leqslant -k$,
2. $\dim \mathrm{Supp}\, \mathcal{H}^k(D_X K) \leqslant -k$.

Let $\mathsf{Perv}(X)$ denote the full subcategory of perverse sheaves in $\mathcal{D}^b_c(X)$. It turns out to be an abelian category whose objects have finite length. Moreover, every simple object is in the form $\mathrm{IC}_Y(\mathcal{L})[\dim Y]$, where $Y \subset X$ is a closed subvariety, and $\mathcal{L}$ is an irreducible local system on an open dense subset of Y.

Note that if Y is smooth, the local system $\mathcal{L}$ (which defines the constant sheaf $\overline{\mathbb{Q}}_\ell$) can be realized as some direct image of a constant sheaf over its finite étale covering $\widetilde{Y}$. This observation is an essential condition to state the BBDG decomposition theorem.

2.4. **BBDG decomposition.** Let X be a variety defined over finite fields of positive characteristics. A constructible and bounded complex in $\mathcal{D}^b_c(X)$ is *semisimple* if it is a direct sum of shifts of simple perverse sheaves, or equivalently, shifts of irreducible intersection cohomology complexes. The original version of the decomposition theorem is in the following. The prototypical references for this are [BBDG18], [Sai89], and [dCM05].

Theorem 2.5 (Beilinson–Bernstein–Deligne–Gabber)**.** *Let $f : X \to Y$ be a proper morphism and fix a local system $\mathcal{L}$ on X. If there is a finite étale covering $\pi : \widetilde{X} \to X$ such that $\pi^*\mathcal{L}$ is a constant sheaf on $\widetilde{X}$ and $\mathrm{IC}_X(\mathcal{L})[\dim X]$ is a simple perverse sheaf, then*

(1) *$f_*(\mathrm{IC}_X(\mathcal{L})[\dim X]) \in \mathcal{D}^b_c(Y)$ is semisimple;*
(2) *$f_*(\mathrm{IC}_X(\mathcal{L})[\dim X])$ is semisimple as a perverse sheaf whenever it lands in $\mathsf{Perv}(Y)$.*

Another statement is relatively neat for the decomposition theorem, deduced by de Cataldo and Migliorini via algebraic Hodge theory.

Theorem 2.6. *Let X be a smooth variety and $f : X \to Y$ be a proper map. If k is a field of characteristic 0, then $\pi_*\underline{k}_X[\dim X]$ is semisimple, where $\underline{k}_X$ denotes the constant sheaf on X.*

3. The Springer representation

In this section, we discuss the construction of Springer representations by following [Lus81]. The ultimate goal is to construct an action of the Weyl group on each étale cohomology group of the Springer fibers. Recall our statement: given a reductive algebraic group $\mathbf{G}$, we are to consider $H^i(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell)$ for each $e \in \mathbf{G}$.

3.1. **The Weyl action on regular semisimple locus.** Recall the definition

$$\widetilde{\mathbf{G}} := \{(e, B) \in \mathbf{G} \times \mathcal{B} \mid e \in \mathbf{B}\}.$$

The first natural projection $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$ is called *Grothendieck–Springer map*. The basic properties of this morphism, such as properness, are essentially used in proving Theorem 3.1 by Lusztig. We then talk about the manifestation of fibers of π over some special locus of $\mathbf{G}$.

Consider the unipotent locus $\mathbf{G}^{\mathrm{uni}} \subset \mathbf{G}$. Through the second projection $\widetilde{\mathbf{G}} \to \mathcal{B}$, the inverse image $\pi^{-1}(\mathbf{G}^{\mathrm{uni}})$ is a vector bundle whose fibers are isomorphic to some open dense subvariety of $\mathbf{G}$. This immediately tells us $\pi^{-1}(\mathbf{G}^{\mathrm{uni}})$ is smooth. Morally, the unipotent singularities of $\mathbf{G}$ are resolved by $\pi|_{\pi^{-1}(\mathbf{G}^{\mathrm{uni}})}$, which is called *Springer resolution of the unipotent*.

Also consider the regular semisimple locus $\mathbf{G}^{\mathrm{rs}} \subset \mathbf{G}$. Recall that an element $x \in \mathbf{G}$ is called regular if $\dim C_{\mathbf{G}}(x) = \operatorname{rank} \mathbf{G}$. And $x \in \mathbf{G}$ is regular semisimple if the connected component of its centralizer $C^\circ_{\mathbf{G}}(x)$ can be realized as a maximal torus. Let T be a representative of some maximal torus in $\mathbf{G}$, and say

$$\widetilde{\mathbf{G}}^{\mathrm{rs}} := \{(e, \mathbf{T}) \in \mathbf{G} \times \mathcal{T} \mid e \in \mathbf{T} \cap \mathbf{G}^{\mathrm{rs}}\}.$$

The action of W is naturally given by

$$\begin{aligned} W \times \widetilde{\mathbf{G}}^{\mathrm{rs}} &\longrightarrow \widetilde{\mathbf{G}}^{\mathrm{rs}} \\ (w, (e, \mathbf{T})) &\longmapsto (e, w\mathbf{T}) \end{aligned}$$

that induces the W-action on $\mathbf{G}^{\mathrm{rs}}$. Again, the restriction of π on the regular semisimple inverse image, say

$$\pi^{\mathrm{rs}} : \widetilde{\mathbf{G}}^{\mathrm{rs}} \simeq \pi^{-1}(\mathbf{G}^{\mathrm{rs}}) \to \mathbf{G}^{\mathrm{rs}},$$

is W-equivariant with respect to the trivial W-action on $\mathbf{G}^{\mathrm{rs}}$. Therefore, W acts freely and properly discontinuously on $\widetilde{\mathbf{G}}^{\mathrm{rs}}$ such that π^{rs} is identified with the quotient. Therefore,

- π^{rs} is a W-Galois (unramified) covering;

- as a local system on $\mathbf{G}^{\mathrm{rs}}$, the direct image $\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell$ admits a W-action.

3.2. **The middle extension.** Back to $\mathbf{G}$, we consider the intersection cohomology complex functor from the category of local systems on $\mathbf{G}^{\mathrm{rs}}$:

$$\begin{aligned}\mathsf{LocSys}(\mathbf{G}^{\mathrm{rs}}) &\longrightarrow \mathsf{Perv}(\mathbf{G})\\ \mathcal{L} &\longmapsto \mathrm{IC}_{\mathbf{G}}(\mathcal{L})\end{aligned}$$

which turns out to be fully faithful. Hence the action of W on $\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell$ induces the W-action on the perverse sheaf $\mathrm{IC}_{\mathbf{G}}(\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)[\dim\mathbf{G}]$ (up to the shift operation). This is the so-called *middle extension of perverse sheaf*, while in 1981, Lusztig didn't use the language of perverse sheaves or intersection cohomology complexes. On the other hand, the following main result of [Lus81] finishes the last step of the middle extension argument.

Theorem 3.1. *Suppose we fix a local system $\mathcal{L}$ on $\mathbf{G}$. Then in* $\mathsf{Perv}(\mathbf{G})$*,*

$$\mathrm{IC}_{\mathbf{G}}(\mathcal{L})[\dim\mathbf{G}] \simeq \pi_*\overline{\mathbb{Q}}_\ell[\dim\mathbf{G}].$$

Proof. Following Theorem 2.3, the intersection complex $\mathrm{IC}_X(\mathcal{L})[d]$ can be uniquely characterized for $X=\mathbf{G}$, $U=\mathbf{G}^{\mathrm{rs}}$ and $d=\dim_k\mathbf{G}$. Here U is dense in X as required because of $\mathbf{G}=\overline{\mathbf{G}^{\mathrm{rs}}}$, the Zariski closure of regular semisimple locus. Note that the morphism $\pi:\widetilde{\mathbf{G}}\to\mathbf{G}$ is proper because $\widetilde{\mathbf{G}}$ is closed in $\mathbf{G}\times\mathcal{B}$ by definition. So the desired isomorphism is easy to construct by applying the *proper base change theorem* to π along the embedding $\mathbf{G}^{\mathrm{rs}}\to\mathbf{G}$. Hence it suffices to verify those properties (1)–(3).

(1) The cohomology sheaf can be interpreted as the derived functor. On the level of stalks, for each $e\in\mathbf{G}$,
$$(R^i\pi_*\overline{\mathbb{Q}}_\ell)_e \simeq \mathcal{H}^i_e(\pi_*\overline{\mathbb{Q}}_\ell)\simeq H^i(\pi^{-1}(e),\overline{\mathbb{Q}}_\ell).$$
Therefore, $\mathcal{H}^k_e(\pi_*\overline{\mathbb{Q}}_\ell[d])=\mathcal{H}^{d+k}_e(\pi_*\overline{\mathbb{Q}}_\ell)=0$ if $d+k<0$. This proves (1).

(2) Note again that $\pi^{-1}(e)\simeq\mathcal{B}_e$ for all Springer fibers over $e\in\mathbf{G}$. Since the stalk $(R^i\pi_*\overline{\mathbb{Q}}_\ell)_e=0$ unless $H^i(\mathcal{B}_e,\overline{\mathbb{Q}}_\ell)\neq 0$, we consider the support where $(R^k\pi_*\overline{\mathbb{Q}}_\ell)_e\neq 0$ for $k\leqslant 2\dim\mathcal{B}_e$. Consequently, for each $e\in\mathbf{G}$,
$$\operatorname{Supp}\mathcal{H}^k_e(\pi_*\overline{\mathbb{Q}}_\ell)\subset\{e\in\mathbf{G}\mid k\leqslant 2\dim\mathcal{B}_e\}$$
in a set-theoretical sense. From a nontrival counting argument by Bala–Carter (see [Car85, Sec 5.9–5.10]) that we choose to omit here, the right set has dimension $n-i$. Hence the containment is equivalent to say $\dim\operatorname{Supp}\mathcal{H}^k(\pi_*\overline{\mathbb{Q}}_\ell[d])<-k$ if $k+d>0$.

(3) By the construction, the projection $\widetilde{\mathbf{G}}\to\mathcal{B}\simeq\mathbf{G}/\mathbf{B}$ has a locally trivial fibration and each of whose fiber is isomorphic to $\mathbf{B}$. It follows that $\widetilde{\mathbf{G}}$ is smooth. Thus, $\overline{\mathbb{Q}}_\ell[d]$ is a self-dual constant sheaf on the smooth variety $\widetilde{\mathbf{G}}$, i.e.,
$$D_{\mathbf{G}}(\overline{\mathbb{Q}}_\ell[d])\simeq\overline{\mathbb{Q}}_\ell[d].$$
Again, since π is proper, the functor $D_{\mathbf{G}}$ commutes with π_*, which implies that
$$D_{\mathbf{G}}(\pi_*\overline{\mathbb{Q}}_\ell[d])\simeq\pi_*D_{\mathbf{G}}(\overline{\mathbb{Q}}_\ell[d])\simeq\pi_*\overline{\mathbb{Q}}_\ell[d].$$
This means that (3) is implied by (2).

Finally, we see $\mathrm{IC}_{\mathbf{G}}(\mathcal{L})[d]$ can only be isomorphic to $\pi_*\overline{\mathbb{Q}}_\ell[d]$. □

In the proof above, we have used the result by Bala–Carter in (2). Thanks to Goresky–MacPherson [GM80], a relatively advanced language to describe this is the "semismallness".

Definitions 3.2. Let X and Y be irreducible varieties of the same dimension n and assume further that X is smooth. Let $f:X\to Y$ be a proper morphism.

- f is called *semismall* if for all $k>0$,
$$\dim\{y\in Y\mid 2\dim f^{-1}(y)\geqslant k\}\leqslant n-k.$$
- f is called *small* if for all $k>0$,
$$\dim\{y\in Y\mid 2\dim f^{-1}(y)\geqslant k\}< n-k.$$

Proposition 3.3. *Suppose $f:X\to Y$ is a proper morphism of algebraic varieties and that X is smooth of dimension d. Fix a local system $\mathcal{L}$ on X.*

(1) *If f is semismall, then $f_*\mathcal{L}[d] \in \mathsf{Perv}(Y)$. Moreover,*
$$f_*\mathcal{L}[d] \simeq \bigoplus_{i\in I} \mathrm{IC}_{Y_i}(\mathcal{L}_i),$$
where all Y_i with $i \in I$ form a stratification on Y, and each $\mathcal{L}_i$ is the irreducible local system on the smooth locus of Y_i.

(2) *If f is small, then $f_*\mathcal{L}[d]$ enjoys all three properties in Theorem 2.3. Furthermore,*
$$f_*\mathcal{L}[d] \simeq \mathrm{IC}_Y((f|_U)_*\mathcal{L}[d]),$$
where $U \subset X$ is an open dense subset such that $f|_U$ is a covering map.

3.3. **The Springer action.** On the middle extension argument, we see

(I) there is a Weyl action on $\mathrm{IC}_{\mathbf{G}}(\mathcal{L})$, since $\mathrm{IC}_{\mathbf{G}}(-)$ is a fully faithful functor;
(II) accordingly, there is a Weyl action on $\pi_*\overline{\mathbb{Q}}_\ell$ by Theorem 3.1 up to shift;
(III) thus, for each $e \in \mathbf{G}$, there is a Weyl action on cohomology sheaves $\mathcal{H}^k_e(\pi_*\overline{\mathbb{Q}}_\ell)$;
(IV) finally, there is a Weyl action on $H^k(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell) \simeq \mathcal{H}^k_e(\pi_*\overline{\mathbb{Q}}_\ell)$.

In particular, when $e = 1$, cohomology groups of the fiber $\mathcal{B}_e = \mathcal{B}$ would carry the so-called *Springer action*, which is the action of W on $H^i(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$.

Lusztig's construction via the middle extension works for all characteristics. This also works for the Lie algebra case when considering $\mathcal{B}_u$ for a nilpotent element $u \in \mathfrak{g}$. However, the reader may see another action arising from the class of maximal torus. To be precise, let $\mathbf{T}$, and $\mathbf{B}$ be a maximal torus and a Borel subgroup of $\mathbf{G}$, respectively. Then it's easy to translate the Weyl action through the isomorphism
$$H^k(\mathbf{G}/\mathbf{T}, \overline{\mathbb{Q}}_\ell) \simeq H^k(\mathbf{G}/\mathbf{B}, \overline{\mathbb{Q}}_\ell)$$
because $\mathbf{G}/\mathbf{T}$ is a vector bundle over $\mathbf{G}/\mathbf{B}$ whose fiber is isomorphic to an open dense subset of $\mathbf{G}$. Historically, this construction gives the *classical Weyl action*. In [BM81], Borho–MacPherson claim that the Springer action of W on $H^i(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$ coincides with the classical Weyl action. The immediate corollary of Borho–MacPherson is that
$$H^*(\mathcal{B}, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell[W]$$
as W-modules. For a detailed argument on this, see [Sho88, Sec 5] or [Spa85].

Caution 3.4. For the case where $\operatorname{char} k = 0$, there is still an isomorphism $H^*(\mathcal{B}) \simeq H^*(\widetilde{\mathbf{G}})$. However, this cannot be deduced from a very similar argument as before. It is because the projection $\widetilde{\mathbf{G}} \to \mathcal{B}$ is not a vector bundle. The correct track to attain the action of W is through the following nontrivial result by Spaltenstein: if C is some conjugacy class of $\mathbf{G}$ containing a strongly regular element $t \in \mathbf{T}$, then
$$H^i(\mathcal{B}) \to H^i(\pi^{-1}(C))$$
is a W-equivariant direct-image morphism for arbitary characteristic.

4. Some prototypical applications

The theory of perverse sheaves and the construction for the Springer resolution have some applications, especially in Lusztig's earlier work on green functions and generalized springer correspondences.

4.1. **Green functions.** Keep the notations as before. Now we concern about Lusztig's original conjectures in [Lus81, Sec 3].

- Since $\mathbf{G}$ is a reductive algebraic group defined over $\mathbb{F}_q$, the finite field of characteristic $p > 0$, the Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ exists.
- For some sake in Kazhdan's argument, assume $p, q \gg 0$.
- Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ be the maximal torus and the Borel subgroup in $\mathbf{G}$. We further assume that they are fixed by F.
- Choose the local system on $\mathbf{G}$ to be $\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell$, and hence we obtain the intersection complex $\mathrm{IC}_{\mathbf{G}}(\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)$.
- The previous statement dictates that $\mathrm{IC}_{\mathbf{G}}(\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}]$ is F-stable, and we fix an isomorphism
$$\phi : F^*\mathrm{IC}_{\mathbf{G}}(\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}] \to \mathrm{IC}_{\mathbf{G}}(\pi^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}].$$

- Let $\mathbf{G}^F$ denote the (finite) F-stable reductive group. For each $w \in W$, there is a corresponding F-stable maximal torus in $\mathbf{G}$, denoted by $\mathbf{T}_w$.

Definition 4.1. The Green function associated with $e \in (\mathbf{G}^{\text{uni}})^F$ is defined as

$$\mathcal{Q}_{\mathbf{T}_w,\mathbf{G}}(e) := \sum_{i \geqslant 0} (-1)^i \text{Tr}(Fw, H^i(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell)).$$

Remark 4.2. The original definition of the Green functions differs from Definition 4.1. It comes from the trace of an alternating some of certain ℓ-adic cohomologies on which $\mathbf{G}^F$ acts. It is then proved by Springer–Kazhdan [Kaz77] that the Green function enjoys an expression above for elements in the unipotent F-stable locus.

Definition 4.3. For the fixed isomorphism ϕ, we define the characteristic function of $\text{IC}_{\mathbf{G}}(\pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}]$ as follows. For each $e \in \mathbf{G}^F$,

$$\chi_\phi(e) := \sum_{i \geqslant 1} (-1)^i \text{Tr}(\phi, \mathcal{H}_e^i(\text{IC}_{\mathbf{G}}(\pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}])).$$

4.2. **Character formula.** On Theorem 3.1, the formula of green functions in Definition 4.1 can be rewritten as

$$\mathcal{Q}_{\mathbf{T}_w,\mathbf{G}}(e) = \sum_{i \geqslant 0} (-1)^i \text{Tr}(Fw, \mathcal{H}_e^i(\text{IC}_{\mathbf{G}}(\pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell))).$$

On the other hand, as we have introduced in Section 1.3 before, we have a decomposition

$$\text{IC}_{\mathbf{G}}(\pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}] \simeq \bigoplus_{\rho \in \text{Irr}(W)} V(\rho) \otimes \rho,$$

where

$$V(\rho) = \text{Hom}_W(\rho, \text{IC}_{\mathbf{G}}(\pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell)[\dim \mathbf{G}]) = \text{IC}_{\mathbf{G}}(\mathcal{H}\text{om}_W(\rho, \pi_*^{\text{rs}}\overline{\mathbb{Q}}_\ell))[\dim \mathbf{G}]$$

is the perverse sheaf associated with ρ. Now for each $e \in (\mathbf{G}^F)^{\text{uni}}$, this decomposition implies

$$\mathcal{Q}_{\rho,\mathbf{G}}(e) = \sum_{i \geqslant 1} (-1)^i \text{Tr}(F, \mathcal{H}_e^i(V(\rho))) = \chi_{\rho,F}(e).$$

On the other hand, if we further assume that $\mathbf{G} = \text{GL}_n(\mathbb{F}_q)$, the Weyl group admits the trivial Frobenius action by F. Hence for each χ,

$$\mathcal{Q}_{\rho,\mathbf{G}}(e) = \frac{1}{|W|} \sum_{w \in W} \text{Tr}(w, \chi) \mathcal{Q}_{\mathbf{T}_w,\mathbf{G}}.$$

4.3. **Generalized Springer resolution for GL_n.** Recall the Springer resolution in Section 3.1 is the morphism $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$. A natural approach to generalize this is by replacing $\mathbf{B}$ in the definition of $\widetilde{G}$ by a parabolic subgroup $\mathbf{P}$ (whose unipotent radical is denoted by $\mathbf{U} = \mathbf{U}(\mathbf{P})$. For each $\mathbf{P}$, we obtain the *generalized Springer resolution*

$$\pi_{\mathbf{P}} : \widetilde{\mathbf{G}}(\mathbf{P}) \to \mathbf{G},$$

where $\widetilde{\mathbf{G}}(\mathbf{P}) := \{(e, \mathbf{P}) \in \mathbf{G} \times \mathbf{G}/\mathbf{P} \mid e \in \mathbf{P}\}$. A unipotent orbit in $\mathbf{G}$ is called *Richardson* if its intersection with $\mathbf{U}$ is dense in $\mathbf{U}$. Note that there is a unique Richardson orbit, say $\mathcal{O}_{\mathbf{P}}$, associated to $\mathbf{P}$, whose closure is exactly the image of $\pi_{\mathbf{P}}$. It turns out that $\pi_{\mathbf{P}}$ is not always birational unless $C_{\mathbf{G}}(e) \subset \mathbf{P}$ for all $e \in \mathcal{O}_{\mathbf{P}}$. In fact, it is a morphism of degree $[C_{\mathbf{G}}(e) : C_{\mathbf{P}}(e)] = [A_{\mathbf{G}}(e) : A_{\mathbf{P}}(e)]$. Hence we obtain the following result, which is particularly useful when $\mathbf{G} = \text{GL}_n(k)$.

Proposition 4.4. [BM81] *The generalized Springer resolution $\pi_{\mathbf{P}}$ is birational if $A_{\mathbf{G}}(e)$ is trivial for all $e \in \mathcal{O}_{\mathbf{P}}$. Furthermore, it induces a semismall resolution of $\overline{\mathcal{O}}_{\mathbf{P}}$.*

4.4. **Computing intersection complex stalks.** As for the considerable case where $\mathbf{G} = \text{GL}_n(k)$ and $k = \mathbb{F}_q$, an algorithm to compute stalks for intersection complexes has been developed by [Lus81] and [BM81] (also see [JMW08] for a concise introduction). In this case, all nilpotent orbits are Richardson and all $A_{\mathbf{G}}(e)$ are trivial. Some setups are given as follows.

- Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of integers of n, that is, $\lambda_1 + \cdots + \lambda_s = n$.
- Let e_i denote the i-th element in the standard basis of $\mathbb{C}^n$, that is, the vector whose i-th coordinate is 1 and others are 0.
- Define $L_i(\lambda)$ as the vector subspace of $\mathbb{C}^n$ spanned by $e_1, \ldots, e_{\lambda_1 + \ldots + \lambda_i}$.

- Accordingly, there is a flag associated to λ, given by
$$0 = L_0^\lambda \subset L_1^\lambda \subset \cdots \subset L_s^\lambda = \mathbb{C}^n.$$
- There is a parabolic subgroup, denoted by $\mathbf{P}(\lambda) \subset \mathrm{GL}_n(k)$, that stabilizes the flag $L_\bullet^\lambda$. All these flags as above form a flag variety $\mathcal{F}(\lambda) = \mathbf{G}/\mathbf{P}(\lambda)$.
- Let $\mathcal{O}_\lambda \subset \mathrm{GL}_n(k)$ be the nilpotent orbit consisting of those nilpotent matrices whose Jordan normal forms have blocks of respective sizes $\lambda_1, \ldots, \lambda_s$.

Proposition 4.5 (Dimension bound)**.** *Keep the notations as before. Let $\lambda' = (\lambda_1', \ldots, \lambda_s')$ be the partition conjugate of λ. Given $e \in \mathbf{G}$, we have $e \in \overline{\mathcal{O}}_\lambda$ if and only if for each $1 \leqslant i \leqslant s$,*
$$\dim(\ker e^i) \geqslant \sum_{k=1}^{i} \lambda_k'.$$
Here the equality holds if and only if $e \in \mathcal{O}_\lambda$.

Theorem 4.6. *For the morphism $\pi_{\mathbf{P}(\lambda')} : \widetilde{\mathbf{G}}(\mathbf{P}(\lambda')) \to \mathbf{G}$, we obtain the following results.*
1. *$\pi_{\mathbf{P}(\lambda')}$ is an isomorphism over $\mathcal{O}_\lambda$;*
2. *$\overline{\mathcal{O}}_\lambda$ is exactly the image of $\pi_{\mathbf{P}(\lambda')}$;*
3. *$\pi_{\mathbf{P}(\lambda')}$ is a resolution of singularities.*

Proof. Note that there is a natural isomorphism
$$\mathbf{T}^*(\mathbf{G}/\mathbf{P}(\lambda)) \simeq \{(e, L_\bullet^\lambda) \in \mathbf{G} \times \mathcal{F}_\lambda \mid e(L_{i+1}^\lambda) \subset L_i^\lambda \text{ for all } i \geqslant 0\}.$$
For each $e \in \operatorname{im} \pi_{\mathbf{P}(\lambda')} \subset \mathbf{G}$, there is a flag of the form $L_\bullet^{\lambda'}$ such that for all $i \geqslant 0$, $e(L_{i+1}^\lambda) \subset L_i^\lambda$. By induction, this implies that $L_i^\lambda \subset \ker e^i$. A dimension calculation shows that $\dim(\ker e^i) \geqslant \sum_{k=1}^i \lambda_k'$. Thus by Proposition 4.5, e lies in $\overline{\mathcal{O}}_\lambda$. This shows that $\operatorname{im} \pi_{\mathbf{P}(\lambda')} \subset \overline{\mathcal{O}}_\lambda$. Similarly, by assuming $e \in \mathcal{O}_\lambda$, we have $\mathcal{O}_\lambda \subset \operatorname{im} \pi_{\mathbf{P}(\lambda')}$. Recall that all such first projections are proper (see the proof of Theorem 3.1), so $\operatorname{im} \pi_{\mathbf{P}(\lambda')} = \overline{\mathcal{O}}_\lambda$. In particular, the morphism $\pi_{\mathbf{P}(\lambda')}$ is an isomorphism over $\mathcal{O}_\lambda$. Again, by the properness, it is a resolution of singularities. ☐

We now consider to apply the BBDG decomposition (see Theorem 2.5) to the proper morphism $\pi_{\mathbf{P}(\lambda')}$. After fixing a (usually constant) local system $\mathcal{L}$ on $\mathcal{O}_\lambda$, we see
$$\pi_{\mathbf{P}(\lambda'),*}\overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_\lambda]$$
is semisimple whose simple direct summands are intersection complexes with respect to the stratification on $\mathcal{O}_\lambda$ (see the previous talks, which we will not cover again). By Theorem 2.3(1), the stalks of the direct image are given by the cohomology of the fibers. One finds the stalks of $\mathrm{IC}_{\mathcal{O}_\lambda}(\mathcal{L})$ by removing the stalks of the other summands. In the case of $\mathbf{G} = \mathrm{GL}_n(k)$, the approach for computing the IC-stalks with $\overline{\mathbb{Q}}_\ell$ coefficients (which is in terms of Green polynomials) has been known since [Lus81].

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