STALKS OF AUTOMORPHIC VOGAN SHEAVES FOR THE STEINBERG PARAMETER OF GL_n

1. The main conjecture

1.1. **Vogan sheaf.** We first fix notation to define the Vogan stack, and then construct the Vogan sheaf with weight.

Notation 1.1. We use the same notation as in David's talk.

- For $n \geqslant 2$, take $G = \mathrm{GL}_n$ over a finite extension $E/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Take coefficient ring $\Lambda = \overline{\mathbb{Q}}_\ell$ with $\ell \neq p$ and fix $q^{1/2} \in \Lambda$.
- Let $T \subset B \subset G$ be the maximal split torus and the standard Borel in G. For $1 \leqslant i \leqslant n-1$, denote by $\alpha_i \coloneqq e_i - e_{i+1} \in \mathbf{X}^*(\hat{T})$ the simple roots of GL_n.
- Let φ be the semisimple L-parameter such that $\varphi(I_F) = 1$ and $\varphi(\mathrm{Fr}) = \delta^{1/2}$, where Fr is the geometric q-Frobenius and
$$\delta^{1/2} \coloneqq \mathrm{diag}(q^{(1-n)/2}, \ldots, q^{(n-1)/2}) \in \hat{T}(\Lambda).$$

Note that the finite Weyl group $W = S_n$ acts on $\delta^{1/2}$ by permuting the diagonal elements, so we get the G-conjugacy class $[\varphi] \in (\hat{T}/\!\!/W)(\Lambda)$.

With Notation 1.1, we construct the Vogan sheaf $\mathcal{L}_{\boldsymbol{k}}$ as follows. Recall from [Han23, §1.3] that the Vogan stack $V_{\hat{G},\varphi}$ at φ parametrizes nilpotent elements $N \in \mathfrak{g}^{\mathrm{ad}\,\varphi(I_F)}$ such that $\mathrm{ad}\,\varphi(\mathrm{Fr}) \cdot N = q^{-1}N$ up to S_φ-conjugacy, where $S_\varphi \coloneqq \mathrm{Cent}_{\hat{G}}(\varphi)$. For $G = \mathrm{GL}_n$, we have

$$\begin{array}{ccc} V_{\hat{G},\varphi} = \mathbb{A}^{n-1}/\hat{T} & \overset{\iota}{\hookrightarrow} & \mathrm{Par}^{\mathrm{unip}}_{\hat{G}} \\ \downarrow & & \\ \mathbb{B}\hat{T} & & \end{array}$$

as a closed substack of the stack of unipotent L-parameters for $\hat{G}$. This diagram depends only on $[\varphi]$. Note that ι factors through $\mathrm{Par}^{\mathrm{ur}}_{\hat{G}} \hookrightarrow \mathrm{Par}^{\mathrm{unip}}_{\hat{G}}$ because of the condition $\varphi(I_F) = 1$.

We then fix $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbf{X}^*(\hat{T})$ and consider its associated weight character
$$\chi_{\boldsymbol{k}} \colon \mathrm{diag}(t_1, \ldots, t_n) \longmapsto t_1^{k_1} \cdots t_n^{k_n}$$
on $\hat{T} = \mathbb{G}_{\mathrm{m}}^n$, valued in $\Lambda^\times$. Viewing this $\chi_{\boldsymbol{k}}$ as a coherent sheaf on $\mathbb{B}\hat{T}$, its pull-push along the diagram above yields a coherent sheaf
$$\mathcal{L}_{\boldsymbol{k}} \coloneqq \iota_* \mathcal{O}(\boldsymbol{k}) \in \mathrm{Coh}(\mathrm{Par}^{\mathrm{unip}}_{\hat{G}}),$$
where $\mathcal{O}(\boldsymbol{k})$ on $\mathbb{A}^{n-1}/\hat{T}$ is the pullback of $\chi_{\boldsymbol{k}}$ from $\mathbb{B}\hat{T}$.

1.2. **The Steinberg stalk.** Assume the categorical local Langlands equivalence. Then $\mathcal{L}_{\boldsymbol{k}}$ corresponds to an automorphic sheaf $\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ on Bun_G. As we are interested in the smooth irreducible representation corresponding to $\mathcal{L}_{\boldsymbol{k}}$, we aim to compute the stalk $i_1^* \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$, where $i_1 \colon \mathrm{Rep}(G(E), \Lambda) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. More generally, one can also consider $i_{\mathrm{b}}^* \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ for basic $\mathrm{b} \in \mathrm{B}(G)_{\mathrm{bas}}$, with $i_{\mathrm{b}} \colon \mathrm{Rep}(G_{\mathrm{b}}(E), \Lambda) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathrm{b}}, \Lambda)$. In the GL_n case, this is called the *Steinberg stalk* as it turns out to be some generalized Steinberg representation π_{I} defined as follows, up to a cohomological degree shift (see Conjecture 1.4).

Definition 1.2. Given a subset $\mathrm{I} \subset \{1, \ldots, n-1\}$ with elements $i_1 < \cdots < i_k$, we obtain the standard parabolic subgroup P_{I} with Levi $\mathrm{GL}_{i_1} \times \mathrm{GL}_{i_2 - i_1} \times \cdots \times \mathrm{GL}_{i_k - i_{k-1}} \times \mathrm{GL}_{n - i_k}$. Define the *generalized Steinberg representation* π_{I} as the unique irreducible quotient of $\mathcal{C}(P_{\mathrm{I}}(E) \backslash G(E), \Lambda)$.

Date: September 8, 2025.

Note that the π_{I}'s are exactly the irreducible representations whose semisimple L-parameter is φ (corresponding to $\delta^{1/2}$). It is a standard fact that the map $\mathrm{I} \mapsto \pi_{\mathrm{I}}$ assigned by Definition 1.2 above is actually a canonical bijection.

We point out that in Zhu's context [Zhu25], the construction of $\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ from $\mathcal{L}_{\boldsymbol{k}}$ is unconditional at the tame level. In this note, we work for simplicity at the unipotent level, and then Zhu's proof yields an equivalence functor $\mathbb{L}^{\mathrm{unip}}$. To apply Zhu's result, we need to change Bun_G to Isoc_G, and replace $\mathrm{D}_{\mathrm{lis}}$ with the sheaf theory on Isoc_G. Moreover, the stalk $i_{\mathrm{b}}^{*}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ must be written as $i_{\mathrm{b}}^{!}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ instead; see [Zhu25, 1.2.1, 1.2.3] for more background.

1.3. **Inner forms.** Pick any integer d and let $\mathrm{b}_{d/n} = \mathrm{b}_d$ be the basic isocrystal of slope d/n. Write $n' \coloneqq n/(d,n)$ and $d' \coloneqq d/(d,n)$ (where we declare $(0,n) = n$). Take $D_{d'/n'}$ as the unique central division algebra over E with Brauer invariant $d'/n' \in \mathbb{Q}/\mathbb{Z}$. Then $G_{\mathrm{b}_d} = \mathrm{GL}_{(d,n)}(D_{d'/n'})$ gives an inner form of G (recall that for $G = \mathrm{GL}_n$ over E, any inner form must be of the form $\mathrm{GL}_m(D)$ with $\deg D = n/m$). Note that G_{b_d} depends only on $d \bmod n$, because the image of Brauer invariant d'/n' in $\mathbb{Q}/\mathbb{Z}$ depends only on $d \bmod n$.

Fact 1.3. There is a canonical bijection between generalized Steinberg representations of G_{b_d} and subsets $\mathrm{I} \subset \{1, \dots, (d,n) - 1\}$.

Due to this fact, we can generalize the previous construction $\mathrm{I} \mapsto \pi_{\mathrm{I}}$ from representations of $G(E)$ to representations of $G_{\mathrm{b}_d}(E)$.

1.4. **The main conjecture.** Fix integers d, n as before. For $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbf{X}^*(\hat{T})$ of degree d, i.e., $k_1 + \cdots + k_n = d$, there are unique integers $m_i \in \mathbb{Z}$ such that

$$\boldsymbol{k} = \boldsymbol{\omega}_d + \sum_{1 \leqslant i \leqslant n-1} m_i \alpha_i.$$

Here $\boldsymbol{\omega}_d \in \mathbf{X}^*(\hat{T})$ is called the *Steinberg weight vector* of G_{b_d}, defined as

$$\boldsymbol{\omega}_d \coloneqq (\lceil d/n \rceil, \lceil 2d/n \rceil - \lceil d/n \rceil, \cdots, d - \lceil (n-1)d/n \rceil).$$

Notice that $\boldsymbol{\omega}_d$ is an n-tuple of degree d. In particular, $\boldsymbol{k} - \boldsymbol{\omega}_d$ is of degree 0.

With the notations above, we state the main conjecture as follows.

Conjecture 1.4 (Hansen)**.** *Fix n, d as before. Suppose $\boldsymbol{k}$ has degree d. For each $1 \leqslant i \leqslant n-1$, define $\delta_i \in \{0, 1\}$ by setting $\delta_i = 1$ if and only if $n' = n/(d,n)$ divides i. Then we expect*

$$i_{\mathrm{b}_d}^{*}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}} = \pi_{\mathrm{I}_{\boldsymbol{k}}}\big[\textstyle\sum_{j \in \mathrm{J}_{\boldsymbol{k}}} (\delta_j - 2m_j)\big]$$

between derived complexes in $\mathrm{Rep}(G_{\mathrm{b}_d}(E), \Lambda)$ that are concentrated in degree 0. Here,

- $\mathrm{I}_{\boldsymbol{k}} \coloneqq \{i \in \{1, \dots, (d,n) - 1\} \mid m_{n'i} \leqslant 0\}$;
- $\mathrm{J}_{\boldsymbol{k}} \coloneqq \{i \in \{1, \dots, n-1\} \mid m_i > 0\}$;
- $\pi_{\mathrm{I}_{\boldsymbol{k}}}$ *is the unique generalized Steinberg representation of $G_{\mathrm{b}_d}(E)$ corresponding to $\mathrm{I}_{\boldsymbol{k}}$.*

In fact, the stalk $i_{\mathrm{b}_d}^{*}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ is identically zero unless $\boldsymbol{k}$ has degree d. To check this, note that the central character $\omega_{\boldsymbol{k}}$ of $\boldsymbol{k}$ on $Z(\hat{G}) = \mathbb{G}_{\mathrm{m}}$ is given by $\omega_{\boldsymbol{k}} \colon z \mapsto z^{k_1 + \cdots + k_n}$; on the other hand, along the Kottwitz map $\kappa_G \colon \mathrm{B}(G) \to \mathbf{X}_*(Z(\hat{G})^{\Gamma})$, we always have $\kappa_G(\mathrm{b}_d) = d \bmod n$ for $G = \mathrm{GL}_n$.

2. Proof of Conjecture 1.4 for $\mathrm{b} = 1$

In this section, we prove Conjecture 1.4 for the special case $\mathrm{b} = 1$, achieved by taking $d = 0$. Note that when $d = 0$, we have the following ingredients in practice.

- $G_{\mathrm{b}_d} = G = \mathrm{GL}_n$.
- $(d,n) = (0,n) = n$, which forces $\mathrm{I}_{\boldsymbol{k}} = \{i \in \{1, \dots, n-1\} \mid m_i \leqslant 0\} = \{1, \dots, n-1\} \setminus \mathrm{J}_{\boldsymbol{k}}$ to hold, and $\delta_i = 1$ for all $1 \leqslant i \leqslant n-1$.
- $\boldsymbol{k} = (k_1, \dots, k_n)$ is of degree $k_1 + \cdots + k_n = 0$ (with this degree condition, we attain $\mu_{\boldsymbol{k}} \coloneqq k_1 e_1 + \cdots + k_n e_n = k_1 \alpha_1 + (k_1 + k_2)\alpha_2 + \cdots + (k_1 + \cdots + k_{n-1})\alpha_{n-1}$ by substituting $\alpha_j = e_j - e_{j+1}$, and thus $m_j = k_1 + \cdots + k_j$).

Therefore, Conjecture 1.4 for $\mathrm{b}=1$ can be simplified into the following statement.

Theorem 2.1. *In* $\mathrm{Rep}(\mathrm{GL}_n(E),\Lambda)$*, for* $\boldsymbol{k}=(k_1,\dots,k_n)$ *of degree* 0*, we have*

$$i_1^*\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}=\pi_{\mathrm{I}_{\boldsymbol{k}}}\big[\textstyle\sum_{j\notin \mathrm{I}_{\boldsymbol{k}}}(1-2m_j)\big],$$

where $m_j=k_1+\cdots+k_j$*.*

The main strategy to prove Theorem 2.1 is first reducing to computation of some RHom-complex of derived sheaves on Steinberg stack by arguing with Jacquet module, and then constructing a free projective resolution to get the cohomology of this RHom-complex.

2.1. **Jacquet module and coherent Springer sheaf.** Recall that we have defined the semisimple L-parameter φ by $\delta^{1/2}$, whose corresponding smooth irreducible representations of $\mathrm{GL}_n(E)$ are exactly π_{I}'s (where the monodromy of the Weil–Deligne parameter is parametrized by I). Thus, the stalk $i_1^*\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ must be a Steinberg representation up to shift. On the other hand, each π_{I} can be characterized among all generalized Steinberg representations by its Jacquet module $\mathbf{r}_G^B(\pi_{\mathrm{I}})$.

Proposition 2.2. *For any* $\mathrm{I}\subset\{1,\dots,n-1\}$*, the Jacquet module of* π_{I} *admits a direct sum decomposition*

$$\mathbf{r}_G^B(\pi_{\mathrm{I}})=\bigoplus_{\sigma}\sigma(\delta^{1/2}),$$

where the direct sum runs over $\sigma\in S_n$ *such that* $\mathrm{I}=\mathrm{P}_\sigma$ *with* $\mathrm{P}_\sigma\coloneqq\{i\in\{1,\dots,n-1\}\mid\sigma^{-1}(i)<\sigma^{-1}(i+1)\}$*.*

Proof. This is essentially proved in [Zel80, §2]; we omit the details. ☐

Example 2.3. Taking $\mathrm{I}=\{1,\dots,n-1\}$ in Proposition 2.2, we get $\mathbf{r}_G^B(\pi_{\mathrm{I}})=\mathbf{r}_G^B(\mathrm{St})=\delta^{1/2}$. This coincides with the usual characterization of the (generic) Steinberg representation.

By Proposition 2.2, to compute the (semi-simplified part of) cohomology of $i_1^*\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$, it suffices to compute that of its Jacquet module. Applying [HHS24, Corollary 2.2.1], we rewrite this Jacquet module via the constant term functor as

$$\mathbf{r}_G^B\, i_1^*\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}=i_1^{*,T}\mathrm{CT}_{B,!}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}.$$

But the right hand side can be computed on spectral side, where automorphic sheaf $\mathrm{CT}_{B,!}\mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$ corresponds to $\mathrm{CT}_B^{\mathrm{Spec}}\mathcal{L}_{\boldsymbol{k}}\coloneqq\mathfrak{p}_*^{\mathrm{Spec}}\mathfrak{q}^{\mathrm{Spec},!}\mathcal{L}_{\boldsymbol{k}}$; the spectral behavior of $i_1^{*,T}$ is the restriction of sheaves on $\mathrm{Par}_{\hat{T}}^{\mathrm{unip}}=\hat{T}\times\mathbb{B}\hat{T}$ to the part with trivial $\mathbb{B}\hat{T}$.

$$\begin{array}{ccccc} & & \mathrm{Par}_{\hat{B}}^{\mathrm{unip}} & & \\ & \swarrow^{\mathfrak{p}^{\mathrm{Spec}}} & & \searrow^{\mathfrak{q}^{\mathrm{Spec}}} & \\ \hat{T}\times\mathbb{B}\hat{T}=\mathrm{Par}_{\hat{T}}^{\mathrm{unip}} & & & & \mathrm{Par}_{\hat{G}}^{\mathrm{unip}}\xleftarrow{\ \iota\ }V_{\hat{G},\varphi}\end{array}$$

Thus, the essential difficulty lies in computing $\mathfrak{q}^{\mathrm{Spec},!}\mathcal{L}_{\boldsymbol{k}}$. To compute this !-pullback, take $V_{\hat{G},\varphi}^{\wedge}$ and $\mathrm{Par}_{\hat{G}}^{\mathrm{unip},\wedge}$ to be the formal completions of $V_{\hat{G},\varphi}$ and $\mathrm{Par}_{\hat{G}}^{\mathrm{unip}}$, respectively. So we have $\hat{\iota}$ and $\hat{\mathfrak{q}}^{\mathrm{Spec}}$ below.

$$\begin{array}{ccccc}\mathrm{Par}_{\hat{B}}^{\mathrm{unip}} & \longleftarrow & V_{\hat{B},\varphi}^{\wedge} & & \\ & \searrow^{\hat{\mathfrak{q}}^{\mathrm{Spec}}} & & \searrow & \\ & & \mathrm{Par}_{\hat{G}}^{\mathrm{unip},\wedge} & \xleftarrow{\ \hat{\iota}\ } & V_{\hat{G},\varphi}^{\wedge}\end{array}$$

Let $V_{\hat{B},\varphi}^{\wedge}$ be the pullback of $V_{\hat{G},\varphi}^{\wedge}$ along $\hat{\mathfrak{q}}^{\mathrm{Spec}}$. The result below describes these formal stacks.

Proposition 2.4 (Xiangqian Yang, see [Yan25, Proposition 3.12])**.**

(1) *As formal stacks, there is an isomorphism*

$$V_{\hat{G},\varphi}^{\wedge}\simeq(\mathrm{Spf}\,\Lambda[v_1,\dots,v_{n-1}][\![u_1,\dots,u_n]\!]/(u_1v_1,\dots,u_{n-1}v_{n-1}))/\hat{T}.$$

(2) *For each $\sigma \in S_n$, denote $\mathrm{P}_\sigma := \{i \in \{1,\dots,n-1\} \mid \sigma^{-1}(i) < \sigma^{-1}(i+1)\}$ and let Q_σ be its complement in $\{1,\dots,n-1\}$. Then*

$$V^\wedge_{\hat{B},\varphi} \simeq \coprod_{\sigma\in S_n} (\operatorname{Spf} \Lambda[v_i]_{i\in \mathrm{P}_\sigma}[\![u_1,\dots,u_n]\!]/(u_i v_i)_{i\in\mathrm{P}_\sigma})/\hat{T}.$$

In all formal stacks above, $\hat{T}$ acts on formal variable u_i trivially with weight 0 and acts on v_i with weight $\alpha_i \in \mathbf{X}^*(\hat{T})$.

Proof. We morally sketch the proof idea in [Yan25] in the special case $G = \mathrm{GL}_2$. When $n = 2$, we have $\delta^{1/2} = \mathrm{diag}(q^{-1/2}, q^{1/2})$. Given a Weil–Deligne parameter (φ, N) where φ is semisimple, it corresponds to a point (x, y) on completed Vogan variety with condition $\operatorname{ad}\varphi(\mathrm{Fr})\cdot N = q^{-1}N$, described by

$$V^\wedge_{\hat{G},\varphi} = \{x = \left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2\end{smallmatrix}\right),\ y = \left(\begin{smallmatrix} 1 & v_1 \\ 0 & 1\end{smallmatrix}\right) \mid u_1 \equiv q^{-1/2},\ u_2 \equiv q^{1/2},\ (qu_1 - u_2)v_1 = 0\}/\hat{T}.$$

Here $\hat{T}$ acts on x trivially, and acts on y by weight α_1. This proves assertion (1) for GL_2.

For assertion (2), up to $\hat{G}$-conjugacy, the Weyl group $W = S_2$ permutes diagonal entries of x, and hence $V^\wedge_{\hat{B},\varphi}$ consists of two components

$$V^\wedge_{\hat{B},\varphi} = X_\emptyset \sqcup X_{\{1\}}.$$

Here $X_\emptyset$ corresponds to $\sigma = \mathrm{id}$ and $\mathrm{Q}_\sigma = \emptyset$, on which the points are (x, y) such that $x \equiv \mathrm{diag}(q^{-1/2}, q^{1/2})$ and $y \in B$; notice that on $X_\emptyset$, we always have $qu_1 - u_2 = 0$ so the variable v_1 is free. Also, $X_{\{1\}}$ corresponds to $\sigma = (12)$ and $\mathrm{Q}_\sigma = \{1\}$, on which $x \equiv \mathrm{diag}(q^{1/2}, q^{-1/2})$ and $y \in B$; but $qu_1 - u_2 \neq 0$ on $X_{\{1\}}$, so the condition forces $v_1 = 0$. This completes the proof for $G = \mathrm{GL}_2$. □

Notation 2.5. To further simplify the notation, write

$$X_\sigma := (\operatorname{Spf} \Lambda[v_i]_{i\in \mathrm{P}_\sigma}[\![u_1,\dots,u_n]\!]/(u_i v_i)_{i\in\mathrm{P}_\sigma})/\hat{T}.$$

Then each X_σ is defined by setting $v_j = 0$ for $j \in \mathrm{Q}_\sigma$ inside $V^\wedge_{\hat{G},\varphi}$; in particular, $X := X_{\mathrm{id}} = V^\wedge_{\hat{G},\varphi}$. So X_σ is viewed as a closed substack via $i_\sigma\colon X_\sigma \hookrightarrow X$. As for the global sections, we take

$$\mathcal{O}_{X_\sigma} = \Lambda[v_i]_{i\in\mathrm{P}_\sigma}[\![u_1,\dots,u_n]\!]/(u_i v_i)_{i\in\mathrm{P}_\sigma}$$

and also set $\mathcal{O}_X/\boldsymbol{u} := \mathcal{O}_{X_{\mathrm{id}}}/(u_1,\dots,u_n) = \Lambda[v_1,\dots,v_{n-1}]$.

Throughout the notes, we use the convention that the automorphic Vogan sheaf $\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ corresponds to the spectral weight sheaf $(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})$ for $\mu_{\boldsymbol{k}} = k_1e_1 + \cdots + k_ne_n = m_1\alpha_1 + \cdots + m_{n-1}\alpha_{n-1}$.

Now, with notations above, the unipotent categorical local Langlands functor $\mathbb{L}^{\mathrm{unip}}$ (which is an equivalence in [Zhu25]) interpolates the equality

$$\mathbb{L}^{\mathrm{unip}}(\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}) = \hat{\imath}_*(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}) \in \mathrm{Coh}(\mathrm{Par}^{\mathrm{unip},\wedge}_{\hat{G}}).$$

On the other hand, to input the condition $\mathrm{b} = 1$ (so that $G_{\mathrm{b}} = G$), recall that the Iwahori compact induction of the trivial representation corresponds to the unipotent coherent Springer sheaf, i.e.,

$$\mathbb{L}^{\mathrm{unip}}(\text{c-Ind}_I^G \mathbb{1}) = \mathrm{CohSpr}^{\mathrm{unip}} := \hat{\mathfrak{q}}^{\mathrm{Spec}}_* \mathcal{O}_{\mathrm{Par}^{\mathrm{unip}}_{\hat{B}}} \in \mathrm{Coh}(\mathrm{Par}^{\mathrm{unip},\wedge}_{\hat{G}}).$$

For our purpose of computing $i_1^*\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$, it suffices to compute

$$\mathrm{RHom}(\text{c-Ind}_I^G\mathbb{1}, \mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}) \cong \mathrm{RHom}(\mathrm{CohSpr}^{\mathrm{unip}}, \hat{\imath}_*(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})),$$

where the isomorphism is due to the full faithfulness of $\mathbb{L}^{\mathrm{unip}}$. However, as a consequence of Proposition 2.4(2), there is a natural isomorphism

$$\hat{\imath}^!(\mathrm{CohSpr}^{\mathrm{unip}}) \cong \bigoplus_{\sigma\in S_n} \mathcal{O}_{X_\sigma}$$

of ind-coherent sheaves on X. Here each $\mathcal{O}_{X_\sigma}$ is viewed as an $\mathcal{O}_X$-module via the closed embedding $i_\sigma\colon X_\sigma \hookrightarrow X$.

To summarize, combining all the arguments above, the computation of $i_1^*\mathcal{L}^{\mathrm{aut}}_{\boldsymbol{k}}$ reduces to computing the direct sum of derived complexes

$$(\dagger) \qquad \bigoplus_{\sigma\in S_n} \mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})).$$

Remark 2.6. In (†), there are $|S_n| = n!$ direct summands, and each summand is determined by $\mathrm{P}_\sigma \subset \{1, \dots, n-1\}$ that has in total $2^{n-1} < n!$ different choices. Indeed, there is an explicit combinatorial formula for the fiber size of the map $\sigma \mapsto \mathrm{P}_\sigma$ at each subset $\mathrm{I} \subset \{1, \dots, n-1\}$. Moreover, this fiber size is equal to the multiplicity of π_{I} appearing in the cohomology $\mathrm{H}^*(i_1^* \mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}})$. (Granting Theorem 2.1, we expect the cohomology of $\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))$ to be concentrated in certain single degree, and so also is that of $i_1^* \mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}}$; but these two degrees are a priori not necessarily the same.)

Suppose $\mathrm{I} = \mathrm{P}_\sigma$ for some $\sigma \in S_n$. On the spectral side, this multiplicity of π_{I} can also be interpreted as the multiplicity of the image of X_σ, as a closed substack of $\mathrm{Par}_{\hat{B}}^{\mathrm{unip},\wedge}$, inside $\mathrm{Par}_{\hat{T}}^{\mathrm{unip},\wedge} = \hat{T} \times \mathbb{B}\hat{T}$ along $\mathfrak{p}^{\mathrm{Spec}}$. On the automorphic side, in the context of Yang–Zhu's proof of torsion vanishing [YZ25], taking the global section of $\hat{T}$-part in $\mathrm{Par}_{\hat{T}}^{\mathrm{unip},\wedge}$ gives a commutative Λ-algebra $\mathcal{O}(\hat{T})$ (also denoted by $\mathcal{R}$ in [Yan25]), which essentially corresponds to the universal unramified character of T. This $\mathcal{O}(\hat{T})$ acts on $\mathrm{CohSpr}^{\mathrm{unip}}$ and hence on (†) above. Using this setting, we identify the multiplicity of π_{I} in $\mathrm{H}^*(i_1^* \mathcal{L}_{\boldsymbol{k}}^{\mathrm{aut}})$ with the length of $\mathrm{H}^*(\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})))$ as an $\mathcal{O}(\hat{T})$-module.

2.2. **A projective resolution.** Continue with the computation of (†). We aim to compute cohomology of $\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))$ for each $\sigma \in S_n$. The technical strategy is to construct a free projective resolution

$$\mathcal{P}_\bullet \longrightarrow \mathcal{O}_{X_\sigma} = \mathcal{O}_X/(v_j)_{j \in \mathrm{Q}_\sigma} \longrightarrow 0,$$

of $\mathcal{O}_X$-modules, satisfying the $\hat{T}$-equivariant condition.

Construction 2.7. Fix an arbitrary subset $\mathrm{Q} \subset \{1, \dots, n-1\}$. Then $\mathrm{Q} = \mathrm{Q}_\sigma$ for some $\sigma \in S_n$. We use the following notations.

- Define $\mathbb{N}^{\mathrm{Q}} \coloneqq \{\boldsymbol{d} = (d_i)_{i \in \mathrm{Q}} \mid d_i \in \mathbb{Z}_{\geqslant 0}\}$;
- For each $\boldsymbol{d} \in \mathbb{N}^{\mathrm{Q}}$, write $|\boldsymbol{d}| \coloneqq \sum_{i \in \mathrm{Q}} d_i$ and define the weight

$$\chi_{\boldsymbol{d}} \coloneqq \sum_{i \in \mathrm{Q}} \lceil d_i/2 \rceil \alpha_i.$$

For $t \geqslant 0$, construct the t-th term of $\mathcal{P}_\bullet$ by

$$\mathcal{P}_t \coloneqq \bigoplus_{\boldsymbol{d} \in \mathbb{N}^{\mathrm{Q}},\, |\boldsymbol{d}| = t} \mathcal{O}_X(\chi_{\boldsymbol{d}}).$$

This $\mathcal{P}_t$ is a free $\mathcal{O}_X$-module with a basis denoted by $\{\mathsf{e}_{\boldsymbol{d}}\}_{|\boldsymbol{d}|=t}$. When $t = 0$, the condition $|\boldsymbol{d}| = 0$ forces $d_i = 0$ for all $i \in \mathrm{Q}$, and thus $\mathcal{P}_0 = \mathcal{O}_X$. For each $\sigma \in S_n$, the projection map $\mathcal{P}_0 \twoheadrightarrow \mathcal{O}_{X_\sigma}$ is given by the natural quotient setting $v_i = 0$ for $i \in \mathrm{Q}_\sigma$.

Next, we construct the differential map $\partial \colon \mathcal{P}_{t+1} \to \mathcal{P}_t$ by giving the image of each $\mathsf{e}_{\boldsymbol{d}}$ as follows:

$$\partial(\mathsf{e}_{\boldsymbol{d}}) = \sum_{i \in \mathrm{Q},\, d_i > 0} (-1)^{\varepsilon(i, \boldsymbol{d})} \vartheta(d_i) \cdot \mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i}.$$

The notations in this formula are explained below.

- Given variables v_i, u_i in $\mathcal{O}_X$, we set $\vartheta_i(r) = v_i$ when r is odd, and set $\vartheta_i(r) = u_i$ when r is even.
- Define $\varepsilon(i, \boldsymbol{d}) \coloneqq \sum_{j \in \mathrm{Q}_{<i}} d_j$, where $\mathrm{Q}_{<i} = \mathrm{Q} \cap \{1, \dots, i-1\}$; we also declare $\varepsilon(1, \boldsymbol{d}) = 0$.
- The $\boldsymbol{e}_i$ is the $|\mathrm{Q}|$-tuple with 1 on its i-th coordinate and 0 elsewhere; set $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i} = 0$ if any coordinate of $\boldsymbol{d} - \boldsymbol{e}_i$ is negative.

Note that the resolution $\mathcal{P}_\bullet$ depends on Q, but we omit Q from the notation.

Lemma 2.8. *The differential map $\partial \colon \mathcal{P}_{t+1} \to \mathcal{P}_t$ in Construction 2.7 satisfies $\partial \circ \partial = 0$.*

Proof. We compute $(\partial \circ \partial)(\mathsf{e}_{\boldsymbol{d}})$. This is a linear combination of elements $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i - \boldsymbol{e}_j}$. When $i = j$, the coefficient of $\mathsf{e}_{\boldsymbol{d} - 2\boldsymbol{e}_i}$ is given by

$$(-1)^{\varepsilon(i, \boldsymbol{d}) + \varepsilon(i, \boldsymbol{d} - \boldsymbol{e}_i)} \vartheta_i(d_i) \vartheta_i(d_i - 1) = (-1)^{\varepsilon(i, \boldsymbol{d}) + \varepsilon(i, \boldsymbol{d} - \boldsymbol{e}_i)} u_i v_i = 0,$$

because we have $u_i v_i = 0$ in $\mathcal{O}_X$. So it only remains to consider the case $i \neq j$. Note that there are two ways to get $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i - \boldsymbol{e}_j}$ from $\mathsf{e}_{\boldsymbol{d}}$, namely through either $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i}$ or $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_j}$. So the coefficient of $\mathsf{e}_{\boldsymbol{d} - \boldsymbol{e}_i - \boldsymbol{e}_j}$

in $(\partial\circ\partial)(\mathsf{e}_{\boldsymbol{d}})$ is computed as

$$(-1)^{\varepsilon(i,\boldsymbol{d})+\varepsilon(j,\boldsymbol{d}-\boldsymbol{e}_i)}\vartheta_i(d_i)\vartheta_j(d_j)+(-1)^{\varepsilon(j,\boldsymbol{d})+\varepsilon(i,\boldsymbol{d}-\boldsymbol{e}_j)}\vartheta_j(d_j)\vartheta_i(d_i).$$

But we always have $\varepsilon(j,\boldsymbol{d}-\boldsymbol{e}_i)=\varepsilon(j,\boldsymbol{d})-\mathbb{1}_{i<j}$, where $\mathbb{1}_{i<j}\in\{0,1\}$ is the characteristic function to test whether $i<j$ is true or not; similarly, $\varepsilon(i,\boldsymbol{d}-\boldsymbol{e}_j)=\varepsilon(i,\boldsymbol{d})-\mathbb{1}_{j<i}$. Thus, for arbitrary i and j, the two exponents of -1 above always differ by 1, so this coefficient equals zero. □

Lemma 2.9. *The resolution $\mathcal{P}_\bullet$ in Construction 2.7 is $\hat{T}$-equivariant.*

Proof. Recall that $\hat{T}$ acts on v_i with weight α_i and acts on u_i with weight 0. Fix $\chi\in\mathbf{X}^*(\hat{T})$. We claim that the following two multiplication maps of $\mathcal{O}_X$-modules are $\hat{T}$-equivariant:

$$v_i\colon\mathcal{O}_X(\chi)\longrightarrow\mathcal{O}_X(\chi-\alpha_i),\qquad u_i\colon\mathcal{O}_X(\chi)\longrightarrow\mathcal{O}_X(\chi).$$

Indeed, it suffices to consider the case $\chi=m\alpha_i$ for some $m\in\mathbb{Z}_{\geqslant0}$, so $\chi-\alpha_i=(m-1)\alpha_i$. To check the first map, for $t\in\hat{T}$, note that $t.(v_ia)=\alpha_i(t)\cdot v_i(t.a)$ for any section a in $\mathcal{O}_X$. So in $\mathcal{O}_X((m-1)\alpha_i)$, we have $t.(v_ia)=\alpha_i^{m-1}(t)\cdot t.(v_ia)=\alpha_i^m(t)\cdot v_i(t.a)$. On the other hand, inside $\mathcal{O}_X(m\alpha)$, we have $v_1(t.a)=v_1(\alpha_i^m(t)\cdot(t.a))=\alpha_i^m(t)\cdot v_i(t.a)$. Comparing these two right hand sides, we identify $t.(v_ia)$ in $\mathcal{O}_X((m-1)\alpha_i)$ and $v_i(t.a)$ in $\mathcal{O}_X(m\alpha_i)$. This gives the $\hat{T}$-equivariant property of the multiplication by v_i, and that by u_i can be checked similarly. So we have verified the claim.

Back to our construction of $\mathcal{P}_\bullet$. When d_i is odd (resp. even), we have $\vartheta_i(d_i)=v_i$ of $\hat{T}$-weight α_i (resp. $\vartheta_i(d_i)=u_i$ of $\hat{T}$-weight 0), and thus $v_i\colon\mathcal{O}_X(\chi_{\boldsymbol{d}})\to\mathcal{O}_X(\chi_{\boldsymbol{d}}-\alpha_i)$ (resp. $u_i\colon\mathcal{O}_X(\chi_{\boldsymbol{d}})\to\mathcal{O}_X(\chi_{\boldsymbol{d}})$) is $\hat{T}$-equivariant by the claim. But when d_i is odd (resp. even), there must be $\lceil(d_i-1)/2\rceil=\lceil d_i/2\rceil-1$ (resp. $\lceil(d_i-1)/2\rceil=\lceil d_i/2\rceil$), so the target of multiplication by v_i (resp. multiplication by u_i) equals $\mathcal{O}_X(\chi_{\boldsymbol{d}}-\alpha_i)=\mathcal{O}_X(\chi_{\boldsymbol{d}-\boldsymbol{e}_i})$ (resp. $\mathcal{O}_X(\chi_{\boldsymbol{d}})=\mathcal{O}_X(\chi_{\boldsymbol{d}-\boldsymbol{e}_i})$), which is the same as in $\mathcal{P}_\bullet$. □

Example 2.10. We illustrate the resolution in Construction 2.7 for GL_2 and GL_3.

(1) For $G=\mathrm{GL}_2$, the Weyl group $W=S_2=\{\mathrm{id},(12)\}$; we have $\mathrm{Q}_{\mathrm{id}}=\emptyset$ and $\mathrm{Q}_{(12)}=\{1\}$. Only the latter case is nontrivial so we take $\mathrm{Q}=\mathrm{Q}_{(12)}$. Then

$$\boldsymbol{d}=d_1\in\mathbb{Z}_{\geqslant0}=\mathbb{N}^{\mathrm{Q}},\qquad\chi_{d_1}=\lceil d_1/2\rceil\alpha_1.$$

So the resolution is 2-periodic, given by

$$\mathcal{P}_t=\mathcal{O}_X(\lceil t/2\rceil\alpha_1).$$

As for the differential map, we compute $\varepsilon(i,\boldsymbol{d})=\varepsilon(i,d_1)=0$, and hence

$$\partial(\mathsf{e}_{\boldsymbol{d}})=\partial(\mathsf{e}_{d_1})=\vartheta_1(d_1)\cdot\mathsf{e}_{d_1-1}.$$

Recall that $\vartheta_1(d_1)=v_1$ for $2\nmid d_1$ and u_1 for $2\mid d_1$. To conclude, the resolution is

$$\begin{array}{ccccccccccc}\cdots\xrightarrow{u_1}&\mathcal{O}_X(2\alpha_1)&\xrightarrow{v_1}&\mathcal{O}_X(\alpha_1)&\xrightarrow{u_1}&\mathcal{O}_X(\alpha_1)&\xrightarrow{v_1}&\mathcal{O}_X&\twoheadrightarrow&\mathcal{O}_{X_{(12)}}&\longrightarrow 0.\\ &\| & &\| & &\| & &\| & & & \\ &\mathcal{P}_3& &\mathcal{P}_2& &\mathcal{P}_1& &\mathcal{P}_0& & & \end{array}$$

As a remark, this recovers the resolution constructed in [BM23, §2] for the PGL_2 case (which is essentially the same as the GL_2 case).

(2) For $G=\mathrm{GL}_3$, the case of $\mathrm{Q}=\{1\}$ or $\{2\}$ is essentially the same as in (1), and that of $\mathrm{Q}=\emptyset$ is trivial. So we only need to consider $\mathrm{Q}=\{1,2\}$. In this case,

$$\boldsymbol{d}=(d_1,d_2)\in\mathbb{Z}^2_{\geqslant0}=\mathbb{N}^{\mathrm{Q}},\qquad\chi_{\boldsymbol{d}}=\lceil d_1/2\rceil\alpha_1+\lceil d_2/2\rceil\alpha_2.$$

Then we have

$$\mathcal{P}_t=\bigoplus_{d_1+d_2=t}\mathcal{O}_X(\lceil d_1/2\rceil\alpha_1+\lceil d_2/2\rceil\alpha_2),$$

where $\mathcal{P}_t$ consists of exactly $t+1$ direct summands, and each of which has a basis $\mathsf{e}_{\boldsymbol{d}}=\mathsf{e}_{(d_1,d_2)}$.

To get the formula of $\partial(e_{\boldsymbol{d}})$, first compute $\varepsilon(1,\boldsymbol{d})=0$ and $\varepsilon(2,\boldsymbol{d})=d_1$; it then follows that

$$\partial(\mathsf{e}_{(d_1,d_2)})=\vartheta_1(d_1)\mathsf{e}_{(d_1-1,d_2)}+(-1)^{d_1}\vartheta_2(d_2)\mathsf{e}_{(d_1,d_2-1)}.$$

In particular, the map $\partial\colon\mathcal{P}_{t+1}\to\mathcal{P}_t$ can be illustrated by a matrix of size $(t+1)\times(t+2)$ with all elements lying in $\{\pm v_i,\pm u_i,0\}$. This map will be significantly simplified if we work in $\mathcal{O}_X/\boldsymbol{u}$ with setting all $u_i=0$.

2.3. **Proof of Theorem 2.1.** We proceed to compute the cohomology

$$\mathrm{H}^*(\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})))$$

of each summand of (†) using the resolution in Construction 2.7. In $\mathrm{IndCoh}(\mathrm{Par}_{\hat{G}}^{\mathrm{unip},\wedge})$, applying the internal Hom functor to

$$\mathcal{P}_\bullet \longrightarrow \mathcal{O}_{X_\sigma} \longrightarrow 0,$$

and then taking the $\hat{T}$-invariant part, we get a cochain complex

$$\begin{aligned}(\mathcal{C}^\bullet)^{\hat{T}} &:= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_\bullet, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))^{\hat{T}} \\ &= \bigoplus_{\boldsymbol{d}\in\mathbb{N}^{\mathrm{Q}},\, |\boldsymbol{d}|=t} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(\chi_{\boldsymbol{d}}), (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))^{\hat{T}} \\ &\cong \bigoplus_{\boldsymbol{d}\in\mathbb{N}^{\mathrm{Q}},\, |\boldsymbol{d}|=t} (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}} - \chi_{\boldsymbol{d}})^{\hat{T}}.\end{aligned}$$

Here the second equality uses the formula of $\mathcal{P}_\bullet$. Note that the differential map in $(\mathcal{C}^\bullet)^{\hat{T}}$ is directly induced from ∂ in $\mathcal{P}_\bullet$.

To further compute $(\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}} - \chi_{\boldsymbol{d}})^{\hat{T}}$, we need the following result.

Lemma 2.11. *Let $\nu = r_1\alpha_1 + \cdots + r_{n-1}\alpha_{n-1}$ for $r_1, \ldots, r_{n-1} \in \mathbb{Z}$. Then we have*

$$\mathcal{O}_X(\nu)^{\hat{T}} = \begin{cases} 0, & \text{if } r_j > 0 \text{ for some } j; \\ v_1^{-r_1} \cdots v_{n-1}^{-r_{n-1}} \Lambda[\![u_n]\!][\![u_j]\!]_{r_j=0}, & \text{otherwise.} \end{cases}$$

In particular, $(\mathcal{O}_X/\boldsymbol{u})(\nu)^{\hat{T}}$ is either isomorphic to Λ or 0, determined by whether all $r_i \leqslant 0$ or not.

Proof. Since the $\hat{T}$-weights of v_i and u_i are respectively α_i and 0, for any monomial of the form $v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \cdot u_1^{b_1} \cdots u_n^{b_n}$ in $\mathcal{O}_X$, after twisting by $\mathcal{O}_X(\nu)$, its $\hat{T}$-weight equals $\sum_{1\leqslant j\leqslant n-1}(r_j + a_j)\alpha_j$. So the $\hat{T}$-invariant condition forces $r_j + a_j = 0$, or equivalently $a_j = -r_j \geqslant 0$. Also, we have conditions $v_j u_j = 0$ in $\mathcal{O}_X$, so we see if $r_j < 0$ (and hence $a_j > 0$), then no u_j is allowed, meaning that $b_j = 0$ for all j; else if $r_j = 0$ (and hence $a_j = 0$), the u_j is free. This completes the proof. □

Corollary 2.12. *Fix $\mathrm{Q} \subset \{1, \ldots, n-1\}$ as before. If $(\mathcal{C}^\bullet)^{\hat{T}} \neq 0$, then there exists $\boldsymbol{d} \in \mathbb{N}^{\mathrm{Q}}$ such that for all $i \in \mathrm{Q}$, we have $\lceil d_i/2 \rceil \geqslant m_i = k_1 + \cdots + k_i$.*

Proof. This follows immediately from the condition $\mu_{\boldsymbol{k}} - \chi_{\boldsymbol{d}} \leqslant 0$ given by Lemma 2.11. □

For the proof of Theorem 2.1, we point out that the case of GL_3 already exhibits all phenomena present for GL_n. Thus, in the following, we only consider $n = 3$. There are $2^{n-1} = 4$ different choices of Q_σ, and we compute the cohomology or the desired $\mathrm{RHom}(\mathcal{O}_{X_\sigma}, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))$ case by case.

Case I. Let $\mathrm{Q}_\sigma = \emptyset$. This uniquely corresponds to $\sigma = \mathrm{id}$, so $X_\sigma = X$. Applying Lemma 2.11 (together with the formula of $(\mathcal{C}^\bullet)^{\hat{T}}$ before), we directly get

$$\mathrm{RHom}(\mathcal{O}_X, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}})) \cong \begin{cases} 0, & \text{if } m_1 > 0 \text{ or } m_2 > 0, \\ \Lambda, & \text{if } m_1 \leqslant 0 \text{ and } m_2 \leqslant 0. \end{cases}$$

Therefore, for $\boldsymbol{k}$ of degree 0 such that $m_1, m_2 \leqslant 0$, we have $\mathrm{I}_{\boldsymbol{k}} = \{1, 2\}$ and $\mathrm{J}_{\boldsymbol{k}} = \emptyset$ by definition. It gives rise to the generic Steinberg representation $\pi_{\{1,2\}} = \mathrm{St}$ without cohomological degree shift.

Case II. Let $\mathrm{Q}_\sigma = \{1\}$, corresponding to $\sigma \in \{(12), (123)\} \subset S_3$. In this case, $\mathcal{O}_{X_\sigma} = \mathcal{O}_X/v_1$. The resolution of $\mathcal{O}_{X_\sigma}$ is essentially the same as Example 2.10(1) for the GL_2 case, namely $\mathcal{P}_\bullet \to \mathcal{O}_{X_\sigma} \to 0$ with $\mathcal{P}_t = \mathcal{O}_X(\lceil t/2 \rceil \alpha_1)$. Applying Lemma 2.11, we compute

$$\begin{aligned}(\mathcal{C}^i)^{\hat{T}} &:= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_i, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))^{\hat{T}} \\ &= (\mathcal{O}_X/\boldsymbol{u})((m_1 - \lceil i/2 \rceil)\alpha_1 + m_2\alpha_2)^{\hat{T}} \\ &\cong \begin{cases} 0, & \text{if } m_1 - \lceil i/2 \rceil > 0 \text{ or } m_2 > 0, \\ \Lambda, & \text{if } m_1 - \lceil i/2 \rceil \leqslant 0 \text{ and } m_2 \leqslant 0. \end{cases}\end{aligned}$$

In particular, if $(\mathcal{C}^i)^{\hat{T}} \neq 0$, then $m_1 \leqslant \lceil i/2 \rceil$; this condition is equivalent to $i \geqslant 2m_1 - 1$ (and we always have $i \geqslant 0$). So the whole complex, which is bounded below, is written as

$$(\mathcal{C}^\bullet)^{\hat{T}} = \left[0 \to \cdots \to 0 \to \Lambda \xrightarrow{u_1} \Lambda \xrightarrow[\sim]{v_1} \Lambda \xrightarrow{u_1} \Lambda \xrightarrow[\sim]{v_1} \cdots \right].$$

Here the first term sits in degree 0 and the first nonzero term sits in degree $2m_1 - 1$. This complex is bounded below. Each multiplication by v_1 is an isomorphism, whereas each multiplication by u_1 is zero as the base ring is $\mathcal{O}_X/\boldsymbol{u}$. Therefore, taking cohomology on the complex $(\mathcal{C}^\bullet)^{\hat{T}}$, we see exactly two possibilities:

(2a) If $m_1 \leqslant 0$, then the cohomology of $(\mathcal{C}^\bullet)^{\hat{T}}$ vanishes at all degrees.

(2b) If $m_1 > 0$ and $m_2 \leqslant 0$, the cohomology is non-vanishing only at the first nonzero term, i.e., $\mathrm{H}^*((\mathcal{C}^\bullet)^{\hat{T}}) = 0$ except for degree $2m_1 - 1$, where

$$\mathrm{H}^{2m_1-1}((\mathcal{C}^\bullet)^{\hat{T}}) = \Lambda.$$

To conclude, we explain the meaning of this cohomology. For $\boldsymbol{k}$ of degree 0 such that $m_1 > 0$ and $m_2 \leqslant 0$, we have $\mathrm{I}_{\boldsymbol{k}} = \{2\}$ and $\mathrm{J}_{\boldsymbol{k}} = \{1\}$ by definition. So this cohomology gives the representation $\pi_{\{2\}}[1 - 2m_1]$, as predicted by Conjecture 1.4.

Case III. Let $\mathrm{Q}_\sigma = \{2\}$, corresponding to $\sigma \in \{(23), (132)\} \subset S_3$. This is the same computation as in Case II, with only swapping the two indices m_1, m_2 and v_1, v_2. Thus, we also get exactly two possibilities of the cohomology of $(\mathcal{C}^\bullet)^{\hat{T}}$, namely:

(3a) If $m_2 \leqslant 0$, then $\mathrm{H}^*((\mathcal{C}^\bullet)^{\hat{T}}) = 0$ at all degrees.

(3b) If $m_1 \leqslant 0$ and $m_2 > 0$, then $\mathrm{H}^*((\mathcal{C}^\bullet)^{\hat{T}}) = 0$ except for degree $2m_2 - 1$, where

$$\mathrm{H}^{2m_2-1}((\mathcal{C}^\bullet)^{\hat{T}}) = \Lambda.$$

Again, for $\boldsymbol{k}$ of degree 0 such that $m_1 \leqslant 0$ and $m_2 > 0$, this corresponds to $\mathrm{I}_{\boldsymbol{k}} = \{1\}$, $\mathrm{J}_{\boldsymbol{k}} = \{2\}$, and hence gives the representation $\pi_{\{1\}}[1 - 2m_2]$ as predicted by Conjecture 1.4.

Case IV (The difficult case)**.** Let $\mathrm{Q}_\sigma = \{1, 2\}$, corresponding to $\sigma \in \{(23), (132)\} \subset S_3$. Following Construction 2.7 and Example 2.10(2), we write down the resolution $\mathcal{P}_\bullet$ of $\mathcal{O}_{X_\sigma} = \mathcal{O}_X/(v_1, v_2)$, that is,

$$\mathcal{P}_t = \bigoplus_{d_1+d_2=t} \mathcal{O}_X(\lceil d_1/2 \rceil \alpha_1 + \lceil d_2/2 \rceil \alpha_2).$$

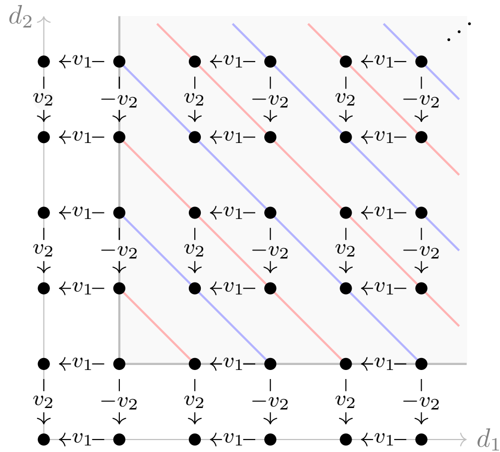
Using Lemma 2.11, whenever $(\mathcal{C}^i)^{\hat{T}} := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_i, (\mathcal{O}_X/\boldsymbol{u})(\mu_{\boldsymbol{k}}))^{\hat{T}} \neq 0$, we must have $\lceil d_1/2 \rceil \geqslant m_1$ and $\lceil d_2/2 \rceil \geqslant m_2$. Thus, the minimal possible values of d_i for $i \in \{1, 2\}$ are given by

$$d_{i,\min} = \begin{cases} 0, & m_i \leqslant 0, \\ 2m_i - 1, & m_i \geqslant 1. \end{cases}$$

We then describe the differential map. Since we eventually work over $\mathcal{O}_X/\boldsymbol{u}$, to simplify the argument, we take all $u_i = 0$ in the formula of $\partial(\mathsf{e}_{\boldsymbol{d}})$ in Construction 2.7. Then the map ∂ over $\mathcal{O}_X/\boldsymbol{u}$ is read as

$$\mathsf{e}_{(d_1,d_2)} \longmapsto \begin{cases} 0, & \text{if } 2 \mid d_1 \text{ and } 2 \mid d_2, \\ v_2 \cdot \mathsf{e}_{(d_1, d_2-1)}, & \text{if } 2 \mid d_1 \text{ and } 2 \nmid d_2, \\ v_1 \cdot \mathsf{e}_{(d_1-1, d_2)}, & \text{if } 2 \nmid d_1 \text{ and } 2 \mid d_2, \\ v_1 \cdot \mathsf{e}_{(d_1-1, d_2)} - v_2 \cdot \mathsf{e}_{(d_1, d_2-1)}, & \text{if } 2 \nmid d_1 \text{ and } 2 \nmid d_2. \end{cases}$$

The following picture depicts the differential map above. The node at coordinate (d_1, d_2) represents the direct summand $\mathcal{O}_X(\lceil d_1/2 \rceil \alpha_1 + \lceil d_2/2 \rceil \alpha_2)$ in $\mathcal{P}_{d_1+d_2}$. Each arrow is given by a multiplication. If there are no arrows between two nodes, then the differential map is zero. When $m_1, m_2 \geqslant 1$, the gray-shaded part in the picture indicates the part of $d_i \geqslant d_{i,\min}$ for $i \in \{1, 2\}$; note that in this case $2 \nmid d_{i,\min}$.



Therefore, taking cohomology on this bicomplex, we see exactly two possibilities again:

(4a) If $m_1 \leqslant 0$ or $m_2 \leqslant 0$, we have $d_{1,\min} = 0$ or $d_{2,\min} = 0$, and then $\mathrm{H}^*((\mathcal{C}^i)^{\hat{T}}) = 0$ at all degrees.

(4b) If $m_1, m_2 > 0$, the cohomology is non-vanishing only at the corner $(d_{1,\min}, d_{2,\min})$. It follows that $\mathrm{H}^*((\mathcal{C}^i)^{\hat{T}}) = 0$ except at the degree $d_{1,\min} + d_{2,\min} = (2m_1 - 1) + (2m_2 - 1)$, where

$$\mathrm{H}^{(2m_1-1)+(2m_2-1)}((\mathcal{C}^i)^{\hat{T}}) = \Lambda.$$

Similar to the explanation given in the three cases before, for $\boldsymbol{k}$ of degree 0 such that $m_1, m_2 > 0$, we have $\mathrm{I}_{\boldsymbol{k}} = \emptyset$ and $\mathrm{J}_{\boldsymbol{k}} = \{1, 2\}$. As $\pi_\emptyset = \mathbf{1}$, the trivial representation of $G(\mathbb{Q}_p)$, this cohomology corresponds to the representation $\mathbf{1}[(1 - 2m_1) + (1 - 2m_2)]$, as desired.

This finishes the computation of cohomology of the derived complex (†) and proves Theorem 2.1 for $G = \mathrm{GL}_3$. The argument for $G = \mathrm{GL}_n$ is essentially the same.

References

[BM23] Alexander Bertoloni Meli. Coherent sheaves for the Steinberg parameter of PGL_2, 2023. Unpublished notes. URL.

[Han23] David Hansen. Beijing notes on the categorical local Langlands conjecture, with an appendix by Adeel Khan, 2023. Last updated on May 2025. URL.

[HHS24] Linus Hamann, David Hansen, and Peter Scholze. Geometric Eisenstein series I: Finiteness theorems, 2024. Preprint. arXiv:2409.07363.

[Yan25] Xiangqian Yang. On Ihara's lemma for definite unitary groups, 2025. Preprint. arXiv:2504.07504.

[YZ25] Xiangqian Yang and Xinwen Zhu. On the generic part of the cohomology of Shimura varieties of abelian type, 2025. Preprint. arXiv:2505.04329.

[Zel80] Andrei V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Annales scientifiques de l'École Normale Supérieure*, 13(2):165–210, 1980. URL.

[Zhu25] Xinwen Zhu. Tame categorical local Langlands correspondence, 2025. Preprint. arXiv:2504.07482.

Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076

Email address: daiwenhan@u.nus.edu