INTEGRAL MODELS OF SHIMURA VARIETIES OF HODGE TYPE

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In this series of lectures, we apply the results on Breuil–Kisin classification of p-divisible groups to construct smooth integral canonical models for Shimura varieties of Hodge type, following [Kis10]. As a preliminary, we will first review the results of Deligne [De82], Blasius [Bla94] and Wintenberger about Hodge cycles on abelian varieties. Then we will cover the main results of [Kis10, §2].

Contents

1. Hodge cycles on abelian varieties

Fix a field k together with a complex embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. Consider a projective smooth variety X over k. There would be natural classical cohomology theories on this setup:

- *de Rham cohomology.*
$$H^i_{\mathrm{dR}}(X) \coloneqq H^i(X, \Omega^\bullet_{X/k}),$$
as a filtered k-vector space of finite dimension, equipped with a descending Hodge filtration, denoted by $F^\bullet H^i_{\mathrm{dR}}(X)$.
- *ℓ-adic cohomology.* For any prime ℓ,
$$H^i_\ell(X) \coloneqq H^i_{\mathrm{et}}(X_{\overline{k}}, \overline{\mathbb{Q}}_\ell),$$
as a $\mathbb{Q}_\ell$-vector space, equipped with a continuous Galois action of $G_k = \mathrm{Gal}(\overline{k}/k)$.
- *Betti cohomology.* For any embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, consider the complex variety $\sigma X \coloneqq X \otimes_{k,\sigma} \mathbb{C}$ and define
$$H^i_\sigma(X) \coloneqq H^i_B((\sigma X)^{\mathrm{an}}, \mathbb{Q}),$$
which is a $\mathbb{Q}$-vector space, equipped with a Hodge structure; namely, admits a Hodge decomposition
$$H^i_\sigma(X) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}_\sigma.$$

These classical cohomology theories are connected via the comparison theorems.

Proposition 1.1. (1) *We have isomorphisms of $\mathbb{C}$-vector spaces*
$$H^i_\sigma(X) \otimes \mathbb{C} \xrightarrow[\sim]{I_\infty} H^i_{\mathrm{dR}}(\sigma X) \xleftarrow[\sim]{\sigma} H^i_{\mathrm{dR}}(X) \otimes_{k,\sigma} \mathbb{C},$$

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where the right isomorphism is induced by σ and hence depends on the choice of σ.

(2) *We have isomorphisms of $\mathbb{Q}_\ell$-vector spaces*

$$H^i_\sigma(X) \otimes \mathbb{Q}_\ell \xrightarrow[\sim]{I_\ell} H^i_{\mathrm{et}}(\sigma X, \mathbb{Q}_\ell) \xleftarrow[\sim]{\sigma} H^i_\ell(X).$$

Again, the right isomorphism is induced by σ.

(3) *All isomorphisms in (1) and (2) above are compatible with additional structures on cohomological theories.*

We then consider their behaviors under *Tate twists*. For an integer $m \geqslant 0$, we have for de Rham cohomology that

$$H^i_{\mathrm{dR}}(X)(m) = H^i_{\mathrm{dR}}(X), \quad F^{p-m} H^i_{\mathrm{dR}}(X)(m) = F^p H^i_{\mathrm{dR}}(X).$$

For ℓ-adic cohomology, if we write $\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$, then

$$H^i_\ell(X)(m) = H^i_\ell(X) \otimes \mathbb{Z}_\ell(1)^{\otimes m}.$$

As for the Betti cohomology,

$$H^i_\sigma(X)(m) = (2\pi i)^m H^i_\sigma(X), \quad (H^i_\sigma(X)(m))^{p-m,q-m} = H^{p,q}_\sigma(X).$$

In fact, as a conclusion, all of these cohomology theories $H^i_?(X)$ with $? \in \{\mathrm{dR}, \ell, \sigma\}$ satisfy the axioms of a Weil cohomology with Tate twists.

We are also interested in cycle class maps:

$$\mathrm{cl}^i_\sigma : \mathrm{CH}^i(X) \otimes \mathbb{Q} \longrightarrow H^{2i}_\sigma(X)(i).$$

The image of the cycle class map of degree i (i.e. with cycles of codimension i) satisfies

$$\operatorname{Im} \mathrm{cl}^i_\sigma \subseteq (H^{2i}_\sigma(X)(i))^{0,0} \cap H^{2i}_\sigma(X)(i).$$

Here the left-hand side is the collection of algebraic cycles, and the right-hand side exactly collects Hodge cycles. We have the following:

- (*Hodge conjecture*) For $k = \mathbb{C}$, the cycle class map is surjective, or equivalently,

$$\operatorname{Im} \mathrm{cl}^i_\sigma = (H^{2i}_\sigma(X)(i))^{0,0} \cap H^{2i}_\sigma(X)(i).$$

Definition 1.2. Write $\mathbb{A}$ for the adelic ring. Let X be a projective smooth k-variety.

(1) Assume $k = \overline{k}$. Define the pair

$$t = (t_{\mathrm{dR}}, t_{\mathrm{et}}) \in H^{2p}_{\mathbb{A}}(X)(p) := H^{2p}_{\mathrm{dR}}(X)(p) \times H^{2p}_{\mathrm{et}}(X)(p),$$

where

$$H^i_{\mathrm{et}}(X) := \prod_\ell{}' H^i_\ell(X) \xrightarrow{\sim} H^i_{\mathrm{et}}(\sigma X) \xleftarrow{\sim} H^i_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

The pair t is called a *Hodge cycle relative to* $\sigma\colon k \hookrightarrow \mathbb{C}$ if

(a) t is rational under the map

$$\begin{aligned} H^{2p}_\sigma(X)(p) &\hookrightarrow H^{2p}_\sigma(X)(p) \otimes (\mathbb{C} \times \mathbb{A}_f) \\ &\xrightarrow{\sim} H^{2p}_{\mathrm{dR}}(X)(p) \otimes_{k,\sigma} \mathbb{C} \times H^{2p}_{\mathrm{et}}(X)(p) \end{aligned}$$

that is given by $t_\sigma \mapsto t$.

(b) t admits the Hodge decomposition, i.e. $t_{\mathrm{dR}} \in F^0 H^{2p}_{\mathrm{dR}}(X)(p)$. Granting (a), this is equivalent to $t_\sigma \in (H^{2p}_\sigma(X)(p))^{0,0}$.

(2) Assume $k = \overline{k}$. The pair $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is called an *absolute Hodge cycle* if it is a Hodge cycle relative to any choice of $\sigma\colon k \hookrightarrow \mathbb{C}$.

(3) For any field k, an *absolute Hodge cycle* on X is an absolute Hodge cycle on $X_{\overline{k}}$ that is fixed by the natural action of G_k.

Here in (1), one may understand the de Rham cohomology and étale cohomology as the archimedean part and finite part of $\mathbb{A}$, respectively. It turns out that $t = (t_{\mathrm{dR}}, (t_\ell)_\ell) \in H^{2p}_{\mathbb{A}}(X)(p)$ is an absolute Hodge cycle if for any $\sigma\colon k \hookrightarrow \mathbb{C}$, there exists $t_\sigma \in H^{2p}_\sigma(X)(p) \cap (H^{2p}_\sigma(X)(p))^{0,0}$ such that

$$I_\infty(t_\sigma) = \sigma t_{\mathrm{dR}}, \quad I_\ell(t_\sigma) = \sigma t_\ell.$$

Example 1.3. (1) Formally, we have that

$$\{\text{algebraic cycles}\} \subseteq \{\text{absolute Hodge cycles}\} \subseteq \{\text{Hodge cycles}\}.$$

If the Hodge conjecture holds, then both containments are to be equalities.

(2) Write $d = \dim_k X$ and consider the diagonal image $\Delta \subseteq X \times X$. Applying the Künneth formula, one obtains

$$H^{2d}(X \times X)(d) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

This leads to a decomposition on the image of cycle class map, read as

$$\mathrm{cl}(\Delta) = \sum_{i=0}^{2d} \pi^i,$$

where each π^i is an absolute Hodge cycle.

The following big theorem of Deligne identifies absolute Hodge cycles with Hodge cycles.

Theorem 1.4 (Deligne)**.** *Assume $k = \overline{k}$ and X is an abelian variety over k. If t is a Hodge cycle on X relative to an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, then it is an absolute Hodge cycle.*

The following two p-adic variants of Theorem 1.4 can be derived via comparison theorems from p-adic Hodge theory, which relates the result of Deligne with more deep intrinsic properties of cohomologies. Let $k \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}$ be a number field. For any prime p, let $\sigma_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ be an embedding, which restricts to k as $\sigma_p\colon k \hookrightarrow \overline{\mathbb{Q}}_p$. Let X be a projective smooth variety over k. Denote $\sigma_p X$ the base change of X over the completion $(\sigma_p(k))^\wedge$.

Proposition 1.5 (p-adic étale versus p-adic de Rham)**.** *There is a functorial isomorphism*

$$I_{\mathrm{dR}}\colon H^i_{\mathrm{et}}((\sigma_p X)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(\sigma_p X) \otimes_{(\sigma_p(k))^\wedge} B_{\mathrm{dR}},$$

compatible with additional structures on both sides.

Definition 1.6. Let $t = (t_{\mathrm{dR}}, (t_p)_p) \in H^{2q}_{\mathbb{A}}(X)(q)$ be an absolute Hodge cycle. It is called *de Rham* if for any p and any $\sigma_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we have

$$I_{\mathrm{dR}}(\sigma_p t_p) = \sigma_p t_{\mathrm{dR}}.$$

Recall that we have isomorphisms

$$\sigma_p\colon H^i_p(X) \xrightarrow{\sim} H^i_{\mathrm{et}}((\sigma_p X)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p),$$
$$\sigma_p\colon H^i_{\mathrm{dR}}(X) \otimes_{k,\sigma_p} (\sigma_p(k))^\wedge \xrightarrow{\sim} H^i_{\mathrm{dR}}(\sigma_p X).$$

Theorem 1.7 (Blasius, Ogus)**.** *Let X be an abelian variety over $\overline{\mathbb{Q}}$. Then every Hodge cycle on X is de Rham.*

Suppose the base change $\sigma_p X$ over $(\sigma_p(k))^\vee$ has a good reduction. Then $\overline{\sigma_p X}$ lies over another unramified extension κ satisfying

$$(\sigma_p(k))^{\wedge,\mathrm{ur}} = W(\kappa)_{\mathbb{Q}} = W(\sigma_p).$$

Then we are able to consider the crystalline cohomology $H^i_{\mathrm{cris}}(\overline{\sigma_p X})$, as a $W(\sigma_p)$-vector space equipped with a Φ-action.

Proposition 1.8 (p-adic étale versus crystalline)**.** *There is a functorial isomorphism*

$$I_{\mathrm{cris}}\colon H^i_{\mathrm{et}}((\sigma_p X)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^i_{\mathrm{cris}}(\overline{\sigma_p X})\otimes_{W(\sigma_p)} B_{\mathrm{cris}},$$

compatible with additional structures on both sides.

Combining Propositions 1.5 and 1.8, we deduce that

$$H^i_{\mathrm{cris}}(\overline{\sigma_p X})\otimes_{W(\sigma_p)} (\sigma_p(k))^\wedge \cong H^i_{\mathrm{dR}}(\sigma_p X).$$

Therefore, $I_{\mathrm{cris}}\otimes 1 = I_{\mathrm{dR}}$.

Definition 1.9. Let $t = (t_{\mathrm{dR}}, (t_p)_p) \in H^{2q}_{\mathbb{A}}(X)(q)$ be a de Rham cycle that is defined over k. Fix an embedding $\sigma_p\colon k \hookrightarrow \overline{\mathbb{Q}}_p$. This t is called *crystalline* at σ_p if

1. X has good reduction at σ_p,
2. $t_{\mathrm{dR}} \in H^{2q}_{\mathrm{cris}}(\overline{\sigma_p X})(q) \hookrightarrow H^{2q}_{\mathrm{dR}}(\sigma_p X)(q)$, and
3. $\Phi(t_{\mathrm{dR}}) = t_{\mathrm{dR}}$.

Corollary 1.10. *Let X be an abelian variety over k with good reduction at σ_p. Let t be a Hodge cycle defined over k. Then t is crystalline at σ_p.*

Sketch of proofs of the theorems. **Step I.** Let $\mathcal{C}$ be the category of projective smooth varieties over k, with $k \hookrightarrow \mathbb{C}$. This induces the category of motives for Hodge, absolute Hodge, de Rham cycles, respectively, denoted by

$$\bigotimes_{\mathrm{H}}\mathcal{C},\quad \bigotimes_{\mathrm{AH}}\mathcal{C},\quad \bigotimes_{\mathrm{dR}}\mathcal{C}.$$

So we have a semisimple Tannakian category for which $\omega_B = H^*_B$ is a fiber functor: for each object $X \in \mathcal{C}$,

$$\mathcal{G}_? = \mathrm{Aut}^{\otimes}(\omega_B, \bigotimes_?\langle X\rangle), \quad ? \in \{\mathrm{H}, \mathrm{AH}, \mathrm{dR}\}.$$

> *Principle A.* Let X be a projective smooth variety over $\mathbb{C}$ (resp. over a number field). Then $\mathcal{G}_{\mathrm{H}} = \mathcal{G}_{\mathrm{AH}}$ (resp. $\mathcal{G}_{\mathrm{dR}} = \mathcal{G}_{\mathrm{AH}}$) if and only if every Hodge cycle (resp. absolute Hodge cycle) in $\bigotimes_?\langle X\rangle$ is absolutely Hodge (resp. de Rham).

In general, we always have the relations

$$\mathcal{G}_{\mathrm{H}} \subseteq \mathcal{G}_{\mathrm{AH}} \subseteq \mathcal{G}_{\mathrm{dR}}.$$

Step II. Let S be a projective smooth geometrically connected variety over k, with $k \hookrightarrow \mathbb{C}$. Let $\pi\colon X \to S$ be a smooth proper morphism over k. Take

$$t_B \in H^0(S_{\mathbb{C}}, R^{2n}\pi_{\mathbb{C},*}\mathbb{Q})(n).$$

> *Principle B.* For the extension $k \subseteq L \subseteq \mathbb{C}$ and a geometric point $s \in S(L) \subseteq S(\mathbb{C})$, let $t_B(s) \in H^{2n}_B(X_S)(n)$ be the restriction. Let $s_0 \in S(k)$. Then
> (i) When $k = \mathbb{C}$, if $t_B(s_0)$ is a Hodge cycle, then $t_B(s)$ is a Hodge cycle as well for each $s \in S(\mathbb{C})$;
> (ii) When $k = \mathbb{C}$, if $t_B(s_0)$ is an absolute Hodge cycle, then $t_B(s)$ is an absolute Hodge cycle as well for each $s \in S(\mathbb{C})$;
> (iii) When $k \subseteq \overline{\mathbb{Q}}$, if $t_B(s_0)$ is a de Rham cycle, then $t_B(s)$ is a de Rham cycle as well for each $s \in S(\mathbb{C})$.

Step III. We now deal with the CM case. Let K be a CM field over $\mathbb{Q}$. Consider the abelian variety $A_\Phi \coloneqq \mathbb{C}^\Phi/\mathcal{O}_K$, which is called the *graph* of Φ. Then, if we take A to be any abelian variety of CM type, then A is isogenic to a quotient of a power of $B = \prod_{\Phi\in S} A_\Phi$. Then it suffices to prove the equalities

$$\mathcal{G}_{\mathrm{H}} = \mathcal{G}_{\mathrm{AH}} = \mathcal{G}_{\mathrm{dR}}$$

for B. Let L be another CM field over $\mathbb{Q}$. The work of Deligne includes results from three aspects:

(1) Cycles of graphs: for any $\Phi \in S$, we have $L \hookrightarrow \mathrm{End}(A_\Phi)$.
(2) For any $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$, the Galois action of σ induces isomorphic graphs, that is, $A_\Phi \simeq A_{\Phi\sigma}$.
(3) Let $T \subseteq S$ be a subset with $|T| = d$. Let $B_T = \prod_{\Phi\in T} A_\Phi$. Suppose L acts on $H^1_B(B_T)$ where each embedding of L occurs with the equal multiplicity. Then

$$\wedge^d_L H^1_B(B_T)(d/2) \subseteq H^d_B(B_T)(d/2).$$

Step IV. Consider the general case where A is not necessarily of CM type. Let $\mathcal{G}_{\mathrm{H}}$ be as above. This together with a cocharacter μ defines a Shimura datum. So we obtain a Shimura variety $\mathbf{Sh}$ of Hodge type. For each open compact subgroup $U \subseteq \mathcal{G}_{\mathrm{H}}(\mathbb{A}_f)$, there is a natural morphism $\pi\colon \mathcal{A} \to \mathbf{Sh}_U$ from the universal abelian variety, such that there is $s_0 \in \mathbf{Sh}_U(\mathbb{C})$ to carry an isogeny $\mathcal{A}_{s_0} \sim A$ (noting that $\mathcal{A}_{s_0}$ is of CM type). In this case, using Principle B and the argument in Step III, we are able to prove the theorems and propositions above for $X = A$.

2. Reductive groups and crystalline representations

Let $S = \operatorname{Spec} R$ with a local ring R. Let M be a finite free R-module. Take $G \subseteq \mathrm{GL}(M)$ as a closed embedding of group schemes, where G is a connected reductive group over S. Consider a decreasing finite length filtration $M^\bullet$ on M, such that $\mathrm{gr}^\bullet M$ is finite flat over R.

Consider $P \subseteq G$, the closed subgroup which respects to $M^\bullet$. Also consider $U \subseteq P$, the closed subgroup which acts trivially on $\mathrm{gr}^\bullet M$. We introduce the following facts about the parabolic subgroup without proof.

Lemma 2.1. (1) *The followings are equivalent.*
  (a) *The filtration $M^\bullet$ admits a splitting such that the corresponding cocharacter $\mu\colon \mathbb{G}_m \to \mathrm{GL}(M)$ factors through G. (Thus, we have a cocharacter on G.)*
  (b) *The subgroup $P \subseteq G$ is a parabolic subgroup with the unipotent radical U, and $\mathrm{gr}^\bullet M$ is induced by a cocharacter $\nu\colon \mathbb{G}_m \to P/U$.*

Moreover, if either of the conditions in (1) *holds, then $M^\bullet$ is called G-split.*

(2) *If R is a field of characteristic* 0*, then $M^\bullet$ is G-split if and only if $\langle M\rangle^\otimes$, the Tannakian category of G-representations generated by M, admits a filtration which induces the given filtration on M.*
(3) *If R is a discrete valuation ring and $K = \operatorname{Frac} R$, then $M^\bullet$ is G-split if and only if the induced filtration on M_K is $G \otimes_R K$-split.*

Let $M^\otimes$ be the direct sum of all R-modules formed from M by taking duals, tensor products, symmetric powers, and exterior powers. We obtain a natural isomorphism $M^\otimes \xrightarrow{\sim} M^{*\otimes}$. If $(s_\alpha) \subseteq M^\otimes$ is a finite collection of Galois invariant tensors, and $G \subseteq \mathrm{GL}(M)$ is the pointwise stabilizer of the s_α, we say that G is the group defined by the tensors s_α.

Proposition 2.2. *Suppose that R is a discrete valuation ring of mixed characteristic, and let $G \subseteq \mathrm{GL}(M)$ be a closed R-flat subgroup whose generic fiber is reductive. Then G is defined by a finite collection of tensors $(s_\alpha) \subseteq M^\otimes$.*

Proof. The proof is similar to that of [De82, Prop. 3.1]. For each finite free R-module W carrying an action of $\mathrm{GL}(M) = \operatorname{Spec} \mathcal{O}_{\mathrm{GL}}$, let W_0 denote W with the trivial $\mathrm{GL}(M)$-action. We have the inclusion of R-schemes $\mathrm{GL}(M) \subseteq \mathrm{End}(M)$, which is fibre by fibre dense. Thus

$$\mathcal{O}_{\mathrm{GL}} = \varinjlim_n \mathrm{Sym}(M \otimes M_0^*) \otimes (\det M)^{-n}.$$

with the transition maps being given by multiplication by $\det \otimes \delta^{-1}$, where $\det \in \mathrm{Sym}(M \otimes M_0^*)$ and $\delta \in \det M$ is some fixed basis vector. Each term in the injective limit is a direct summand of the next term, so it suffices to find a collection of tensors $(s_\alpha) \subseteq \mathcal{O}_{\mathrm{GL}}$ defining G.

For any finite projective R-module W with an action of $\mathrm{GL}(M)$, the $\mathcal{O}_{\mathrm{GL}^-}$ comodule structure on W gives a $\mathrm{GL}(M)$-equivariant map $W \to W_0 \otimes_R \mathcal{O}_{\mathrm{GL}}$. This map is injective and its cokernel is a direct summand, a section being induced by the identity section $\mathcal{O}_{\mathrm{GL}} \to R$. Hence it suffices to find elements defining G in any representation of $\mathrm{GL}(M)$ on a finite projective R-module.

Now let $I \subseteq \mathcal{O}_{\mathrm{GL}}$ denote the ideal of G. Then G is the scheme-theoretic stabilizer of I. Let $W \subseteq \mathcal{O}_{\mathrm{GL}}$ be a finite rank, $\mathrm{GL}(M)$-stable, saturated R-submodule such that $W \cap I$ contains a set of generators of I. Then G is the stabilizer of $W \cap I \subseteq W$. If $r = \mathrm{rank}_R W \cap I$, then $L = \wedge^r(W \cap I) \subseteq \wedge^r W$ is a line, and G is the stabilizer of L.

Since G has reductive generic fibre the quotient map $(\wedge^r W)^* \to L^*$ has a G equivariant splitting over the generic point $\eta \in \operatorname{Spec} R$. Hence there exists a G-stable line $\tilde{L}^* \subseteq (\wedge^r W)^*$ which maps isomorphically to L^* over η. Now G acts trivially on $L \otimes_R \tilde{L}^*$ as this is true over η, and the stabilizer of $L \otimes_R \tilde{L}^* \subseteq (\wedge^r W) \otimes_R (\wedge^r W)^*$ is equal to G. □

Now let k be a perfect field of characteristic p and $W = W(k)$ the Witt ring. Take $K_0 = W_{\mathbb{Q}}$ the fractional field, and K a finite totally ramified extension over K_0. Denote $G_K = \mathrm{Gal}(\overline{K}/K)$ (which is not $G \otimes_R K$). Take $\mathsf{Rep}_{G_K}^{\mathrm{cris},\circ}$ the category of G_K-stable $\mathbb{Z}_p$-lattices in a fixed crystalline representation of G_K. Choose $L \in \mathsf{Rep}_{G_K}^{\mathrm{cris},\circ}$.

Consider the reductive group $G \subseteq \mathrm{GL}(L)$. Then by Proposition 2.2, there exists a finite collection $(s_\alpha) \subseteq L^{\otimes}$ that defines G. Also, the G_K action $G_K \to \mathrm{GL}(L)$ on L factors through $G(\mathbb{Z}_p)$ if and only if these tensors are G_K-invariant by definition.

Fix a uniformizer $\pi \in \mathcal{O}_K$, and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for π. We set $\mathfrak{S} = W[\![u]\!]$ equipped with a Frobenius φ which acts as the usual Frobenius on W and sends u to u^p. Let $\mathsf{Mod}_{\mathfrak{S}}^{\varphi}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

$$1 \otimes \varphi \colon \varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)].$$

For $i \in \mathbb{Z}$, we set

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}) = (1 \otimes \varphi)^{-1}(E(u)^i \mathfrak{M}) \cap \varphi^*(\mathfrak{M}).$$

Recall that there exists a fully faithful tensor functor

$$\mathfrak{M} \colon \mathsf{Rep}_{G_K}^{\mathrm{cris},\circ} \longrightarrow \mathsf{Mod}_{\mathfrak{S}}^{\varphi}$$

which is compatible with the formation of symmetric and exterior powers. Moreover, we have the following theorem as a reminder.

Theorem 2.3. *If L is in $\mathsf{Rep}_{G_K}^{\mathrm{cris},\circ}$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $\mathfrak{M} = \mathfrak{M}(L)$, then*

(1) *There are canonical isomorphisms*

$$D_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathfrak{M}/u\mathfrak{M}[1/p], \quad D_{\mathrm{dR}}(V) \xrightarrow{\sim} \varphi^*(\mathfrak{M}) \otimes_{\mathfrak{S}} K,$$

where the map $\mathfrak{S} \to K$ is given by $u \mapsto \pi$. The first isomorphism is compatible with Frobenius, and the second maps $\mathrm{Fil}^i \varphi^*(\mathfrak{M}) \otimes_W K_0$ onto $\mathrm{Fil}^i D_{\mathrm{dR}}(V)$ for $i \in \mathbb{Z}$.

(2) *There is a canonical isomorphism*

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M}.$$

(3) *If k'/k is an algebraic extension of fields, then there exists a canonical φ equivariant isomorphism*

$$\mathfrak{M}(L|_{G_{K'}}) \xrightarrow{\sim} \mathfrak{M}(L) \otimes_{\mathfrak{S}} \mathfrak{S}',$$

where $\mathfrak{S}' = W(k')[\![u]\!]$ and $G_{K'} = \mathrm{Gal}(\overline{K} \cdot W(k')_{\mathbb{Q}}/K \cdot W(k')_{\mathbb{Q}})$.

Now we go back to the collection $(s_\alpha) \subseteq L^{\otimes}$. View the tensors s_α as morphisms $s_\alpha \colon \mathbb{1} \to L^{\otimes}$ in $\mathsf{Rep}_{G_K}^{\mathrm{cris},\circ}$. Applying the functor $\mathfrak{M}$, we obtain morphisms $\tilde{s}_\alpha \colon \mathbb{1} \to \mathfrak{M}(L)^{\otimes}$ in $\mathsf{Mod}_{\mathfrak{S}}^{\varphi}$.

Theorem 2.4. *Let L be in $\mathsf{Rep}_{G_K}^{\mathrm{cris},\circ}$ and $G \subseteq \mathrm{GL}(L)$ a reductive $\mathbb{Z}_p$-subgroup defined by a finite collection of G_K-invariant tensors $(s_\alpha) \subseteq L^{\otimes}$.*

(1) *If $\mathfrak{M} = \mathfrak{M}(L)$, then $(\tilde{s}_\alpha) \subseteq \mathfrak{M}^{\otimes}$ defines a reductive subgroup of $\mathrm{GL}(\mathfrak{M})$.*

(2) *If k is separably closed, then there is an $\mathfrak{S}$-linear isomorphism*

$$\mathfrak{M} \xrightarrow{\sim} L \otimes_{\mathbb{Z}_p} \mathfrak{S}$$

which takes the tensor $\tilde{s}_\alpha$ to s_α. In particular, the subgroup $G_{\mathfrak{S}} \subseteq \mathrm{GL}(\mathfrak{M})$ defined by $(\tilde{s}_\alpha)$ is isomorphic to $G \times_{\mathbb{Z}_p} \mathfrak{S}$.

Proof. Using Theorem 2.3(3), it suffices to prove the theorem while assuming $k = k^{\mathrm{sep}}$. Moreover, the second statement implies the first. Set $\mathfrak{M}' = L \otimes_{\mathbb{Z}_p} \mathfrak{S}$, which induces the collection $(s_\alpha) \subseteq \mathfrak{M}'^{\otimes}$. Also set

$$P = \underline{\mathrm{Isom}}_{\mathfrak{S}}((\mathfrak{M}, (\tilde{s}_\alpha)), (\mathfrak{M}', (s_\alpha))).$$

Then the fibers of P are either empty or a torsor under G.

Claim. P is a G-torsor, i.e. P is flat over $\mathfrak{S}$ with non-empty fibers.

The claim implies the proposition since a torsor under a reductive group is étale locally trivial, while the ring $\mathfrak{S}$ is strictly Henselian as k is separably closed, so any G torsor over $\mathfrak{S}$ is trivial.

Step I. $P_{\mathfrak{S}_{(p)}}$ is a G-torsor. Since $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is faithfully flat over $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}$ is faithfully flat over $\mathfrak{S}_{(p)}$, it suffices to show that $P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a G-torsor. However the isomorphism in Theorem 2.3(2) shows that $P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a trivial G-torsor.

Step II. P_{K_0} is a G-torsor, where we regard K_0 as a $\mathfrak{S}$-algebra via $u \mapsto 0$. This follows from Theorem 2.3(1), which implies the existence of a canonical isomorphism

$$B_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_W \mathfrak{M}/u\mathfrak{M}.$$

Step III. $P_{\mathfrak{S}[1/pu]}$ is a G-torsor. Let $U \subseteq \operatorname{Spec} \mathfrak{S}[1/up]$ denote the maximal open subset over which P is flat with non-empty fibres. By Step I, we know this subset is non-empty, since it contains the generic point. In particular, the complement of U in Spec $\mathfrak{S}[1/up]$ contains finitely many closed points.

Let $x \in \operatorname{Spec} \mathfrak{S}[1/up]$ be a closed point. If $x \notin U$, we consider two cases. If $|u(x)| < |\pi|$, then since the s_α are Frobenius invariant, we have $P_{\mathfrak{S}[1/p]} \xrightarrow{\sim} \varphi^*(P_{\mathfrak{S}[1/p]})$ in a formal neighborhood of x. Hence $P_{\mathfrak{S}}[1/p]$ cannot be a G-torsor at $\varphi(x)$, since φ is a faithfully flat map on $\mathfrak{S}$. Repeating the argument we find $\varphi(x), \varphi^2(x), \ldots \notin U$, which gives a contradiction. Similarly, if $|u(x)| \geqslant |\pi|$, consider a sequence of points $x_0, x_1, \ldots$ with $x_0 = x$, and $\varphi(x_{i+1}) = x_i$. For $i \geqslant 1$, we have $P_{\mathfrak{S}[1/p]} \xrightarrow{\sim} \varphi^*(P_{\mathfrak{S}[1/p]})$ in a formal neighborhood of x_i, so we find that $x_i \notin U$ for $i \geqslant 1$.

Step IV. $P_{\mathfrak{S}[1/p]}$ is a G-torsor. By Step III, it suffices to show that the restriction of P to $K_0[\![u]\!]$ is a G-torsor. For any $\mathfrak{N}$ in $\mathsf{Mod}_{\mathfrak{S}}^{\varphi}$ there is a unique φ-equivariant isomorphism

$$\mathfrak{N} \otimes_{\mathfrak{S}} K_0[\![u]\!] \xrightarrow{\sim} K_0[\![u]\!] \otimes_{K_0} \mathfrak{N}/u\mathfrak{N}[1/p]$$

lifting the identity map on $\mathfrak{N}/u\mathfrak{N} \otimes \mathcal{O}_{K_0} K_0$, which is functorial in $\mathfrak{N}$ (see, for example, [Kis06, 1.2.6]). Applying this to $\mathfrak{M}$ and the morphisms $\tilde{s}_\alpha$ shows that the restriction of P to $K_0[\![u]\!]$ is isomorphic to $P_{K_0} \otimes_{K_0} K_0[\![u]\!]$, which is a G-torsor by Step II.

Step V. P is a G-torsor. Let U be the complement of the closed point in Spec $\mathfrak{S}$. By Steps I and IV we know that $P|_U$ is a G-torsor. By a result of Colliot-Thélène and Sansuc [CS79, Thm. 6.13], P extends to a G-torsor over $\mathfrak{S}$ and, as we remarked above, any such torsor is trivial. Hence $P|_U$ is trivial, and there is an isomorphism $\mathfrak{M}|_U \xrightarrow{\sim} \mathfrak{M}'|_U$ taking $\tilde{s}_\alpha$ to s_α. Since any vector bundle over U has a canonical extension to $\mathfrak{S}$, obtained by taking its global sections, this isomorphism extends to $\mathfrak{S}$. This implies that P is the trivial G-torsor and completes the proof of the proposition. □

Corollary 2.5. *With the assumptions of 2.4, suppose that G is connected and k is finite. Then there exists an isomorphism $\mathfrak{M} \xrightarrow{\sim} L \otimes_{\mathbb{Z}_p} \mathfrak{S}$ which takes the tensor $\tilde{s}_\alpha$ to s_α. In particular, the subgroup $G_{\mathfrak{S}} \subseteq \mathrm{GL}(\mathfrak{M})$ defined by $(\tilde{s}_\alpha)$ is isomorphic to $G \times_{\mathrm{Spec}\mathbb{Z}_p} \operatorname{Spec} \mathfrak{S}$.*

Proof. As in Theorem 2.4 we set $\mathfrak{M}' = L \otimes_{\mathbb{Z}_p} \mathfrak{M}$, and we denote by $P \subseteq \underline{\mathrm{Hom}}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{M}')$ the subscheme of isomorphisms between $\mathfrak{M}$ and $\mathfrak{M}'$ which take $\tilde{s}_\alpha$ to s_α. Then P is a G-torsor by 2.4. Since G is connected and k is finite, any such torsor is trivial [Sp79, 4.4], and the corollary follows. □

Corollary 2.6. *Let L be a G_K-stable lattice in a crystalline representation V, $\mathfrak{M} = \mathfrak{M}(L)$ and $(s_\alpha) \subseteq L^\otimes$ a collection of G_K-invariant tensors which define a reductive subgroup G of $\mathrm{GL}(L)$. Then we have the following.*

(1) *If we view $(s_\alpha) \subseteq \mathrm{Fil}^0 D_{\mathrm{cris}}(V)^\otimes$ via the p-adic comparison isomorphism*
$$B_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} B_{\mathrm{cris}} \otimes_{\mathcal{O}_{K_0}} D_{\mathrm{cris}}(V),$$
then $(s_\alpha) \subseteq (\mathfrak{M}/u\mathfrak{M})^\otimes \subseteq D_{\mathrm{cris}}(V)^\otimes$.

(2) *If k^{sep} denotes a separable closure of k, then there exists a $W(k^{\mathrm{sep}})$-linear isomorphism*
$$L \otimes_{\mathbb{Z}_p} W(k^{\mathrm{sep}}) \xrightarrow{\sim} \mathfrak{M}/u\mathfrak{M} \otimes_{W(k)} W(k^{\mathrm{sep}})$$
taking s_α to s_α. In particular, $(s_\alpha) \subseteq (\mathfrak{M}/u\mathfrak{M})^\otimes$ defines a reductive subgroup G' of $\mathrm{GL}(\mathfrak{M}/u\mathfrak{M})$, which is a pure inner form of G.

(3) *If k is finite and G is connected, then there exists a W-linear isomorphism*
$$L \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathfrak{M}/u\mathfrak{M}$$
taking s_α to s_α. In particular, $(s_\alpha) \subseteq (\mathfrak{M}/u\mathfrak{M})^\otimes$ defines a reductive subgroup $G' \subseteq \mathrm{GL}(\mathfrak{M}/u\mathfrak{M})$, which is isomorphic to $G \times_{\mathbb{Z}_p} W$.

Proof. (1) and (2) follow from 2.4; in fact (1) holds for any G_K-invariant tensors, without assuming that G is reductive. To see that G' is a pure inner form of G in (2), note that specializing the torsor P which appears in the proof of 2.4 at $u = 0$ gives a class in $H^1(\mathrm{Spec}\, W, G)$, and G' can be obtained from G by twisting by this class.

Finally, (3) follows from Corollary 2.5 once we remark that $s_\alpha \in D_{\mathrm{cris}}(V)^\otimes$ is equal to
$$\tilde{s}_\alpha|_{u=0} \colon \mathbb{1} \longrightarrow (\mathfrak{M}/u\mathfrak{M})^\otimes \hookrightarrow D_{\mathrm{cris}}(V)^\otimes,$$
the final inclusion being given by the first isomorphism of Theorem 2.3(1). The equality is a formal consequence of the functoriality of this isomorphism. □

Corollary 2.7. *Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$, and if $p = 2$ assume that $\mathscr{G}^*$ is connected. Let $L = T_p\mathscr{G}^*$, $\mathfrak{M} = \mathfrak{M}(L) = \mathfrak{M}(\mathscr{G})$, and $(s_\alpha) \subseteq L^\otimes$ be a collection of G_K-invariant tensors defining a reductive subgroup $G \subseteq \mathrm{GL}(L)$. Then*

(1) *There is a canonical φ-equivariant isomorphism $\varphi^*(\mathfrak{M}/u\mathfrak{M}) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_0)(W)$, where $\mathscr{G}_0 = \mathscr{G} \otimes_{\mathcal{O}_K} k$.*

(2) *There exists a $W(k^{\mathrm{sep}})$-linear isomorphism*
$$L \otimes_{\mathbb{Z}_p} W(k^{\mathrm{sep}}) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_0)(W) \otimes_W W(k^{\mathrm{sep}})$$
taking s_α to $\varphi^*(s_\alpha) \in \mathbb{D}(\mathscr{G}_0)(W)^\otimes$. In particular, $(\varphi^*(s_\alpha)) \subseteq \mathbb{D}(\mathscr{G}_0)(W)^\otimes$ defines a reductive subgroup $G_W \subseteq \mathrm{GL}(\mathbb{D}(\mathscr{G}_0)(W))$ which is an inner form of G.

(3) *If G is connected and k is finite, then there exists a W-linear isomorphism*
$$L \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_0)(W)$$
taking s_α to $\varphi^*(s_\alpha) \in \mathbb{D}(\mathscr{G}_0)(W)^\otimes$. In particular, $(\varphi^*(s_\alpha)) \subseteq \mathbb{D}(\mathscr{G}_0)(W)^\otimes$ defines a reductive subgroup $G_W \subseteq \mathrm{GL}(\mathbb{D}(\mathscr{G}_0)(W))$ which is isomorphic to $G \times_{\mathbb{Z}_p} W$.

(4) *The filtration $\mathrm{Fil}^1\mathbb{D}(\mathscr{G}_0)(k) \subseteq \mathbb{D}(\mathscr{G}_0)(k)$ is given by a cocharacter*
$$\mu_0 \colon \mathbb{G}_m \longrightarrow G_W \otimes_W k.$$

3. Deformation theory

Let k be a perfect field of characteristic p. Let $\mathcal{G}_0$ be a p-divisible group over k. Take $M_0 = \mathbb{D}(\mathcal{G}_0)(W)$ with $W = W(k)$ the Witt ring. Fix a cocharacter $\mu\colon \mathbb{G}_m \to \mathrm{GL}(M_0)$ such that $\mu_0 \equiv \mu \bmod p$ gives rise to the Hodge filtration on $\mathbb{D}(\mathcal{G}_0)(k) = M_0 \otimes_W k$. According to the Grothendieck–Messing deformation theory, we have $\mathcal{G}$ a p-divisible group over W that lifts $\mathcal{G}_0$.

Let $U^\circ \subseteq \mathrm{GL}(M_0)$ be the opposite unipotent deformation defined by μ. Let R be the complete local ring at the identity of U°. Then

$$R \cong W[\![t_1, \ldots, t_n]\!], \quad n = \dim_W U^\circ,$$

equipped with a Frobenius action $\varphi\colon t_i \mapsto t_i^p$ for $1 \leqslant i \leqslant n$. Put $M \coloneqq M_0 \otimes_W R$ and there is a filtration on M, written the first piece as

$$\mathrm{Fil}^1 M = (\mathrm{Fil}^1 M_0) \otimes_W R.$$

Also, for each tautological R-point $u \in U^\circ(R)$, the composition

$$\Phi\colon M = M_0 \otimes_W R \xrightarrow{\varphi\otimes\varphi} M \xrightarrow{\ u\ } M$$

is semi-linear. The work of Faltings shows that there is a p-divisible group $\mathcal{G}_R$ over R such that

$$\mathcal{G}_R \otimes_R (R/(t_1, \ldots, t_n)) \simeq \mathcal{G}$$

and $\mathcal{G}_R$ is a versal deformation of $\mathcal{G}_0$. Moreover, there is an isomorphism

$$\mathbb{D}(\mathcal{G}_R)(R) \simeq M$$

which is compatible with the actions of Frobenii and filtrations. Whenever R is formally smooth, there exists an integral connection

$$\nabla\colon M \longrightarrow M \otimes \Omega^1_R$$

such that $\varphi^* M \to M$ is parallel.

Let $G_W \subseteq \mathrm{GL}(M_0)$ be a connected reductive group defined by a finite collection of φ-invariant tensors $(s_\alpha) \subseteq M_0^\otimes$, such that the Hodge filtration on $\mathbb{D}(\mathcal{G}_0)(k)$ is $G_W \otimes_W k$-split. Then we may take $\mu\colon \mathbb{G}_m \to G_W$ lifting μ_0. Denote $U_G^0 \subseteq G_W = G$ the opposite unipotent deformation given by μ. Then R_G is a complete local ring at the identity of U_G^0. We may choose the t_i such that

$$R_G \simeq R/(t_{r+1}, \ldots, t_n) = W[\![t_1, \ldots, t_r]\!], \quad r = \mathrm{rank}_W(\mathcal{G}/\mathrm{Fil}^0\mathcal{G}),$$

where $\mathcal{G} = \mathrm{Lie}(G)$. Take a totally ramified extension K over $K_0 = W[1/p]$.

Proposition 3.1. *Suppose that $p > 2$ or $\mathcal{G}_0^*$ is connected. Let $\varpi\colon R \to \mathcal{O}_K$ be a map of W-algebras and $\mathcal{G}_\varpi$ the induced p-divisible group over $\mathcal{O}_K$. Then ϖ factors through R_G if and only if $\mathcal{G}_\varpi$ is G_W-adapted, i.e., there is a collection of φ-invariants, say $(\tilde{s}_\alpha) \subseteq \mathbb{D}(\mathcal{G}_\varpi)(S)^\otimes$, lifting $(s_\alpha) \subseteq \mathbb{D}(\mathcal{G}_0)(W)^\otimes$, such that*

1. *If $s_{\alpha,\mathcal{O}_K}$ denotes $\tilde{s}_\alpha$ in $\mathbb{D}(\mathcal{G}_\varpi)(\mathcal{O}_K)^\otimes$, then*
$$(s_{\alpha,\mathcal{O}_K}) \subseteq \mathrm{Fil}^0(\mathbb{D}(\mathcal{G}_\varpi)(\mathcal{O}_K)^\otimes).$$
2. *The collection $(\tilde{s}_\alpha)$ deforms a reductive group $G_S \subseteq \mathrm{GL}(\mathbb{D}(\mathcal{G}_\varpi)(S))$.*

Proof. We first prove the "only if" part. If $\varpi\colon R_G \to \mathcal{O}_K$ to a map $\tilde{\varpi}\colon R_G \to S$. Set $\tilde{s}_\alpha = \tilde{\varpi}(s_\alpha \otimes 1)$. Then $\tilde{s}_\alpha$ satisfy conditions (1) and (2). We only need to check that $\tilde{\varpi}(s_\alpha \otimes 1)$ are φ-invariant. For this, take

$$M_S \coloneqq \mathbb{D}(\mathcal{G}_\varpi)(S) = M_{R_G} \otimes S$$

with the Frobenius action inherited. Then

$$\varphi_S^*(M_S) = \varphi^*\tilde{\varpi}^* M_{R_G} \xrightarrow[\varepsilon]{\sim} \varphi_{R_G}^*\tilde{\varpi}^* M_{R_G} \xrightarrow{\tilde{\varpi}^*(\varphi\otimes 1)} \tilde{\varpi}^* M_{R_G}.$$

Since each s_α is φ-invariant, we deduce

$$\tilde{\varpi}^*(\varphi \otimes 1) \circ \varepsilon(\tilde{s}_\alpha) = \tilde{\varpi}^*(\varphi \otimes 1)(s_\alpha \otimes 1) = \tilde{s}_\alpha.$$

Conversely, we prove the "if" part. Suppose we obtain $(\tilde{s}_\alpha)$ that satisfies (1) and (2). Let $\varpi_0\colon R \to W$ be the natural projection that gives $\varpi \times \varpi_0\colon R \to \mathcal{O}_K \times_k W$. Denote by $\mathcal{G}_{\varpi\times\varpi_0}$ the p-divisible group over $\mathcal{O}_K \times_k W$ induced by it.

Assume first that $p > 2$. Then the surjective map $W[u] \to \mathcal{O}_K \times_k W$ sending u to $(\pi, 0)$ induces a map $\widehat{S} \to \mathcal{O}_K \times_k W$. Let $G_{\widehat{S}} = G_S \otimes_S \widehat{S}$. It turns out there is a $G_{\widehat{S}}$-split filtration on $\mathbb{D}(\mathscr{G}_\varpi(\widehat{S}))$ which simultaneously lifts the filtration on $\mathbb{D}(\mathscr{G}_\varpi(\mathcal{O}_K))$ and the chosen filtration on $\mathbb{D}(\mathscr{G})(W)$. Since the kernel of $\widehat{S} \to \mathcal{O}_K \times_k W$ is equipped with topologically nilpotent divided powers, such a filtration corresponds to a p divisible group $\mathscr{G}_{\tilde{\varpi}}$ over $\widehat{S}$, deforming $\mathscr{G}_{\varpi\times\varpi_0}$. Since R is a versal deformation ring for $\mathscr{G}_0, \mathscr{G}_{\tilde{\varpi}}$ is induced by a map $\tilde{\varpi}\colon R \to \widehat{S}$ lifting $\varpi \times \varpi_0$.

We may identify

$$\mathbb{D}(\mathscr{G}_{\tilde{\varpi}})(\widehat{S}) = \mathbb{D}(\mathscr{G}_\varpi)(\widehat{S}) = \mathbb{D}(\mathscr{G}_\varpi(S)) \otimes_S \widehat{S}$$

with $M_{\widehat{S}} := M_R \otimes_R \widehat{S} = M_0 \otimes_W \widehat{S}$, and we view $\tilde{s}_\alpha$ as elements of $M_{\widehat{S}}^\otimes$. Consider the composite

$$\varphi^*(M_{\widehat{S}}) \xrightarrow[\varepsilon]{\sim} \tilde{\varpi}^*\varphi^*(M_R) \xrightarrow{\tilde{\varpi}^*(\varphi\otimes 1)} \tilde{\varpi}^*(M_R) = M_{\widehat{S}}.$$

The map $\theta\colon M_0 \to M_{\widehat{S}} = M_0 \otimes_W \widehat{S}$ is induced by an element of $U^\circ(\widehat{S}[1/p])$. Hence, viewing $\tilde{s}_\alpha$ and $s_\alpha \otimes 1$ in $(M_{\widehat{S}} \otimes_{\widehat{S}} K_0[\![u]\!])^\otimes$, and applying [Kis10, 1.5.6], we find that $\tilde{s}_\alpha = s_\alpha \otimes 1$ and that θ is induced by a point of $U_G^\circ(K_0[\![u]\!]) \cap U^\circ(\widehat{S}[1/p]) = U_G^\circ(\widehat{S}[1/p])$. In particular, each of the two maps in [Kis10, 1.5.10] sends $s_\alpha \otimes 1$ to $s_\alpha \otimes 1$. For ε this holds as $\nabla_{\widehat{S}}(s_\alpha \otimes 1) = \nabla_{\widehat{S}}(\tilde{s}_\alpha) = 0$, while $\tilde{\varpi}^*(\varphi \otimes 1) \circ \varepsilon$ has this property since $\tilde{s}_\alpha$ is φ-invariant. It follows that

$$\varpi^*(\varphi \otimes 1)\colon M_0 \xrightarrow{m \mapsto m\otimes 1} \tilde{\varpi}^*\varphi^*(M_R) \longrightarrow \tilde{\varpi}^* M_R = M_0 \otimes_W \widehat{S}$$

has the form $m \mapsto A\varphi(m)$ for some $A \in U_G^\circ(\widehat{S})$. This means that $\tilde{\varpi}$ factors through R_G, and hence so does ϖ.

Finally suppose that $\mathscr{G}_0^*$ is connected. Then using results of Zink, we can repeat the above argument with S in place of $\widehat{S}$, even when $p = 2$: Consider the map $S \to \mathcal{O}_K \times_k W$ sending u to $(\pi, 0)$, and choose a G_S-split filtration on $\mathbb{D}(\mathscr{G}_\varpi)(S)$ which lifts the filtrations on $\mathbb{D}(\mathscr{G})(W)$ and $\mathbb{D}(\mathscr{G}_\varpi)(\mathcal{O}_K)$. In the terminology of [Zi01] this filtration gives $\mathbb{D}(\mathscr{G}_\varpi)(S)$ the structure of an S-window over S, and hence gives rise to a p-divisible group $\mathscr{G}_{\tilde{\varpi}}$ over S which deforms $\mathscr{G}_{\varpi\times\varpi_0}$. By [Zi02, Corollary 97] the canonical isomorphism $\mathbb{D}(\mathscr{G}_{\tilde{\varpi}})(S) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_\varpi)(S)$ respects filtrations. The rest of the argument is as in the case $p > 2$. □

Corollary 3.2. *Suppose $p > 2$ or $\mathcal{G}_0^*$ is connected. Let K'/K be a finite extension and $\varpi\colon R \to \mathcal{O}_{K'}$ a map of W-algebras inducing a p-divisible group $\mathcal{G}_\varpi$ over $\mathcal{O}_{K'}$. Let $L = T_p\mathcal{G}_\varpi^*(-1)$, and $(s_{\alpha,\mathrm{et}}) \subseteq L^\otimes$ a family of $G_{K'}$-invariant tensors defining a reductive subgroup of $\mathrm{GL}(L)$, such that under the p-adic comparison isomorphism*

$$L \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \xrightarrow{\sim} M_0 \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}},$$

$s_{\alpha,\mathrm{et}}$ maps to $s_\alpha \in M_0^\otimes$. Then ϖ factors through R_G.

4. Integral canonical models for Shimura varieties of Hodge type

We first introduce the Shimura datum (G, X). Let G be a reductive group over $\mathbb{Q}$ and X a conjugacy class of maps of algebraic groups over $\mathbb{R}$, read as

$$h\colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \longrightarrow G_\mathbb{R}.$$

On $\mathbb{R}$-points, such a map induces a map of real groups $\mathbb{C}^\times \to G(\mathbb{R})$. We require that (G, X) satisfy the following conditions:

(1) For $\mathfrak{g} = \mathrm{Lie}\, G_\mathbb{R}$, the composite

$$\mathbb{S} \longrightarrow G_\mathbb{R} \longrightarrow G_\mathbb{R}^{\mathrm{ad}} \longrightarrow \mathrm{GL}(\mathfrak{g})$$

defines a Hodge structure of type $(-1, 1), (0, 0), (1, -1)$.

(2) $h(i)$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.
(3) G^{ad} has no factors whose real points form a compact group.

Let $K = K_p K^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. This leads to an algebraic variety $\mathbf{Sh}_K(G, X)$ over the reflex field $E = E(G, X)$. Then a theorem of Baily–Borel asserts that

$$\mathbf{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

Lemma 4.1. *Let $i\colon (G_1, X_1) \hookrightarrow (G_2, X_2)$ be an embedding of Shimura data and $K_{2,p} \subseteq G_2(\mathbb{Q}_p)$ be an open compact subgroup. Let $K_{1,p} := K_{2,p} \cap G_1(\mathbb{Q}_p)$, with $K_1 = K_{1,p}K^{1,p} \subseteq G_1(\mathbb{A}^p_f)$. Then there exists a compact open subgroup $K_2 = K_{2,p}K^{2,p} \subseteq G_2(\mathbb{A}_f)$ with $K_1 \subseteq K_2$, such that i induces an embedding*

$$\mathbf{Sh}_{K_1}(G_1, X_1) \hookrightarrow \mathbf{Sh}_{K_2}(G_2, X_2).$$

Fix a finite-dimensional $\mathbb{Q}$-vector space V and $\psi\colon V \times V \to \mathbb{Q}$ a perfect alternating form. Take $G = \mathrm{GSp}(V, \psi)$ and $X = S^{\pm}$ the Siegel double space. From these, we obtain $\mathbf{Sh}_K(G, X)$ over $E = \mathbb{Q}$, a moduli space of polarized abelian varieties, where (G, X) is a Shimura datum of Hodge type, i.e., there exists an embedding $i\colon (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$. Fix compact open subgroups $K \subseteq G(\mathbb{A}_f)$ and $K' \subseteq \mathrm{GSp}(\mathbb{A}_f)$, such that $K \subseteq K'$. Also, i induces a morphism

$$\mathbf{Sh}_K(G, X) \longrightarrow \mathbf{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$$

of algebraic varieties over $E = E(G, X)$. Let $(s_{\alpha,B}) \subseteq V^{\otimes}$ be a finite collection of tensors defining $G \subseteq \mathrm{GSp}(V, \psi) \subseteq \mathrm{GL}(V)$. Let $f\colon \mathcal{A} \to \mathbf{Sh}_K(G, X)$ be a pullback of the universal abelian scheme. Denote

$$\mathcal{V}_B := R^1 f_{\mathbb{C},*}\underline{\mathbb{Q}}, \qquad \mathcal{V}_{\mathrm{dR},\mathbb{C}} = R^1 f_{\mathbb{C},*}\Omega^{\bullet}_{\mathcal{A}/\mathbf{Sh}_K(G,X)}.$$

We choose collections $(s_{\alpha,B}) \subseteq \mathcal{V}_B^{\otimes}$ and $(s_{\alpha,\mathrm{dR}}) \subseteq \mathcal{V}^{\otimes}_{\mathrm{dR},\mathbb{C}}$. Now let $\kappa \supset E$ be a field of characteristic 0, and $\overline{\kappa}$ an algebraic closure of κ. Fix an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ and an embedding of E-algebras $\sigma\colon \overline{\kappa} \hookrightarrow \mathbb{C}$. Let $x \in \mathbf{Sh}_K(G, X)(\kappa)$ and denote by $\mathcal{A}_x$ the corresponding abelian variety over κ. Denote by $H^1_B(\mathcal{A}_x(\mathbb{C}), \mathbb{Q})$ the Betti cohomology of $\mathcal{A}_x(\mathbb{C})$. Write $H^1_{\mathrm{dR}}(\mathcal{A}_x)$ for its de Rham cohomology and $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\kappa}}) = H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\kappa}}, \mathbb{Q}_p)$ for the p-adic étale cohomology of $\mathcal{A}_{x,\overline{\kappa}} = \mathcal{A}_x \otimes_\kappa \overline{\kappa}$. The embedding σ induces isomorphisms

$$H^1_{\mathrm{dR}}(\mathcal{A}_x) \otimes_{\kappa,\sigma} \mathbb{C} \xrightarrow{\sim} H^1_B(\mathcal{A}_x(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(\mathcal{A}_{x,\overline{\kappa}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}.$$

Let $s_{\alpha,B,x}$ be the fibre of $s_{\alpha,B}$ at x (regarded as a $\mathbb{C}$-valued point via σ), and denote by $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes} \otimes_{\kappa,\sigma} \mathbb{C}$ and $s_{\alpha,\mathrm{et},x} \in H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\kappa}})^{\otimes}$ the images of $s_{\alpha,B,x}$ under these two isomorphisms.

Lemma 4.2. *The action of* $\mathrm{Gal}(\overline{\kappa}/\kappa)$ *on* $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\kappa}}, \mathbb{Q}_p)$ *fixes each* $s_{\alpha,\mathrm{et},x}$ *and factors through* $G(\mathbb{Q}_p)$*. Moreover we have* $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes}$.

Proof. Let $\mathbf{Sh}_{K^p}(G, X) = \lim_{H_p} \mathbf{Sh}_{H_pK^p}(G, X)$, where H_p runs over compact open subgroups of K_p, and similarly for $\mathbf{Sh}_{K'^p}(\mathrm{GSp}, S^{\pm})$.

The action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\kappa}}, \mathbb{Q}_p)$ is induced by the map $\mathrm{Gal}(\overline{\kappa}/\kappa) \to K'_p$, obtained by pulling back to $\overline{\kappa}$ the $K_{p'}$-torsor $\mathbf{Sh}_{K'^p}(\mathrm{GSp}, S^{\pm}) \to \mathbf{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$. On the other hand, we have a commutative, K_p-equivariant diagram

$$\begin{array}{ccc} \mathbf{Sh}_{K^p}(G, X) & \longrightarrow & \mathbf{Sh}_{K'^p}(\mathrm{GSp}, S^{\pm}) \\ \downarrow & & \downarrow \\ \mathbf{Sh}_K(G, X) & \longrightarrow & \mathbf{Sh}_{K'}(\mathrm{GSp}, S^{\pm}) \end{array}$$

which shows that the restriction of $\mathbf{Sh}_{K'^p}(\mathrm{GSp}, S^{\pm})$to $\mathbf{Sh}_K(G, X)$ descends to a K_p-torsor. This shows that the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $H^1_{\text{ét}}$ $(\mathcal{A}_{x,\overline{\kappa}}, \mathbb{Q}_p)$ is induced by a map $\mathrm{Gal}(\overline{\kappa}/\kappa) \to K_p \subseteq G(\mathbb{Q}_p)$. In particular this action fixes each $s_{\alpha,\mathrm{et},x}$.

To see the final statement note that, by a result of Deligne [De82, 2.11], the Hodge cycle $(s_{\alpha,\mathrm{dR},x}, s_{\alpha,\mathrm{et},x})$ is an absolute Hodge cycle, for each α. In particular, this implies [De82, 2.7] that $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes} \otimes_\kappa \overline{\kappa}$. Moreover, since an absolute Hodge cycle is determined by either its de Rham or étale component, $\mathrm{Gal}(\overline{\kappa}/\kappa)$ fixes $s_{\alpha,\mathrm{dR},x}$ as it fixes $s_{\alpha,\mathrm{et},x}$. Hence $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes}$. □

Now we come to the construction of integral models. Let $i\colon (G,X) \hookrightarrow (\mathrm{GSp}(V,\psi), S^{\pm})$ as before. Assume G is unramified over $\mathbb{Q}_p$, i.e. there exists a reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ such that $G_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = G_{\mathbb{Q}_p}$. Let $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ and $K = K_pK^p$, where $K^p \subseteq G(\mathbb{A}_f^p)$ is an open compact subgroup. The goal now is to find a smooth integral canonical model $\mathscr{S}_K(G,X)$ over $\mathcal{O}_{(v)}$ for some place $v \mid p$ of $\mathcal{O} \subseteq E(G,X)$. We will need the following.

Lemma 4.3. *Let W be a $\mathbb{Q}_p$-vector space and $i\colon G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(W)$ a closed embedding of algebraic groups. If $p = 2$, assume that $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ has no factors of type B.*[1] *Suppose that $G_{\mathbb{Z}_p}$ is a reductive group over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$. Then there exists a $\mathbb{Z}_p$-lattice $W_{\mathbb{Z}_p}$ in W such that i is induced by a closed imbedding*

$$i_{\mathbb{Z}_p}\colon G_{\mathbb{Z}_p} \longrightarrow \mathrm{GL}(W_{\mathbb{Z}_p}).$$

Proof. Denote $\mathbb{Z}_p^{\mathrm{ur}}$ a strict henselization of $\mathbb{Z}_p$, and write $\mathbb{Q}_p^{\mathrm{ur}} = \mathbb{Z}_p^{\mathrm{ur}}[1/p]$. Write $W^{\mathrm{ur}} = W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}}$ and $G_{\mathbb{Z}_p^{\mathrm{ur}}} = G_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}}$. Then $G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$ is a bounded subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p^{\mathrm{ur}})$ in the sense that any regular function on $G_{\mathbb{Z}_p^{\mathrm{ur}}}$ is bounded on $G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$. Let L be any $\mathbb{Z}_p^{\mathrm{ur}}$-lattice in W^{ur}. The boundedness implies that $\bigcup_{g \in G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})} g \cdot L$ is a $\mathbb{Z}_p^{\mathrm{ur}}$-lattice in W^{ur}. Hence

$$W_{\mathbb{Z}_p^{\mathrm{ur}}} = \sum_{\gamma \in G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}}) \rtimes \Gamma} \gamma \cdot L$$

is a $\mathbb{Z}_p^{\mathrm{ur}}$-lattice in W^{ur}, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. Then it is equipped with a natural $G_{\mathbb{Z}_p^{\mathrm{ur}}}$-action, which induces $i_{\mathbb{Z}_p^{\mathrm{ur}}}\colon G_{\mathbb{Z}_p^{\mathrm{ur}}} \to \mathrm{GL}(W_{\mathbb{Z}_p^{\mathrm{ur}}})$. Since $W_{\mathbb{Z}_p^{\mathrm{ur}}}$ is Γ-stable, $i_{\mathbb{Z}_p^{\mathrm{ur}}}$ arises from a $\mathbb{Z}_p$-lattice $W_{\mathbb{Z}_p}$ of W by étale descent. The map $i_{\mathbb{Z}_p^{\mathrm{ur}}}$ is compatible with the descent data on the source and target, as this can be checked on generic fibers, so it descends to a map $i_{\mathbb{Z}_p}\colon G_{\mathbb{Z}_p} \to \mathrm{GL}(W_{\mathbb{Z}_p})$. Finally, $i_{\mathbb{Z}_p}$ is a closed embedding by Prasad–Yu [PY06, 1.3]. □

Remark 4.4. If $p = 2$, Kisin assumed that $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ has no factors of type B. For a Shimura datum (G,X) of Hodge type, by Deligne's classification, factors of type B of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ have simply connected derived subgroup, for which Prasad–Yu [PY06, 1.3] applies successfully.

Now by Lemma 4.3, there is a lattice $V_{\mathbb{Z}}$ of V such that $i_{\mathbb{Q}_p}$ is induced by an embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p})$. Fix such a choice of $V_{\mathbb{Z}}$. Since $G_{\mathbb{Z}_p}$ has generic fiber $G \otimes_{\mathbb{Q}} \mathbb{Q}_p$, flat base change implies that the closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is a reductive subgroup $G_{\mathbb{Z}_{(p)}}$ such that $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = G_{\mathbb{Z}_p}$.

Let $(s_\alpha) \subseteq V^{\otimes}_{\mathbb{Z}_{(p)}}$ be a finite collection of tensors defining $G_{\mathbb{Z}_{(p)}} \subseteq \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. Let $K'_p \subseteq \mathrm{GSp}(\mathbb{Q}_p)$ be the stabilizer of $V_{\mathbb{Z}_p}$, which is a maximal compact subgroup of $\mathrm{GSp}(\mathbb{Q}_p)$ (but is not hyperspecial in general). By Lemma 4.1 we may choose $K' = K'_pK'^p$ so that i induces an embedding

$$\mathbf{Sh}_K(G,X) \longrightarrow \mathbf{Sh}_{K'}(\mathrm{GSp}, S^{\pm}).$$

We may assume that ψ induces an inclusion $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^*$ into the dual lattice $V_{\mathbb{Z}}^* \subseteq V_{\mathbb{Q}}$. Let $d = |V_{\mathbb{Z}}^*/V_{\mathbb{Z}}|$ and write $2g = \dim_{\mathbb{Q}} V$. We attain an embedding $\mathbf{Sh}_K(\mathrm{GSp}, S^{\pm}) \hookrightarrow \mathcal{A}_{g,d,K'}$ where the target is the moduli space over $\mathbb{Q}$ of abelian varieties with a polarization of degree d and a K'^p-level structure. It has a natural integral model, and we get an embedding of $\mathbb{Z}_{(p)}$-schemes, read as

$$\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm}) \longrightarrow \mathcal{A}_{g,d,K'}.$$

By the theory of moduli spaces of Mumford, for any $\mathbb{Z}_{(p)}$-scheme T,

$$\mathcal{A}_{g,d,K'}(T) = \{(A, \lambda, \varepsilon^p_{K'})\}/\sim,$$

[1] This restriction, which arises from the necessary restriction in the result of Prasad–Yu [PY06, 1.3] used in the proof, is one of the reasons for the restrictions in our results when $p = 2$.

where

- A is an abelian scheme over T,
- $\lambda\colon A \to A^*$ is a polarization of degree d, and
- $\varepsilon^p_{K'} \in \Gamma(T, \underline{\mathrm{Isom}}(V_{\hat{\mathbb{Z}}^p}, \hat{V}^p(A))/K'^p)$, where $\hat{V}^p(A) = \varprojlim_{p\nmid n} A[n]$.

Denote by $\mathscr{S}^-_K(G,X)$ the closure of $\mathbf{Sh}_K(G,X)$ in $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})_{\mathcal{O}_{(v)}}$. From now on we make the following assumption when $p = 2$:

($\diamond$) If $p = 2$, then the abelian variety over any characteristic p point of $\mathscr{S}^-_K(G,X)$ has connected p-divisible group.

Proposition 4.5. *Let $x \in \mathscr{S}^-_K(G,X)$ be a closed point with residue field of characteristic p, and write $\hat{U}_x := \mathscr{S}^-_K(G,X)^\wedge_x$ for the completion of $\mathscr{S}^-_K(G,X)$ at x. Then the irreducible components of $\hat{U}_x$ are formally smooth over $\mathcal{O}_{(v)}$.*

Proof. Let $k = k(x)$ and $\mathcal{G}_0$ be the p-divisible group over k associated to x. Let F/E be a finite extension and $\tilde{x} \in \mathscr{S}^-_K(G,X)(F)$ a point specializing to x. Write $W = W(k)$ and take the $\mathrm{Gal}(\overline{E}/F)$-invariant tensors $s_{\alpha,\mathrm{et},\tilde{x}}$ (or $s_{\alpha,p,\tilde{x}}$. These tensors give rise to φ-invariant tensors $(s_{\alpha,0,\tilde{x}}) \subseteq \mathbb{D}(\mathcal{G}_0)(W)^{\otimes}$ which defines the reductive group $G_W \subseteq \mathrm{GL}(\mathbb{D}(\mathcal{G}_0)(W))$ such that the Hodge filtration on $\mathbb{D}(\mathcal{G}_0)(W) \otimes_W k$ is $G_W \otimes k$-split. Let R be the versal deformation ring of $\mathcal{G}_0$. From this we obtain a formally smooth quotient R_{G_W} of R.

Let $\hat{U}'_x = \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})^\wedge_x$ be the completion at x. Let $j\colon \hat{U}'_x \to \mathrm{Spf}\, R$ be the induced map defining the p-divisible group over $\hat{U}'_x$ which arises from the universal family of polarized abelian schemes over $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$. Then j is a closed embedding since a polarization on a deformation of $\mathcal{G}_0$ is determined by its restriction to $\mathcal{G}_0$.

We claim that the composite

$$Z \hookrightarrow \hat{U}_x \hookrightarrow \hat{U}'_x \hookrightarrow \mathrm{Spf}\, R$$

factors through $\mathrm{Spf}\, R_{G_W}$. Granting the claim, since Z and R_{G_W} have the same dimension over W, we have the isomorphism $Z \xrightarrow{\sim} \mathrm{Spf}\, R_{G_W}$. As $\tilde{x}$ was an arbitrary point of $\mathscr{S}^-_K(G,X)$ lifting x, this proves the proposition.

To prove the claim, by Corollary 3.2, it suffices to check that for any finite extension F'/F in $\overline{E}$ and $\tilde{x}' \in \mathbf{Sh}_K(G,X)(F')$ lying in $Z(F'_v)$, the tensor $s_{\alpha,\mathrm{et},\tilde{x}'}$ maps to $s_{\alpha,0,\tilde{x}}$ under the p-adic comparison theorem. A result of Blasius and Wintenberger [Bla94] asserts that under the p-adic comparison isomorphism,

$$I_{\mathrm{dR}}(s_{\alpha,\mathrm{et},\tilde{x}'}) = s_{\alpha,\mathrm{dR},\tilde{x}'}$$

So it suffices to check that the isomorphism

$$H^1_{\mathrm{cris}}(A_x/W) \otimes_{F'} F'_v \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_{\tilde{x}'}) \otimes_{F'} F'_v$$

takes $s_{\alpha,0}$ to $s_{\alpha,\mathrm{dR},\tilde{x}'}$. Equivalently, we are to check that the composite

$$I\colon H^1_{\mathrm{dR}}(A_{\tilde{x}'}) \otimes_{F'} F'_v \xrightarrow{\sim} H^1_{\mathrm{cris}}(A_x/W) \otimes_{F'} F'_v \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_{\tilde{x}'}) \otimes_{F'} F'_v$$

takes $s_{\alpha,\mathrm{dR},\tilde{x}}$ to $s_{\alpha,\mathrm{dR},\tilde{x}'}$. By Berthelot–Ogus [BO83, 2.9], I is given by parallel transport of Gauss–Manin connection. Since the generic fiber Z_η of Z is connected and $s_{\alpha,\mathrm{dR}}|_{Z_\eta}$ is parallel, we see $I(s_{\alpha,\mathrm{dR},\tilde{x}}) = s_{\alpha,\mathrm{dR},\tilde{x}'}$. This completes the proof. ☐

Let X be an $\mathcal{O}_{(v)}$-scheme. We say X has the *extension property* if for any regular, formally smooth $\mathcal{O}_{(v)}$-scheme S, a map $S \otimes E \to X$ extends to S.

Theorem 4.6. *For $K = K_pK^p$, let $\mathscr{S}_K(G,X)$ denote the normalization of $\mathscr{S}^-_K(G,X)$, and set*

$$\mathscr{S}_{K_p}(G,X) = \varprojlim_{K^p} \mathscr{S}_{K_pK^p}(G,X),$$

where $K^p \subseteq G(\mathbb{A}^p_f)$ runs over sufficiently small compact open subgroups of $G(\mathbb{A}^p_f)$. Then, under the assumption ($\diamond$),

(1) *$\mathscr{S}_{K_p}(G,X)$ is an inverse limit of smooth $\mathcal{O}_{(v)}$-schemes with finite étale transition maps, whose restriction to E may be $G(\mathbb{A}_f^p)$-equivariantly identified with $\mathbf{Sh}_{K_p}(G,X)$, i.e.*

$$\mathscr{S}_{K_p}(G,X)\otimes E\cong \mathbf{Sh}_{K_p}(G,X).$$

(2) *$\mathscr{S}_{K_p}(G,X)$ has the extension property, and in particular depends only on (G,X) and K_p, and noto on the symplectic embedding i.*

Proof. (1) follows directly from Proposition 4.5. For (2), suppose that S is regular and formally smooth over $\mathcal{O}_{(v)}$. A morphism $S\otimes E\to \mathscr{S}_{K_p'}(\mathrm{GSp},S^{\pm})$ can be extended to the height 1 primes by [Mil92, Prop 2.13] and then to all of S by a result of Faltings [Mo98, 3.6]. Hence a morphism $S\otimes E\to \mathbf{Sh}_{K_p}(G,X)$ extends to a map $S\to \mathscr{S}^{-}_{K_p}(G,X)$ and this map lifts to $\mathscr{S}_{K_p}(G,X)$ since S is formally smooth; equivalently, the following diagram commutes:

$$\begin{array}{ccc} S & \longrightarrow & \mathscr{S}^{-}_{K_p}(G,X) \\ & \searrow & \uparrow \\ & & \mathscr{S}_{K_p}(G,X). \end{array}$$

This completes the proof of (2). □

Corollary 4.7. *Let $\mathcal{V}^{\circ}_{\mathrm{dR}}=R^1f_*\Omega^{\bullet}_{\mathcal{A}/\mathscr{S}_{K_p}(G,X)}$ be the vector bundle on $\mathscr{S}_{K_p}(G,X)$ by pulling back the de Rham cohomology of the universal abelian scheme $\mathcal{A}$ over $\mathscr{S}_{K_p'}(\mathrm{GSp},S^{\pm})$. Then the section $s_{\alpha,\mathrm{dR}}\in\mathcal{V}^{\otimes}_{\mathrm{dR}}$ extends to $G(\mathbb{A}_f^p)$-invariant sections of $(\mathcal{V}^{\circ}_{\mathrm{dR}})^{\otimes}$ over $\mathcal{O}_{(v)}$.*

We comment on recent nontrivial improvements around Theorem 4.6.

- By Kim–Madapusi Pera [KMP16], the assumption (⋄) can be removed. Involving the use of deformation theory, such a result depends on the following ingredients:
  - (i) The Vasin–Zink parity, which implies the Faltings purity.
  - (ii) The classification of p-divisible groups over some 2-adic discrete valuation ring, by Kim and Lavi.
- By Y. Xu [Xu20], we are able to prove

$$\mathscr{S}_K(G,X)\xrightarrow{\sim}\mathscr{S}^{-}_K(G,X)\subseteq\mathscr{S}_{K'}(\mathrm{GSp},S^{\pm}).$$

The following gives more details in Y. Xu's work. Write

$$S^{-}_{K,K'}(G,X)\coloneqq S^{-}_K(G,X)\subseteq S_{K'}(\mathrm{GSp},S^{\pm}).$$

Lemma 4.8. *Either of the following two statements hold:*

(1) *either there is a sufficiently small open compact subgroup K' such that*

$$\mathscr{S}_K(G,X)\xrightarrow{\sim}\mathscr{S}^{-}_{K,K'}(G,X),$$

(2) *or there are distinct points $x,x'\in\mathscr{S}_K(G,X)(k)$, which have the same image in $\mathscr{S}_{K'}(\mathrm{GSp},S^{\pm})$ for all $K'\supseteq K$.*

Moreover, in case (2), $s_{\alpha,\ell,x}=s_{\alpha,\ell,x'}$ for $\ell\neq p$.

We also consider the ℓ-adic tensors with $\ell=p$. For any finite extension F of E, $x\in\mathscr{S}_K(G,X)(k)$, and its lifting $\tilde{x}\in\mathscr{S}_K(G,X)(F)$, the isomorphism

$$H^1_{\mathrm{et}}(\mathcal{A}_{\tilde{x},\overline{F}})\otimes_{\mathbb{Q}_p}B_{\mathrm{cris}}\xrightarrow{\sim}H^1_{\mathrm{cris}}(\mathcal{A}_x/W)\otimes_W B_{\mathrm{cris}}$$

takes $s_{\alpha,p,\tilde{x}}$ to $s_{\alpha,\mathrm{cris},\tilde{x}}=s_{\alpha,0,\tilde{x}}$. By the result of Kisin, we have

- The tensor $s_{\alpha,\mathrm{cris},\tilde{x}}$ depends only on x, and hence we can only concern about $s_{\alpha,\mathrm{cris},x}$.
- Both $x,x'\in\mathscr{S}_K(G,X)(k)$ have the same image in $\mathscr{S}_{K,K'}(G,X)$. Then $x=x'$ if and only if $s_{\alpha,\mathrm{cris},x}=s_{\alpha,\mathrm{cris},x'}$.

The general sense is that crystalline collections overdetermines the point x. This is relatively clear when $\ell = p$, and indeed, it also holds for $\ell \neq p$. Therefore, it suffices to show that

Lemma 4.9. *$s_{\alpha,\ell,x} = s_{\alpha,\ell,x'}$ if and only if $s_{\alpha,\mathrm{cris},x} = s_{\alpha,\mathrm{cris},x'}$.*

Obtaining this, we are able to apply the CM lifting on $\mathscr{S}_K(G, X)$ by Kisin.

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