Seminar on Relative Langlands Duality (2023 Fall)

HYPERSPHERICAL HAMILTONIAN VARIETIES

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Recently, Ben-Zvi–Sakellaridis–Venkatesh proposed a duality in the relative Langlands program [BZSV], in which the main player is a class of Hamiltonian G-varieties M called *hyperspherical varieties*. These are the expanded notes based on two seminar talks about the structure theory of their new proposal.

We try to first introduce the motivations of their work and then describe the connection between symplectic geometry and representation theory at an explicit level. The notes also contain a vague discussion on certain technicalities for the cornerstone theory of relative Langlands duality. We primarily focus on providing a refined overview of the background, as an addendum of their paper, but it may result in a lack of rigor.

The main references besides [BZSV] are [Gan23, §9–§13], [BZ23, §11–§17], and [GW23, §1–§6]. As for more backgrounds of [BZSV] beyond the present notes, [GW09, GN10, Zhu17, Zhu18, BZCHN23] are particularly recommended. We claim responsibility for all mistakes while disclaiming any originality.

Contents

1. Motivations

1.1. **Background on periods and L-functions.** Over a global field F, let G be a reductive group containing a subgroup H. Denote $[\mathsf{H}] = \mathsf{H}(F)\backslash\mathsf{H}(\mathbb{A}_F) \subset \mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}_F) = [\mathsf{G}]$ the locally symmetric spaces as automorphic quotients. A fundamental question in the automorphy theory is to study the H-period integral. More precisely, let π be a (tempered) cuspidal automorphic

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representation of $\mathsf{G}(\mathbb{A}_F)$ and define the *period* for any $\varphi \in \pi$ as the integral

$$\mathcal{P}_{\mathsf{H}}(\varphi) \coloneqq \int_{[\mathsf{H}]} \varphi(\boldsymbol{h})\,\mathrm{d}\boldsymbol{h}.$$

Some of the central themes of the (relative) Langlands program are as follows:

(a) *Characterize the nonvanishing of certain automorphic periods* $\mathcal{P}_{\mathsf{H}}(\varphi)$.
(b) *Whenever* $\mathcal{P}_{\mathsf{H}}(\varphi) \neq 0$, *relate it to certain (special value of) L-function.*

There is a corresponding problem of (a) at the local level, called the H-*distinction problem*:

(Dist) *Classify irreducible smooth representations of* G *which possess nonzero* H-*invariant periods.*

Our first goal of this introductory section is to explain why *hyperspherical variety* of [BZSV] is the natural object to investigate when we aim to attack both (a) and (b). To begin with (b), the notion of *spherical varieties* arises from trying to construct a generic recipe of understanding when a subgroup H leads to a useful period. Moreover, to integrate the considerations of (a) and (b) into a unified problem, drawing inspiration from the historical work of others, [BZSV] establishes the concept of *hyperspherical varieties*. In a sequel, hyperspherical varieties serve as the key object in addressing local problems around (Dist).

Given a representation $\mathtt{V}$ of $\mathsf{G}^\vee$, one can associate it an automorphic L-function $L(\pi, \mathtt{V}, s)$ with the variable $s \in \mathbb{C}$ by considering the characteristic polynomials of conjugacy classes in $\mathsf{G}^\vee$ that act on $\mathtt{V}$. Since the $\mathsf{G}^\vee$-conjugacy classes carry the information of Hecke eigenvalues (which further correspond to conjugacy classes of $\mathtt{V}^\vee$), it can be reasonable to match $L(\pi, \mathtt{V}, s)$ with $L(\rho, \mathtt{V}, s)$, the L-function for a Galois representation ρ depending on $\mathtt{V}$.

Conjecturally, at the *global* level, the classical Langlands correspondence asserts the following picture, in which the automorphic L-function $L(\pi, \mathtt{V}, s)$ serves an intermediate role, linking up the automorphic period $\mathcal{P}_{\mathsf{H}}(\varphi)$ and (the special value of) the Galois L-function $L(\rho, \mathtt{V}, s)$.

$$\begin{tikzcd}
(\textit{Automorphic side}) & (\textit{Galois side}) \\
\boxed{\text{Automorphic periods } \mathcal{P}_{\mathsf{H}}(\varphi)} \arrow[r, leftrightarrow] \arrow[d, phantom, "\wr"] & \boxed{\text{Galois } L\text{-functions } L(\rho, \mathtt{V}, s)} \arrow[d, phantom, "\wr"] \\
\boxed{\text{Data on } [\mathsf{H}] \subset [\mathsf{G}]} \arrow[r, leftrightarrow] \arrow[d, hook] & \boxed{\text{Data on } \mathtt{V}} \arrow[d, hook] \\
\boxed{\text{Data on } \mathsf{G}} \arrow[r, leftrightarrow, dashed] & \boxed{\text{Data on } \mathsf{G}^\vee}
\end{tikzcd}$$

Question 1.1. *Is there a conjectural correspondence between data on* G *and data on* $\mathsf{G}^\vee$*, realizing the classical Langlands correspondence at a certain restricted level? How do we describe the objects carrying the information about* G *or* $\mathsf{G}^\vee$*?*

Here comes the spoiler:

> In [BZSV], we can enlarge the (global) correspondence between $\mathcal{P}_{\mathsf{H}}(\varphi)$ and $L(\rho, \mathtt{V}, s)$ to that between *hyperspherical Hamiltonian* G-*varieties* and *hyperspherical Hamiltonian* $\mathsf{G}^\vee$-*varieties*. The rigorous correspondence is realized via the automorphic quantization and the spectral quantization on both sides respectively.

Rather, before explaining such a rough answer to Question 1.1, note that there can be many explicit evidences of the correspondence between automorphic periods and Galois L-functions.

Example 1.2 (Waldspurger formula and Whittaker period formula)**.**

(1) Let E be a quadratic extension of the number field F. Take

$$\mathsf{H} = \mathrm{Res}_{E/F}\,\mathbb{G}_m \subset \mathrm{PGL}_2 = \mathsf{G}$$

as reductive groups over F, where $\mathrm{Res}_{E/F}$ denotes the Weil restriction of scalar. It turns out that H is the maximal torus in G. Morally, Waldspurger [Wal85] proved the formula

$$\mathcal{P}_{\mathsf{H}}(\varphi)^2 \sim L(\pi_E, 1/2),$$

up to some local factors. Here $L(\pi_E, s)$ stands for the standard L-function of the base-change π_E of π to $\mathrm{PGL}_{2,E}$.

(2) Let F be a number field. Set $\mathsf{G} = \widetilde{\mathrm{SL}}_2$ and thus $\mathsf{G}(\mathbb{A}_F)$ is the two-fold metaplectic cover of $\mathrm{SL}_2(\mathbb{A}_F)$. In SL_2, take the subgroup $\mathsf{N} = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\}$ and let $\mathsf{H} = \mathsf{H}_\chi$ be the subgroup of matrices in N twisted by some fixed character χ of $F\backslash\mathbb{A}_F$. Then for the cuspidal automorphic representation π of $\mathsf{G}(\mathbb{A}_F)$, the global Whittaker period is written as

$$\mathcal{P}_{\mathsf{H}}(\varphi) = \int_{[\mathsf{H}]} \varphi(\boldsymbol{h})\,\mathrm{d}\boldsymbol{h} = \int_{F\backslash\mathbb{A}_F} \varphi\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \cdot \overline{\chi(h)}\,\mathrm{d}h$$

for any $\varphi \in \pi$. Roughly, up to some local factors, the *Whittaker period formula* dictates that

$$\mathcal{P}_{\mathsf{H}}(\varphi)^{\square} \sim L(\pi, 1/2),$$

where the "doubling period" $\mathcal{P}_{\mathsf{H}}(\varphi)^{\square}$ resembles $\mathcal{P}_{\mathsf{H}}(\varphi)^2$ in terms of (1). This formula (cf. [Qiu13]) on metaplectic SL_2 generalizes (1), relating the Fourier coefficients of half-integral-weight modular forms to the central L-values of integral-weight modular forms.

Innocently, since we are able to define the period for each pair $[\mathsf{H}] \subset [\mathsf{G}]$, it seems ad hoc when we opt for specific periods only in terms of Example 1.2. However, people are primarily interested in certain nice periods in the sense that their (conjectural) corresponding L-functions have nice analytic properties, such as admitting a decomposition of the Euler product. Also, we may expect conversely a general principle that *nice properties of L-functions come from realizations as periods*. This is why we care about which period to be realized.

1.2. **Multiplicity-freeness and spherical varieties.** To attack the H-distinction problem (Dist), one may naturally consider first decomposing an automorphic representation of G after restricting to H, and then detecting the properties of H-invariant periods defined by those restricted subrepresentations. But when we work with infinite-dimensional representations of G, such a process hardly makes sense. Fortunately, the meaning of the first step, namely decomposing restricted representations of G, can be assigned in a precise way, by using the Plancherel decomposition for unitary representations of G restricted to H. Such a phenomenon is called the *branching law*.

The local Gan–Gross–Prasad conjecture, which is a theorem now, considers such branching laws for certain pairs (G, H) of classical groups over a local field F. We choose one situation to depict with more details as follows.

Example 1.3 (Multiplicity one property)**.** Let Π and π be irreducible admissible automorphic representations of $\mathrm{SO}_{n+1}(F)$ and $\mathrm{SO}_n(F)$, respectively. The question of interest for Gan–Gross–Prasad is the understanding of

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SO}_n(F)}(\Pi, \pi) &\cong \mathrm{Hom}_{\mathrm{SO}_n(F)}(\Pi \otimes \pi^\vee, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathrm{SO}_{n+1}(F)\times\mathrm{SO}_n(F)}(\mathcal{S}(X), \Pi^\vee \otimes \pi),\end{aligned}$$

where $X = \mathrm{SO}_n(F)\backslash(\mathrm{SO}_{n+1}(F) \times \mathrm{SO}_n(F))$. In the context of H-distinction problem (Dist), note that π appears in Π if and only if $\mathrm{Hom}_{\mathrm{SO}_n(F)}(\Pi, \pi) \neq 0$. For this, historically, the first important result proved is the *multiplicity one property* (cf. [AGRS10, SZ12])

$$\mathrm{mult}(\Pi, \pi) := \dim_F \mathrm{Hom}_{\mathrm{SO}_n(F)}(\Pi, \pi) \leqslant 1.$$

Before the full multiplicity one theorem was proved, even finite dimensionality of the multiplicity spaces was not known, which were later answered in greater generality by Sakellaridis–Venkatesh [SV17].

Motivated by Example 1.3, we make the definition of multiplicity-free property. Recall first that a *homogeneous* G*-space* is of the form $\mathsf{H}\backslash\mathsf{G}$; it is naturally equipped with the G-variety structure whenever G is an algebraic variety.

Definition 1.4 (Spherical varieties)**.** A homogeneous G-variety $X = \mathsf{H}\backslash\mathsf{G}$ is called a *spherical variety* if it satisfies the *multiplicity-free* condition, i.e., any irreducible character of H appearing in some irreducible representation of G has multiplicity at most 1.

Write B for the Borel subgroup of G and $X(\mathsf{T})^+$ for the group of dominant characters with respect to B. Then the multiplicity-free condition in Definition 1.4 is equivalent to the following:

- ⋄ For each $\lambda \in X(\mathsf{T})^+$, the coordinate ring $\mathbb{F}[X]$, as a G-module, satisfies
$$\mathrm{mult}_\lambda(\mathbb{F}[X]) \coloneqq \dim_{\mathbb{F}} \mathrm{Hom}_{\mathsf{G}}(\mathbb{F}[X](\lambda), \mathbb{F}[X]) \leqslant 1,$$
where $\mathbb{F}[X](\lambda)$ is the simple G-module of the highest weight λ.

Remark 1.5. A priori people define an arbitrary G-variety X to be spherical if the Borel subgroup $\mathsf{B} \subset \mathsf{G}$ has an orbit as an open subvariety of X; or alternatively, say X has finitely many B-orbits. These alternative descriptions can be equivalent to the multiplicity-free condition whenever X is a homogeneous space.

1.3. **Theta correspondence and Adams' conjecture.** Our goal of this subsection is to claim that:

> The relative Langlands duality in [BZSV], using the quantization, encompasses *theta correspondence* (or the *Howe duality*) and *Adams' conjecture*, which are results in the style of local-global compatibility.

Such an idea is enlightened by Wee Teck Gan's CUHK talk in 2023.

Respectively, let V and W be a quadratic space and a symplectic space over a global field F. We will see in §2.2.3 that there is a family of Weil representations of $\mathrm{O}(V) \times \mathrm{Sp}(W)$, written as $\{\Omega_{V,W,\psi}\}$ and characterized by global characters $\psi \colon F\backslash\mathbb{A}_F \to \mathbb{S}^1$. Since there is a natural map
$$\mathsf{i}_{V,W} \colon \mathrm{O}(V) \times \mathrm{Sp}(W) \longrightarrow \mathrm{Sp}(V \otimes W),$$
one may expect to pullback a certain representation from $\mathrm{Sp}(V \otimes W)$ along $\mathsf{i}_{V,W}$ to realize $\Omega_{V,W,\psi}$. However, such a pullback is exclusively available from $\mathrm{Mp}(V \otimes W)$, the metaplectic cover of $\mathrm{Sp}(V\otimes W)$, which is still seen as a craggy enigma so far. Considering $\mathcal{C}([\mathrm{O}(V)\times\mathrm{Sp}(W)])$, the space of smooth functions on the automorphic quotient, there would be an equivariant map
$$\Omega_{V,W,\psi} \xrightarrow{\theta} \mathcal{A}(\mathrm{Mp}(V \otimes W)) \longrightarrow \mathcal{C}([\mathrm{O}(V) \times \mathrm{Sp}(W)])$$
given by the "formation of theta series" (cf. [Gan23, §3–§4]).

Construction 1.6 (Theta lifting)**.** For each $\phi \in \Omega_{V,W,\psi}$, the image of $\theta(\phi)$ along the right map above (which will be made explicit in Construction 2.10) is a function on $[\mathrm{O}(V) \times \mathrm{Sp}(W)]$, and thus can be used as a kernel function to transfer functions on $[\mathrm{O}(V)]$ to those on $[\mathrm{Sp}(W)]$.

More precisely, suppose $\sigma \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}(W))$ is a cuspidal representation of $\mathrm{Sp}(W)(\mathbb{A}_F)$. Then for $\phi \in \Omega_{V,W,\psi}$ and $f \in \sigma$, we define
$$\theta(\phi, f)(\boldsymbol{h}) \coloneqq \int_{[\mathrm{Sp}(W)]} \theta(\phi)(g\boldsymbol{h}) \cdot \overline{f(g)}\, \mathrm{d}g,$$
for $g \in \mathrm{Sp}(W_{\mathbb{A}_F})$ and $\mathrm{d}g$ denoting the Tamagawa measure. Then we set
$$\Theta(\sigma) \coloneqq \langle \theta(\phi, f) \colon \phi \in \Omega_{V,W,\psi},\ f \in \sigma \rangle \subset \mathcal{A}(\mathrm{O}(V)).$$
This is an $\mathrm{O}(V_{\mathbb{A}_F})$-submodule of the space of automorphic forms on $\mathrm{O}(V)$ and we call it the *global theta lifting* of σ. One may also switch the positions of $\mathrm{Sp}(W)$ and $\mathrm{O}(V)$ to define the global theta lifting of $\mathcal{A}_{\mathrm{cusp}}(\mathrm{O}(V))$ that lands in $\mathcal{A}(\mathrm{Sp}(W))$.

The first basic property about theta lifting, according to [Gan23], is the following finiteness result. Here we consider $\mathsf{G} = \mathrm{O}(V) \times \mathrm{Sp}(W)$ as before.

Proposition 1.7 (Finiteness, by Howe [How89] and Kudla [Kud86])**.** *Given the notations above, for any $\pi \in \mathsf{Irr}(\mathrm{O}(V))$ and $\sigma \in \mathsf{Irr}(\mathrm{Sp}(W))$,*

(1) *$\Theta(\pi) \subset \mathcal{A}(\mathrm{Sp}(W))$ has finite length, and*
(2) *$\dim \mathrm{Hom}_{\mathsf{G}}(\Omega_{V,W,\psi}, \pi \otimes \sigma) < \infty$.*

The Howe duality theorem refines the finiteness results of Proposition 1.7. It was first shown by Howe [How89] in the archimedean case, by Waldspurger [Wal90] in the p-adic case with $p \neq 2$, and in general in [Min08, GT16, GS17].

Theorem 1.8 (Howe duality)**.** *The Weil representation $\Omega_{V,W,\psi}$ is* strongly multiplicity-free*, i.e., for any $\pi \in \mathsf{Irr}(\mathrm{O}(V))$ and $\sigma_1, \sigma_2 \in \mathsf{Irr}(\mathrm{Sp}(W))$,*

(1) *$\dim \mathrm{Hom}_{\mathsf{G}}(\Omega_{V,W,\psi}, \pi \otimes \sigma_i) \leqslant 1$, and*
(2) *if $\mathrm{Hom}_{\mathsf{G}}(\Omega_{V,W,\psi}, \pi \otimes \sigma_i) \neq 0$ for $i = 1, 2$, then $\sigma_1 \cong \sigma_2$.*

On the other hand, we introduce Adams' conjecture in another flavor. Note that for $\mathsf{G} = \mathrm{O}(V) \times \mathrm{Sp}(W)$ we have
$$\mathsf{G}^\vee = \mathrm{O}(2m) \times \mathrm{SO}(2n+1),$$
with $m = \dim V$ and $n = \dim W$.

Conjecture 1.9 (Adams, [Ada89])**.** *Suppose $\Pi = \pi \otimes \sigma \in \mathsf{Irr}(\mathsf{G})$ is of Arthur type with A-parameter $\psi \colon W_F' \times \mathrm{SL}_2 \to \mathsf{G}^\vee$. If Π occurs in the Weil representation $\Omega_{V,W,\psi}$, namely $\mathrm{Hom}_{\mathsf{G}}(\Omega_{V,W,\psi}, \Pi) \neq 0$, then there exists a spherical variety X with $X^\vee = \mathrm{O}(2m)$, such that the A-parameter ψ factors through $X^\vee \times \mathrm{SL}_2$ as follows:*

$$\begin{tikzcd}
W_F' \times \mathrm{SL}_2 \arrow[rr, "\psi"] \arrow[dr] & & \mathsf{G}^\vee = \mathrm{O}(2m) \times \mathrm{SO}(2n+1) \\
& X^\vee \times \mathrm{SL}_2 \arrow[ur, "\iota_X"] \arrow[r] & \mathrm{O}(2n) \times \mathrm{SO}(2n+1-2m). \arrow[u, hook]
\end{tikzcd}$$

From the insight of Sakellaridis–Venkatesh [SV17], we can see the desired maps ι_X in Adams' conjecture are encoded by the geometry of spherical varieties (see also §3.5), which admits a description of multiplicity-freeness appeared in Howe duality. If there were a larger context encompassing Theorem 1.8 and Conjecture 1.9, then there would be a hidden connection between the geometry of spherical varieties and Weil representation. In [BZSV], we consider further hyperspherical varieties, which are spherical varieties satisfying a stronger condition. Hopefully, the Howe duality can be understood as a phenomenon of relative Langlands duality. Moreover, we expect that the L-parameters and A-parameters in the usual sense can be classified by the geometry of hyperspherical Hamiltonian G-varieties.

1.4. **Derived geometric Satake and Hamiltonian actions.** This subsection mainly refers to [BZ23, §11–§17]. Let G be a reductive group over an algebraically closed field $\mathbb{F}$. Recall the definition of *loop group* and *positive loop group* associated with G as
$$L\mathsf{G}(R) \coloneqq \mathsf{G}(R(\!(t)\!)) \quad \text{and} \quad L^+\mathsf{G}(R) \coloneqq \mathsf{G}(R[\![t]\!])$$
for a k-algebra R. Let $\mathrm{Gr}_{\mathsf{G}} = L\mathsf{G}/L^+\mathsf{G}$ be the affine Grassmannian associated with G.

Let $\mathsf{Shv}(-)$ be a "topological" sheaf theory on $\mathbb{F}$-schemes. For examples, we can use Betti sheaves if $\mathbb{F} = \mathbb{C}$, or D-modules if $\mathbb{F}$ is of characteristic 0, or étale $\overline{\mathbb{Q}}_\ell$-sheaves for ℓ not equal to the characteristic of $\mathbb{F}$. Let e be the coefficient field of our sheaf theory, which is always an algebraically closed field of characteristic 0. Let $\mathsf{G}^\vee$ be the dual group over e. Recall the geometric Satake equivalence.

Theorem 1.10 (Geometric Satake equivalence)**.** *There is an equivalence of* e*-linear symmetric monoidal categories*

$$(\mathsf{Perv}_{L^+\mathsf{G}}(\mathrm{Gr}_{\mathsf{G}}), *) \cong (\mathsf{Rep}(\mathsf{G}^\vee), \otimes).$$

The monoidal structure on $\mathsf{Perv}_{L^+\mathsf{G}}(\mathrm{Gr}_{\mathsf{G}})$ comes from the convolution structure on

$$\mathsf{Hecke}_{\mathsf{G}} := L^+\mathsf{G}\backslash L\mathsf{G}/L^+\mathsf{G} = B(L^+\mathsf{G}) \times_{B(L\mathsf{G})} B(L^+\mathsf{G}).$$

However, this does not explain the commutativity of the convolution product. It only gives a $\mathbb{E}_1$-structure. To get a symmetric monoidal category, we need an $\mathbb{E}_3$-structure. We still need an $\mathbb{E}_2$-structure. This can be seen as follows. If our base field is $\mathbb{C}$, and we take Betti sheaf theory. We have

$$\mathrm{Gr}_{\mathsf{G}} = \mathrm{Map}((D, D^*), (B\mathsf{G}, *))$$

where $D = \operatorname{Spec} \mathbb{C}[\![t]\!]$ and $D^* = \operatorname{Spec} \mathbb{C}(\!(t)\!)$ are respectively the formal disk and the punctured formal disk. Up to homotopy, we can rewrite this as

$$\mathrm{Gr}_{\mathsf{G}} \simeq \mathrm{Map}((\mathbb{R}^2, \mathbb{R}^2 - D_0), (B\mathsf{G}, *)) = \Omega^2(B\mathsf{G}) = \Omega\mathsf{G} \simeq \Omega\mathsf{G}_c,$$

where $D_0 = \{x \in \mathbb{R}^2 \colon |x| < 1\}$ is the unit dick, and G_c is the compact real form of G. In fact, the homotopy equivalence $\mathrm{Gr}_{\mathsf{G}} \simeq \Omega\mathsf{G}_c$ can be made to be a homeomorphism. This shows that the affine Grassmannian is homeomorphic to a double loop space and hence naturally has a $\mathbb{E}_2$-product on the underlying topological space. This is the origin of the missing $\mathbb{E}_2$-structure. In the algebraic zoo, we certainly cannot use the argument above. The solution is to use the factorization property of Gr_{G}. Basically, it allows you to vary and collapse points on a curve C over $\mathbb{F}$.

One may expect that the geometric Satake equivalence can be upgraded to an equivalence of derived categories. This is possible, but the answer is not the naive one taking $\mathsf{QC}(B\mathsf{G})$, the category of quasi-coherent sheaves, on the right-hand side.

Theorem 1.11 (Derived Satake equivalence)**.** *There is an equivalence of* ($\mathbb{E}_3$-)*monoidal categories*

$$\mathsf{Shv}(\mathsf{Hecke}_G) \cong \mathsf{QC}^!((\mathrm{pt} \times_{\mathsf{G}^\vee} \mathrm{pt})/\mathsf{G}^\vee) \cong (\mathrm{Sym}(\mathfrak{g}^\vee[2])\text{-}\mathsf{mod})^{\mathsf{G}^\vee}$$

where $\mathsf{QC}^!$ *is a modified version of* QC*.*

We can view derived Satake equivalence as an equivalence of local line operators of the 4-dimensional *topological field theories* (TFT) $\mathcal{A}_{\mathsf{G}}$ and $\mathcal{B}_{\mathsf{G}^\vee}$. For this, we shall restrict to the Betti setting. We work with ∞-categories in the rest of the present section. Roughly a TFT is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathsf{Bord}_{2,4}^{\square} \longrightarrow \mathcal{C}$$

where

- $\mathsf{Bord}_{2,4}^{\square}$ is the 2-category where objects are 2-manifolds, morphisms are bordisms between 2-manifolds (i.e. 3-manifolds with boundaries) and 2-morphisms are 2-bordisms between 2-manifolds (i.e. 4-manifolds with corners). The notation $\square$ is certain conditions on the bordisms (oriented, etc.).
- $\mathcal{C}$ is a symmetric monoidal 2-category (usually taken to be $\mathsf{dgCat}_{\mathbb{C}}$, the $(\infty, 2)$-category of $\mathbb{C}$-linear cocomplete stable ∞-categories).

We take $\mathcal{C} = \mathsf{dgCat}_{\mathbb{C}}$. Then $\mathcal{Z}(M^2)$ is a $\mathbb{C}$-linear dg-category. Monoidality of $\mathcal{Z}$ ensures that $\mathcal{Z}(\emptyset) = \mathsf{Vect}_{\mathbb{C}}$ is the category of $\mathbb{C}$-vector spaces.

A meta-version of geometric Langlands should say that there is an equivalence of two TFTs

$$\mathcal{A}_{\mathsf{G}} \cong \mathcal{B}_{\mathsf{G}^\vee},$$

which will be described later.

Consider the 2-sphere S^2. By definition $\mathcal{Z}(S^2)$ is a dg-category. We can endow $\mathcal{Z}(S^2)$ with a natural $\mathbb{E}_3$-monoidal structure as follows. Consider the bordism defined by a 3-dimensional ball with two small balls inside removed. It can be viewed as a bordism from $S^2 \sqcup S^2$ to S^2. Thus it defines an algebra structure on S^2 and the structure turns out to be $\mathbb{E}_3$. Hence $\mathcal{Z}(S^3)$ inherits a natural $\mathbb{E}_3$-monoidal structure. The monoidal category $\mathcal{Z}(S^2)$ is called the category of *line operators* of $\mathcal{Z}$.

Given a 2-manifold M^2 and a point $x \in M^2$, we have an action of $\mathcal{Z}(S^2)$ on the category $\mathcal{Z}(M^2)$ defined as follows. Consider the identity bordism $M^2 \times I$ on M. Digging a hole at the point $(x, \frac{1}{2}) \in M^2 \times I$ defines a bordism from $M \sqcup S^2$ to M. It induces the action map $\mathcal{Z}(S^2) \otimes \mathcal{Z}(M^2) \to \mathcal{Z}(M^2)$. These actions vary locally constantly on the surface M. As a result, we obtain an action of the *factorization homology* $\int_{M^2} \mathcal{Z}(S^2)$ on $\mathcal{Z}(M^2)$. The factorization homology $\int_{M^2} \mathcal{Z}(S^2)$ is defined as the "quotient" of $\bigotimes_{x \in M^2} \mathcal{Z}(S^2)$ by the relation that the objects vary locally constantly on the surface.

Similarly, the 3-sphere S^3, viewed as a bordism between empty spaces, is mapping to a complex in $\mathsf{Vect}_{\mathbb{C}} = \mathrm{End}_{\mathsf{dgCat}_{\mathbb{C}}}(\mathsf{Vect}_{\mathbb{C}})$. It carries a natural $\mathbb{E}_4$-algebra structure as above. The $\mathbb{E}_4$-algebra $\mathcal{Z}(S^3)$ is called the algebra of *local operators* of $\mathcal{Z}$. If M^3 is a 3-manifold, we have an action of $\int_{M^2} \mathcal{Z}(S^3)$ on $\mathcal{Z}(M^3)$. The relation between local operators and line operators is

$$\mathcal{Z}(S^3) = \mathrm{Hom}_{\mathcal{Z}(S^2)}(\mathcal{Z}(D^3), \mathcal{Z}(D^3)) = \mathrm{End}_{\mathcal{Z}(S^2)}(1_{\mathcal{Z}(S^2)}).$$

1.4.1. *$\mathcal{A}$-side.* The $\mathcal{A}$-side should take a Riemannian surface C to the category

$$\mathcal{A}_{\mathsf{G}}(C) = \mathsf{Shv}(\mathrm{Bun}_{\mathsf{G}}(C)).$$

We have $\mathcal{A}_{\mathsf{G}}(S^2) = \mathsf{Shv}(\mathrm{Bun}_{\mathsf{G}}(\mathbb{P}^1)) \cong \mathsf{Shv}(\mathsf{Hecke}_{\mathsf{G}})$ because $\mathbb{P}^2$ and $D \times_{D^*} D$ are "homotopic". It follows that the line operators on the $\mathcal{A}$-side are exactly the spherical Hecke category.

Local operators for $\mathcal{A}_{\mathsf{G}}$ are given by

$$\mathcal{A}_{\mathsf{G}}(S^3) = \mathrm{End}_{\mathsf{Shv}(\mathsf{Hecke}_{\mathsf{G}})}(1) = H^*_{L^+\mathsf{G}}(\mathrm{pt}) = H^*_{\mathsf{G}}(\mathrm{pt}) = \mathrm{Sym}(\mathfrak{h}^\vee[-2])^W.$$

1.4.2. *$\mathcal{B}$-side.* The $\mathcal{B}$-side can be made much more explicit. We will ignore the difference between QC and $\mathsf{QC}^!$ in the sequel. It should take a Riemannian surface C to the category

$$\mathcal{B}_{\mathsf{G}^\vee}(C) = \mathsf{QC}(\mathrm{LS}_{G^\vee}(C))$$

where $\mathrm{LS}_{G^\vee}(C)$ is the moduli stack of $\mathsf{G}^\vee$-local systems on C. Apply to S^2. Note that S^2 is glued from two disks along a circle. Using that $\mathrm{Map}_{\mathrm{lc}}(S^1, B\mathsf{G}^\vee) = \mathsf{G}^\vee/\mathsf{G}^\vee$, we have

$$\mathrm{LS}_{G^\vee}(S^2) = \mathrm{Map}_{\mathrm{lc}}(\mathrm{pt} \times_{S^1} \mathrm{pt}, B\mathsf{G}^\vee) = B\mathsf{G}^\vee \times_{\mathsf{G}^\vee/\mathsf{G}^\vee} B\mathsf{G}^\vee = (\mathrm{pt} \times_{\mathsf{G}^\vee} \mathrm{pt})/\mathsf{G}^\vee.$$

The underlying classical stack of $\mathrm{LS}_{G^\vee}(S^2)$ is simply $B\mathsf{G}$. If we choose some coordinates, we can write

$$(\mathrm{pt} \times_{\mathsf{G}^\vee} \mathrm{pt})/\mathsf{G}^\vee = (\mathrm{pt} \times_{\mathfrak{g}^\vee} \mathrm{pt})/\mathsf{G}^\vee = \mathrm{Spec}(\wedge^\bullet(\mathfrak{g}^{\vee,*}[1]))/\mathsf{G}^\vee.$$

Local operators for $\mathcal{B}_{\mathsf{G}^\vee}$ are given by

$$\mathcal{B}_{\mathsf{G}^\vee}(S^3) = \mathrm{End}_{\wedge^\bullet(\mathfrak{g}^{\vee,*}[1])}(\mathbb{C})^{\mathsf{G}^\vee} = \mathrm{Sym}(\mathfrak{g}^\vee[-2])^{\mathsf{G}^\vee} = \mathrm{Sym}(\mathfrak{h}^\vee[-2])^W.$$

Therefore the local operators on the $\mathcal{A}$-side and $\mathcal{B}$-side are equal. Koszul duality shows that

$$\wedge^\bullet(\mathfrak{g}^{\vee,*}[1])\text{-}\mathsf{mod} \cong \mathrm{Sym}(\mathfrak{g}^\vee[-2])\text{-}\mathsf{mod}.$$

Let C be a Riemann surface. Fix a point $x \in C$. We should have an action of $(\mathrm{Sym}(\mathfrak{g}^\vee[-2])\text{-}\mathsf{mod})^{\mathsf{G}^\vee}$ on $\mathcal{B}_{\mathsf{G}^\vee}(C)$. Or equivalently, the category $\mathcal{B}_{\mathsf{G}^\vee}(C)$ is linear over $\mathfrak{g}^{\vee,*}[2]/\mathsf{G}^\vee$. We first consider the non-derived version. Restricting to a point $x \in C$, we obtain a morphism

$$\mathrm{LS}_{\mathsf{G}^\vee}(C) \longrightarrow B\mathsf{G}^\vee.$$

Hence we get a tensor action of $\mathsf{QC}(\mathsf{BG})$ on $\mathsf{QC}(\mathrm{LS}_{\mathsf{G}^\vee})(C)$. Integrating over C, we obtain an action of $\int_C \mathsf{QC}(B\mathsf{G}^\vee)$ on $\mathsf{QC}(\mathrm{LS}_{\mathsf{G}^\vee}(C))$. In fact, the factorization homology $\int_C \mathsf{QC}(B\mathsf{G}^\vee)$ is equal to $\mathsf{QC}(\mathrm{LS}_{\mathsf{G}^\vee}(C))$. The same story works for the $\mathcal{A}$-side. The result is the spectral action of $\mathsf{QC}(\mathrm{LS}_{\mathsf{G}^\vee}(C))$ on $\mathsf{Shv}(\mathrm{Bun}_{\mathsf{G}}(C))$.

In the derived setting, we obtain an action of the factorization homology

$$\int_C \mathcal{B}_{\mathsf{G}^\vee}(S^2) \hookrightarrow \mathcal{B}_{\mathsf{G}^\vee}(C).$$

A result of Beraldo shows that $\int_C \mathcal{B}_{\mathsf{G}^\vee}(S^2)$ is equal to the deformation quantization of the shifted cotangent bundle $T^*[1]\mathrm{LS}_{\mathsf{G}^\vee}(C)$. Hence the line operators detect the singular support of sheaves on $\mathrm{LS}_{\mathsf{G}^\vee}(C)$.

1.4.3. *Hamiltonian spaces.* Symplectic geometry enters the story naturally as an $\mathbb{E}_n$-algebra can be viewed as a deformation quantization of a $\mathbb{P}_n$-algebra, which is roughly an n-shifted Poisson algebra if n is odd. We can view $\mathfrak{g}^{\vee,*}[2]/\mathsf{G}^\vee$ as a 3-shifted symplectic stack. This is because $T^*(\mathsf{BG}^\vee) = \mathfrak{g}^{\vee,*}[-1]/\mathsf{G}^\vee$ is a symplectic stack.

Hamiltonian spaces over G are related to boundary theories in TFTs. Let $\mathcal{Z}$ be a TFT, a boundary theory is a morphism $\mathcal{T}$ between the trivial TFT and $\mathcal{Z}$. For example, a boundary theory for the trivial 4-dimensional TFT is equivalent to a 3-dimensional TFT. Hence $\mathcal{T}(S^2)$ will be a $\mathbb{E}_3$-algebra. On the level of cohomology, it is just a graded Poisson algebra. Hence $\operatorname{Spec} H^*(\mathcal{T}(S^2))$ is a graded Poisson variety. The Poisson product has degree -2. On the other hand, from a graded symplectic space M (where the symplectic form has degree 2), there is a 3-dimensional TFT called the Rozansky–Witten theory associated with M.

Now consider the case that $\mathcal{Z}$ is not trivial. Then $\mathcal{T}(S^2)$ is an $\mathbb{E}_3$-algebra in the category $\mathcal{Z}(S^2)$. This defines an affine morphism

$$\operatorname{Spec}(\mathcal{T}(S^2)) \longrightarrow \operatorname{Spec}(\mathcal{Z}(S^2))$$

compatible with Poisson products, where $\operatorname{Spec}(\mathcal{Z}(S^2))$ is the 1-affine spectrum.

Apply the above discussion to the case $\mathcal{Z} = \mathcal{B}_{\mathsf{G}^\vee}$. From a boundary theory $\mathcal{T}$ of $\mathcal{B}_{\mathsf{G}^\vee}$, we get an affine morphism

$$M/\mathsf{G}^\vee \longrightarrow \mathfrak{g}^{\vee,*}[2]/\mathsf{G}^\vee$$

which is compatible with Poisson structures. This is same to a graded Hamiltonian G-space. Conversely, from a graded Hamiltonian $\mathsf{G}^\vee$-action on a symplectic space, we can construct a Rozansky–Witten boundary theory for $\mathcal{B}_{\mathsf{G}^\vee}$.

2. Hamiltonian spaces and quantization

In the following, we work over an algebraically closed field $\mathbb{F}$ of characteristic 0. Referring to [Gan23, §11] and [GW23], we plan to dedicate this section to revealing the hidden connection between

$$\boxed{\textit{Symplectic Geometry}} \overset{?}{\longleftarrow\!\!\!-\!\!\!-\!\!\!\longrightarrow} \boxed{\textit{Representation Theory}}.$$

2.1. **Hamiltonian G-varieties.**

2.1.1. *Symplectic manifolds.* In classical mechanics, the phase space of a classical system (i.e. the moduli of all possible states of the given system) is modeled by a symplectic manifold (M, ω), where M is a smooth variety over $\mathbb{F}$ and ω is a non-degenerate closed symplectic 2-form on M. The symplectic form gives an identification

$$\iota_\omega : TM \xrightarrow{\sim} T^*M$$

of tangent and cotangent bundles of M.

Denote by $\mathbb{F}(M)$ the space of rational $\mathbb{F}$-valued functions on M. The space $\mathbb{F}^\infty(M)$ of smooth $\mathbb{F}$-valued functions on M is called the *space of observables* of the system. Any function $f \in \mathbb{F}^\infty(M)$ gives a 1-form $\mathrm{d}f$. By contraction with ω we get a vector field $\mathfrak{X}_f$ on M. The symplectic form ω induces a Poisson bracket on $\mathbb{F}^\infty(M)$ (namely a Lie bracket which is a derivation in each variable) via

$$\{f_1, f_2\} \coloneqq \omega(\mathfrak{X}_{f_1}, \mathfrak{X}_{f_2}),$$

making $(\mathbb{F}^\infty(M), \{\cdot,\cdot\})$ a Poisson algebra. In general, a manifold M for which $\mathbb{F}^\infty(M)$ is equipped with a Poisson algebra structure is called a *Poisson manifold.* Geometrically, a Poisson manifold admits a foliation whose leaves are symplectic manifolds. Many constructs and results in symplectic geometry continue to hold in the setting of Poisson manifolds and our discussion will happen in this broader framework.

2.1.2. *Moment maps and Hamiltonian* G*-varieties.* Let M be a symplectic variety. The group G acts on M by symplectomorphisms, i.e. the action preserves the symplectic form ω.

Definition 2.1. A *moment map* for M is a G-equivariant morphism

$$\mu\colon M \longrightarrow \mathfrak{g}^*$$

satisfying the condition:

- For $\mathfrak{X} \in \mathfrak{g}$, the identity
$$\omega(\widetilde{\mathfrak{X}}, -) = \mathrm{d}\mu_{\mathfrak{X}}$$
holds, where $\widetilde{\mathfrak{X}}$ is the vector field defined by differentiating the G-action on M and $\mu_{\mathfrak{X}}$ is the function on M defined by μ composing with $\mathfrak{X}\colon \mathfrak{g}^* \to \mathbb{G}_a$.

The condition above can be rewritten as follows:

- We have the following maps
$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathfrak{X}\mapsto\widetilde{\mathfrak{X}}"] \arrow[dr, dashed] & \{\text{Vector fields on } M\} \\
& \mathcal{O}(M) \arrow[u, "f\mapsto\mathfrak{X}_f"']
\end{tikzcd}$$
compatible with Lie brackets or Poisson brackets. Giving a moment map is equivalent to giving a G-equivariant lifting $\mathfrak{g} \to \mathcal{O}(M)$ compatible with Lie brackets.

Definition 2.2. A *Hamiltonian* G*-space* is a symplectic G-variety M equipped with a G-equivariant moment map $\mu\colon M \to \mathfrak{g}^*$.

Example 2.3 (Examples of Hamiltonian spaces)**.**

(1) (*Cotangent bundle*). Let X be a smooth variety together with a G-action. The cotangent bundle $M = T^*X$ carries a natural symplectic structure defined as follows: Let θ be the tautological 1-form on M. Then $\omega = -\mathrm{d}\theta$ is a symplectic form on M. Then M is naturally a Hamiltonian G-space. The moment map $\mu\colon M \to \mathfrak{g}^*$ is dual to the action map $\mathfrak{g} \to TX$.

(2) Let (V, ω) be a symplectic vector space. Take $\mathsf{G} = \mathrm{Sp}(V)$. Then V is naturally a Hamiltonian G-space. The moment map is defined by
$$\begin{aligned}\mu\colon V &\longrightarrow \mathfrak{sp}^*(V)\\ m &\longmapsto (Z \mapsto \tfrac{1}{2}\omega(\mathfrak{Z}m, m))\end{aligned}$$
for $m \in V$ and $\mathfrak{Z} \in \mathfrak{sp}(V)$.

(3) (*Whittaker space*). Assume G is reductive. Let U be a maximal unipotent subgroup of G. Let $\psi\colon \mathsf{U} \to \mathbb{G}_a$ be a generic additive character. Define M as the left quotient by U of the preimage of $\mathrm{d}\psi$ under $T^*\mathsf{G} \to \mathfrak{u}^*$. It is a twist of the cotangent bundle of $\mathsf{U}\backslash\mathsf{G}$. As suggested by the construction, the Whittaker space captures the Whittaker models studied in representation theory.

(4) Consider $T^*\mathsf{G}$ with $\mathsf{G} \times \mathsf{G}$-action via left and right multiplication (denoted by $\mathsf{G}_l \times \mathsf{G}_r$). Then $T^*\mathsf{G} \cong \mathfrak{g}^* \times \mathsf{G}$ with action $(g_l, g_r) \cdot (\xi, g) = (\mathrm{Ad}(g_l^{-1})\xi, g_l^{-1} g g_r)$. The moment map is
$$\begin{aligned}\mu\colon \mathfrak{g}^* \times \mathsf{G} &\longrightarrow \mathfrak{g}_l^* \times \mathfrak{g}_r^*\\ (\xi, g) &\longmapsto (-\xi, \mathrm{Ad}(g)\xi).\end{aligned}$$

(5) (*Coadjoint orbit*). Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. It carries a natural Symplectic structure defined as follows: Let $\xi \in \mathcal{O}$ be a point. There is an isomorphism $T_\xi \mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\xi$ where $\mathfrak{g}_\xi$ is the centralizer of ξ in $\mathfrak{g}$. Define a bilinear form ω_ξ on $\mathfrak{g}$ by the formula

$$\omega_\xi(\mathfrak{X}, \mathfrak{Y}) = \xi([\mathfrak{X}, \mathfrak{Y}]), \quad \forall \mathfrak{X}, \mathfrak{Y} \in \mathfrak{g}.$$

Then ω_ξ descends to a symplectic form on $\mathfrak{g}/\mathfrak{g}_\xi$. Varying ξ along $\mathcal{O}$, we obtain a non-degenerate 2-form $\omega_{\mathcal{O}}$ on $\mathcal{O}$ which can be checked to be closed. Therefore $\mathcal{O}$ is naturally a symplectic variety with G-action. The moment map for $\mathcal{O}$ is simply the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

2.1.3. *Grading on Hamiltonian* G*-spaces.* In most of the examples, the Hamiltonian spaces we encountered admit natural gradings. Following [BZSV], we write $\mathbb{G}_{gr} = \mathbb{G}_m$ for the multiplicative group used for grading.

Definition 2.4. A Hamiltonian G-space M is *graded* if it is endowed with a $\mathbb{G}_{gr}$-action commuting with G-action, preserving the symplectic form ω up to square character (i.e. $\omega(\lambda\cdot\mathfrak{X}, \lambda\cdot\mathfrak{Y}) = \lambda^2\omega(\mathfrak{X}, \mathfrak{Y})$), and compatible with the grading on $\mathfrak{g}^*$ given by scalar multiplying by square character.

Using the notation in [BZSV], if we write $M^\vee$ for the dual Hamiltonian variety of M, then Definition 2.4 means that M is equipped with a commuting $\mathbb{G}_{gr}$ action of weight 2 on $M^\vee \to \mathfrak{g}^*$.

Example 2.5 (Graded Hamiltonian spaces)**.** The examples in Example 2.3 can be upgraded to graded Hamiltonian spaces as follows:

(1) The grading on $M = T^*X$ is given by square character acting on fibers.
(2) The grading on V is given by the usual scalar multiplying.
(3) The Whittaker space is a vector bundle over $\mathsf{U}\backslash\mathsf{G}$. We let $\lambda \in \mathbb{G}_{gr}$ acts by the left multiplying by $\lambda^{2\rho^\vee}$ on $\mathsf{U}\backslash\mathsf{G}$ and scalar multiplying by λ^2 on fibers.

2.2. **Hamiltonian reduction and quantization.** The upcoming context in this subsection is mostly copied from [GW23, §3] and [Gan23, §11].

2.2.1. *Hamiltonian reduction.* Symplectic reduction, also known as Hamiltonian reduction à la Marsden–Weinstein [Lan95], is constructed as the procedure that transfers the essential physical information carried by a Hamiltonian G-variety (M, ω) along the moment map $\mu\colon M \to \mathfrak{g}^*$, just so the representability of M as a phase space of classical mechanic systems is preserved. Following the convention of [BZSV], we use the notion of *Hamiltonian reduction* (resp. *induction*) instead of *symplectic reduction* (resp. *induction*) in the present notes, to emphasize the dependence on the Hamiltonian structure (i.e., on the moment map).

The philosophy behind the construction of [BZSV] postulates that many standard operations in symplectic geometry correspond to standard operations in representation theory (see Proposition 3.4 later for an explicit realization). We shall review two of the most pertinent ones, say Hamiltonian reduction and Hamiltonian induction, which correspond respectively to the formation of coinvariant spaces (or, more generally, multiplicity spaces) and induction of representations.

Definition 2.6 (Hamiltonian reduction)**.** Let M be a Hamiltonian G-space with moment map μ. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint G-orbit. The fiber product $\mu^{-1}(\mathcal{O}) = M \times_{\mathfrak{g}^*} \mathcal{O}$ inherits a G-action. The *Hamiltonian reduction* is defined to be the quotient symplectic stack[1]

$$M /\!\!/\!\!/_{\mathcal{O}} \mathsf{G} := \mu^{-1}(\mathcal{O}) /\!\!/ \mathsf{G} = M \times^{\mathsf{G}}_{\mathfrak{g}^*} \mathcal{O}.$$

The symplectic form on $M /\!\!/\!\!/_{\mathcal{O}} \mathsf{G}$ is inherited from M: Let $\omega^\flat$ be the non-degenerate 2-form on $M /\!\!/\!\!/_{\mathcal{O}} \mathsf{G}$ determined by the condition

$$p^*\omega^\flat = i^*\omega - \mu^*\omega_{\mathcal{O}},$$

[1]We only use this construction when the action is free, and hence we can take the GIT quotient.

where $p\colon \mu^{-1}(\mathcal{O}) \to M /\!\!/\!\!/_{\mathcal{O}} \mathsf{G}$ is the projection, $i\colon \mu^{-1}(\mathcal{O}) \to M$ is the inclusion, and $\omega_{\mathcal{O}}$ is the symplectic form on $\mathcal{O}$ defined in Example 2.3 (5). Then $(M /\!\!/\!\!/_{\mathcal{O}} \mathsf{G}, \omega^{\flat})$ is a symplectic space. If $\mathcal{O} = \{0\}$ is the trivial orbit, we abbreviate

$$M /\!\!/\!\!/ \mathsf{G} := M /\!\!/\!\!/_{\{0\}} \mathsf{G}.$$

Example 2.7. If X is a smooth G-variety, then

$$T^*X /\!\!/\!\!/ \mathsf{G} = T^*(X/\mathsf{G}).$$

Example 2.8 (Twisted cotangent bundle)**.** Let X be a graded G-variety. Let $\Psi \to X$ be a $\mathsf{G} \times \mathbb{G}_{gr}$-equivariant $\mathbb{G}_a$-torsor, where G acts on $\mathbb{G}_a$ trivially and $\mathbb{G}_{gr}$ acts on $\mathbb{G}_a$ via the square character. We get a Hamiltonian $\mathbb{G}_a$-space $T^*\Psi$ with a moment map $T^*\Psi \to \mathfrak{g}_a^*$. Note that the moment map is equivariant for the *trivial* $\mathsf{G} \times \mathbb{G}_{gr}$-action on $\mathfrak{g}_a^*$. It follows that

$$T^*(X, \Psi) := T^*\Psi /\!\!/\!\!/_{\{1\}} \mathbb{G}_a$$

is a graded Hamiltonian G-space.

Apply this construction to the case $X = \mathsf{U}\backslash\mathsf{G}$ where U is a unipotent subgroup in G and $\Psi = \mathsf{U}_0\backslash\mathsf{G}$ where U_0 is the kernel of a generic character $\psi\colon \mathsf{U} \to \mathbb{G}_a$. We recover the Whittaker bundle in Example 2.3 (3).

2.2.2. *The rough idea of quantization.* Recall our philosophy that associates an object from symplectic geometry with a group representation (and vice-versa). Morally,

> The (*geometric*) *quantization* is the process that arises a unitary representation of G from an arbitrary Hamiltonian G-space M.

Namely, it is the functor

$$\begin{aligned} \mathcal{H}\colon \mathsf{Sp}_{\mathsf{G}}^{\dagger}(\mathbb{F}) &\longrightarrow \mathsf{Vect}(\mathbb{C}) \\ M &\longmapsto \mathtt{V}_M \end{aligned}$$

from the (groupoid) category of finite-dimensional symplectic G-manifolds over $\mathbb{F}$ satisfying the Lagrangian involution condition $\dagger$ (which is a specific compatibility), to the category of finite-dimensional $\mathbb{C}$-vector spaces.

More precisely, regarding (M, ω) as the phase space of classical mechanics, suppose we want to pass to quantum mechanics. This basically means replacing M by $\mathbb{P}(\mathtt{V}_M)$, the projectivization of a Hilbert space $\mathtt{V}_M$ associated to M. The smooth functions on $\mathbb{P}(\mathtt{V}_M)$ are Hermitian operators on $\mathtt{V}_M$, so we can replace $\mathbb{F}^{\infty}(M)$ by $\mathrm{Herm}(\mathtt{V}_M)$ after the quantization process. In fact, the quantum spectra of the classical observables $\mathbb{F}^{\infty}(M)$ are given by the eigenvalues of elements in $\mathrm{Herm}(\mathtt{V}_M)$.

Since M is a Hamiltonian G-space, the natural G-action on M induces a G-action via unitary operators on $\mathtt{V}_M$. In a sequel, the Hilbert space $\mathtt{V}_M$ is a unitary representation of G. The following context is about the precise construction of $\mathtt{V}_M$.

2.2.3. *Weil representation as quantization.* We first introduce Weil representation as a heuristic example. The rough idea is as follows. Suppose Ξ is a symplectic vector space equipped with the symplectic form ω. Every symplectic structure on Ξ is isomorphic to one of the form $\Xi = \mathbb{X} \oplus \mathbb{X}^*$. The subspace $\mathbb{X}$ is not unique, and a choice of $\mathbb{X}$ is called a *polarization*. The subspaces that give such an isomorphism are called *Lagrangians*, and the natural projection $\Xi \to \mathbb{X}^*$ along $\mathbb{X}$ is a Lagrangian fibration. The *Weil representation* is exactly the unitary representation of $\mathrm{Sp}(\Xi)$ realized on $L^2(\mathbb{X}^*)$, which is also a quantization of the Hamiltonian $\mathrm{Sp}(\Xi)$-space Ξ.

Let V be a finite-dimensional quadratic space over $\mathbb{F}$, born with the quadratic form $q : V \to \mathbb{F}$ and associated with the symmetric bilinear form $\langle v_1, v_2 \rangle_V := q(v_1 + v_2) - q(v_1) - q(v_2)$.[2] Let W be a finite-dimensional symplectic vector space over $\mathbb{F}$, equipped with the symplectic form $\langle \cdot, \cdot \rangle_W$. Then the tensor product space $\Xi = V \otimes W$ inherits a natural symplectic form $\langle \cdot, \cdot \rangle_{\Xi} =$

[2] For simplicity, we shall assume that $\mathrm{disc}(V)$ is trivial in $\mathbb{F}^{\times}/\mathbb{F}^{\times 2}$.

$\langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_W$. Since the isometry groups of V, W, Ξ are respectively $\mathrm{O}(V), \mathrm{Sp}(W), \mathrm{Sp}(\Xi)$, we get a natural map

$$\mathsf{i}_{V,W} \colon \mathrm{O}(V) \times \mathrm{Sp}(W) \longrightarrow \mathrm{Sp}(V \otimes W) = \mathrm{Sp}(\Xi)$$

and the restriction of $\mathsf{i}_{V,W}$ to $\mathrm{O}(V)$ or $\mathrm{Sp}(W)$ is injective. The images of $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ along $\mathsf{i}_{V,W}$ are mutual centralizers of each other in $\mathrm{Sp}(\Xi)$.

Moreover, there is a unique nonlinear metaplectic double cover of $\mathrm{Sp}(\Xi)$, denoted by $\mathrm{Mp}^{\ddagger}(\Xi)$, which further extends by pushing out along $\mu_2 \hookrightarrow \mathbb{S}^1$ as follows:

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, hook] & \mathrm{Mp}^{\ddagger}(\Xi) \arrow[r] \arrow[d] & \mathrm{Sp}(\Xi) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{S}^1 \arrow[r] & \mathrm{Mp}(\Xi) \arrow[r] & \mathrm{Sp}(\Xi) \arrow[r] & 1. \end{tikzcd}$$

It turns out that the images of $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ are also mutual centralizers of each other in $\mathrm{Mp}(\Xi)$. The main reason for considering the metaplectic groups is that if $\mathbb{F}$ is a local field, $\mathrm{Mp}(\Xi)$ has a finite family of distinguished smooth genuine representations, which are the *Weil representations* $\{\omega_{\Xi,\psi}\}_{\psi}$. Each Weil representation in this family is parametrized by a nontrivial additive character $\psi\colon \mathbb{F} \to \mathbb{S}^1$, with the provision that

$$\omega_{\Xi,\psi} \cong \omega_{\Xi,\psi'} \iff \psi'(-) = \psi(a^2 \cdot (-)) \text{ for some } a \in \mathbb{F}^{\times}.$$

We sketch the idea of doing this:

> *A quantization of the* $\mathrm{Sp}(W) \times \mathrm{O}(V)$*-Hamiltonian space* $\Xi = V \otimes W$ *is exactly a Weil representation* $\Omega_{V,W,\psi}$*, as explained in the following.*

Suppose now $\mathbb{F}$ is the prescribed global field F. Each nontrivial adelic character $\psi\colon F\backslash\mathbb{A}_F \to \mathbb{S}^1$ uniquely characterizes an abstract Weil representation ω_ψ of $\mathrm{Mp}(\Xi_{\mathbb{A}_F})$, which can be realized on $\mathcal{S}(\mathbb{X}^*_{\mathbb{A}_F})$. A basic property is that $\mathrm{Mp}(\Xi_{\mathbb{A}_F})$ splits uniquely over $\mathrm{Sp}(W_F)$, which allows us to consider the space $\mathcal{A}(\mathrm{Mp}(\Xi))$ of automorphic forms on $\mathrm{Mp}(\Xi_{\mathbb{A}_F})$. As a result, it admits a natural equivariant map

$$\begin{aligned} \theta\colon \mathcal{S}(\mathbb{X}^*_{\mathbb{A}_F}) &\longrightarrow \mathcal{A}(\mathrm{Mp}(\Xi)) \\ \phi &\longmapsto \sum_{x^* \in \mathbb{X}^*_F} (\omega_\psi(-)(\phi))(x^*). \end{aligned}$$

given by "averaging over rational points". This map is called the *formation of theta series* as the function $\theta(\phi)(-)$ is the automorphic incarnation of theta functions in the classical sense.

Now given the morphism $\mathsf{i}_{V,W}\colon \mathrm{O}(V) \times \mathrm{Sp}(W) \to \mathrm{Sp}(\Xi)$, we wonder whether it can be lifted to $\mathrm{Mp}(\Xi)$ both locally and globally. This has to do with pulling back a Weil representation $\omega_{\Xi,\psi}$ of $\mathrm{Sp}(\Xi)$ to a representation of $\mathrm{O}(V) \times \mathrm{Sp}(W)$ along $\mathsf{i}_{V,W}$. The technicalities in addressing this question are omitted and we summarize the results here:

(i) When $2 \mid \dim_{\mathbb{F}} V$, the desired lift, denoted by $\mathsf{i}^{\bullet;\bullet}_{V,W}$, exists both locally and globally, fitting in the pullback diagram

$$\begin{tikzcd} \mathrm{O}(V) \times \mathrm{Sp}(W) \arrow[r, "\mathsf{i}^{\bullet;\bullet}_{V,W}"] \arrow[d, equal] & \mathrm{Mp}(\Xi) \arrow[d] \\ \mathrm{O}(V) \times \mathrm{Sp}(W) \arrow[r, "\mathsf{i}^{\circ;\circ}_{V,W}"] & \mathrm{Sp}(\Xi) \end{tikzcd}$$

in which we write $\mathsf{i}^{\circ;\circ}_{V,W} = \mathsf{i}_{V,W}$ before the lifting.

(ii) When $2 \nmid \dim_{\mathbb{F}} V$, the morphism $\mathsf{i}_{V,W}$ fails to lift on $\mathrm{Sp}(W)$ rather than on $\mathrm{O}(V)$, i.e., we exclusively have the following pullback diagram

$$\begin{tikzcd} \mathrm{O}(V) \arrow[r, "\mathsf{i}^{\bullet;\circ}_{V,W}"] \arrow[d, hook] & \mathrm{Mp}(\Xi) \arrow[d] \\ \mathrm{O}(V)\times\mathrm{Sp}(W) \arrow[r, "\mathsf{i}^{\circ;\circ}_{V,W}"] & \mathrm{Sp}(\Xi) \end{tikzcd}$$

in which the left vertical morphism is the natural subgroup embedding.

To avoid redundant arguments, we assume $2 \mid \dim_{\mathbb{F}} V$ from now on. Note that $\mathsf{i}^{\bullet;\bullet}_{V,W}$ in (i) is uniquely characterized by the identity map of $\mathrm{O}(V)\times\mathrm{Sp}(W)$ as the pullback. However, if we choose to twist the identity map on $\mathrm{O}(V)$-component by the global character $\psi\colon F\backslash\mathbb{A}_F \to \mathbb{S}^1$, the lift of $\mathsf{i}^{\circ;\circ}_{V,W}$ would not be canonical, even if the lift $\mathsf{i}^{\circ;\bullet}_{V,W}$ on $\mathrm{Sp}(W)$-component is unique and independent of ψ. If this is the case to be described, we define

$$\mathsf{i}^{\bullet;\circ}_{V,W,\psi}\colon \mathrm{O}(V) \longrightarrow \mathrm{Mp}(\Xi)$$

to be the pullback in (i), with changing the identity map on $\mathrm{O}(V)$ to the twist-by-ψ map.

Remark 2.9. One can describe this splitting concretely using the *Schrödinger model* (cf. [Gan23, §2.4]). Choose a symplectic polarization $W = \mathsf{W}\oplus\mathsf{W}^*$ so that $\Xi^\dagger = V\otimes\mathsf{W}^*$ is a maximal isotropic subspace of $\Xi = V\otimes W$. According to the construction of Schrödinger models, we are in case able to realize the Weil representation $\omega_{\Xi,\psi}$ on the space $\mathcal{S}(\Xi^\dagger)$ of Schwarz–Bruhat functions on $\Xi^\dagger$. In this model, the action of $\mathsf{i}^{\bullet;\circ}_{V,W,\psi}(\mathrm{O}(V))$ is geometric:

$$(h\phi)(-) = \phi(h^{-1}\cdot(-)), \quad \forall h \in \mathsf{i}^{\bullet;\circ}_{V,W,\psi}(\mathrm{O}(V)),\ \phi\in\mathcal{S}(\Xi^\dagger).$$

Construction 2.10 (Splitting symplectic Weil representation on metaplectic cover)**.**

(1) In the local setting, we simply construct the Weil representation

$$\Omega_{V,W,\psi} \coloneqq \omega_{\Xi,\psi}\circ \mathsf{i}^{\bullet;\bullet}_{V,W,\psi}$$

of $\mathrm{O}(V)\times\mathrm{Sp}(W)$ via pulling back the Weil representation of $\mathrm{Mp}(\Xi)$.

(2) In the global setting, one obtains by restriction of functions an equivariant map

$$\Omega_{V,W,\psi} = \mathcal{S}(\Xi^\dagger_{\mathbb{A}_F}) \xrightarrow{\ \theta\ } \mathcal{A}(\mathrm{Mp}(\Xi)) \xrightarrow{\mathsf{i}^{\bullet;\bullet,*}_{V,W,\psi}} \mathcal{C}([\mathrm{O}(V)\times\mathrm{Sp}(W)]),$$

where the target equals to the space of smooth functions on $[\mathrm{O}(V)]\times[\mathrm{Sp}(W)] = \mathrm{O}(V_F)\backslash\mathrm{O}(V_{\mathbb{A}_F})\times\mathrm{Sp}(W_F)\backslash\mathrm{Sp}(W_{\mathbb{A}_F})$.

Remark 2.11 (Anomaly, cf. [GW23, Remark 3.7]). Note that the Weil representation is not a representation of $\mathrm{Sp}(\Xi)$ but of the metaplectic cover $\mathrm{Mp}(\Xi)$. In the language of [BZSV], this is because Ξ has *anomaly* (which can be detected via Betti or étale cohomology), and anomalous varieties are at present excluded from the expectations of the duality of hyperspherical varieties. This phenomenon is one of the deep mysteries of nature.

2.2.4. *Quantization of Hamiltonian reduction.* Since there would be some natural obstruction to quantizing the Poisson variety $\mathfrak{g}^*$, the strategy to determine the quantization of a Hamiltonian reduction $M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}$ (cf. Definition 2.6) is to work with the space of functions. It follows from the same philosophy in algebraic geometry that motivates us to consider sheaves on schemes.

Given the Hamiltonian G-variety (M,ω) we write $\mathcal{H}(M,\omega) = (\rho_M, \mathsf{V}_M)$ as the resulting quantization along the functor $\mathcal{H}$. Here ρ_M is the unitary representation landing in $\mathrm{Herm}(\mathsf{V}_M)$. Indeed, ρ_M is exactly the quantization of $\mu^*\colon \mathbb{F}^\infty(\mathfrak{g}^*)\to\mathbb{F}^\infty(M)$, which is the pullback of the moment map $\mu\colon M\to\mathfrak{g}^*$. Consider that, under the equivariant G-actions,

$$\mathbb{F}^\infty(M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}) = \mathbb{F}^\infty((M\times_{\mathfrak{g}^*}\mathcal{O})/\!\!/\mathsf{G}) = \mathbb{F}^\infty(M\times_{\mathfrak{g}^*}\mathcal{O})^{\mathsf{G}} = (\mathbb{F}^\infty(M)\otimes_{\mathbb{F}^\infty(\mathfrak{g}^*)}\mathbb{F}^\infty(\mathcal{O}))^{\mathsf{G}}.$$

If $\mathsf{V}_{M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}}$ is the Hilbert space quantizing $M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}$, then one must have $\mathrm{Herm}(\mathsf{V}_{M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}}) = \mathcal{H}(\mathbb{F}^\infty(M/\!\!/\!\!/_{\mathcal{O}}\mathsf{G}))$. Then we take the following heuristic computation (in which we have used the duality of vector spaces liberally):

$$\begin{aligned}\mathcal{H}(\mathbb{F}^\infty(M/\!\!/\!\!/_{\mathcal{O}}\,\mathsf{G})) &= \mathcal{H}((\mathbb{F}^\infty(M)\otimes_{\mathbb{F}^\infty(\mathfrak{g}^*)}\mathbb{F}^\infty(\mathcal{O}))^{\mathsf{G}})\\ &= (\mathrm{Herm}(\mathtt{V}_M)\otimes_{\mathcal{H}(\mathbb{F}^\infty(\mathfrak{g}^*))}\mathrm{Herm}(\mathtt{V}_{\mathcal{O}}))^{\mathsf{G}}\\ &= (\mathtt{V}_M^*\otimes(\mathtt{V}_M\otimes\mathtt{V}_{\mathcal{O}}^*)_{\mathsf{G}}\otimes\mathtt{V}_{\mathcal{O}})^{\mathsf{G}}\\ &= \mathrm{Hom}_{\mathsf{G}}(\mathtt{V}_{\mathcal{O}}^*\otimes\mathtt{V}_M,(\mathtt{V}_M\otimes\mathtt{V}_{\mathcal{O}}^*)_{\mathsf{G}})\\ &= \mathrm{Herm}((\mathtt{V}_M\otimes\mathtt{V}_{\mathcal{O}}^*)_{\mathsf{G}}).\end{aligned}$$

Hence, we arrive at the conclusion: the quantization of the Hamiltonian reduction $(M\times_{\mathfrak{g}^*}\mathcal{O})/\!\!/\mathsf{G}$ is

$$\mathcal{H}(M/\!\!/\!\!/_{\mathcal{O}}\,\mathsf{G}) = (\mathtt{V}_M\otimes\mathtt{V}_{\mathcal{O}}^*)_{\mathsf{G}}.$$

This space is essentially the (dual of the) multiplicity space of the irreducible representation $\mathtt{V}_{\mathcal{O}}$ in $\mathtt{V}_M$.

2.3. **Theta correspondence via quantization of Hamiltonian reduction.** Recall from §1.3 that the relative Langlands duality hopefully encompasses both Howe duality (Theorem 1.8) and Adams' conjecture (Conjecture 1.9). However, the theory of theta correspondence does not arise from a spherical variety and so does not really fit into the framework of Adams' conjecture. But it does not go very far from these.

We use the same notations as in §2.2.3 and copy the following context from [Gan23, §10.4, §11.9]. Write $\mathfrak{o}(V) = \mathrm{Lie}\,\mathrm{O}(V)$ and pick a coadjoint orbit $\mathcal{O}\subset\mathfrak{o}(V)^*$. This corresponds to choosing an (irreducible) representation π of $\mathrm{O}(2m)$. The Hamiltonian reduction

$$M/\!\!/\!\!/_{\mathcal{O}}\,\mathsf{G} = (M\times_{\mathfrak{o}(V)^*}\mathcal{O})/\!\!/\,\mathrm{O}(V)$$

amounts to extracting the multiplicity space of the π-isotropic component of the Weil representation $\Omega_{V,W,\psi}$. As one has the commuting action of $\mathrm{Sp}(W)$, $M/\!\!/\!\!/_{\mathcal{O}}\,\mathsf{G}$ is a symplectic $\mathrm{Sp}(W)$-variety, whose quantization is the multiplicity space

$$\Theta(\pi) = (\Omega_{V,W,\psi}\otimes\pi^\vee)_{\mathrm{O}(V)},$$

namely the big theta lifting of π (see Construction 1.6).

2.4. **Hamiltonian induction.** Let $\mathsf{H}\subset\mathsf{G}$ be a subgroup. Hamiltonian induction will take a Hamiltonian H-space to a Hamiltonian G-space. Under geometric quantizations, Hamiltonian induction corresponds to the (L^2-)induction of representations.

Definition 2.12 (Hamiltonian induction)**.** Let S be a Hamiltonian H-space. The Hamiltonian induction is defined as

$$\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S) := (S\times T^*\mathsf{G})/\!\!/\!\!/\,\mathsf{H}.$$

Here $T^*\mathsf{G}$ is considered as a Hamiltonian H-space via the left multiplication. The G-action on $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ is induced from the right multiplication on $T^*\mathsf{G}$.

We can rewrite the Hamiltonian induction as

$$\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S) = S\times_{\mathfrak{h}^*}^{\mathsf{H}} T^*\mathsf{G}.$$

Using the identification $T^*\mathsf{G} = \mathfrak{g}^*\times\mathsf{G}$ from Example 2.3 (4), we can further rewrite the Hamiltonian induction as

$$\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S) = (S\times_{\mathfrak{h}^*}\mathfrak{g}^*)\times^{\mathsf{H}}\mathsf{G}$$

where H acts on $\mathfrak{g}^*$ via $h\colon \xi\mapsto \mathrm{ad}(h^{-1})\xi$. The moment map on $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ is induced from $\mathfrak{g}^*\times^{\mathsf{H}}\mathsf{G}\to\mathfrak{g}^*$. We see that $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ is a fiber bundle over $\mathsf{H}\backslash\mathsf{G}$.

If S is a graded Hamiltonian H-space, we can endow $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ with a natural grading using the diagonal action of $\mathbb{G}_{gr}$ on $S\times T^*\mathsf{G}$.

2.4.1. *Frobenius reciprocity.* There is a "Frobenius reciprocity" for Hamiltonian inductions, parallel to the usual Frobenius reciprocity in representation theory. It should take place in the category of Hamiltonian spaces and Lagrangian correspondences. For two symplectic spaces M and N, a *Lagrangian correspondence* from M to N is a correspondence

$$M^\circ \longleftarrow L \longrightarrow N,$$

where M° is the variety M equipped with opposite symplectic form, and $L \hookrightarrow M^\circ \times N$ is a Lagrangian subspace. The composition of two Lagrangian correspondence should be taken as a fiber product. However, due to the issues of non-transversal intersections, we should really consider the category of shifted symplectic spaces in the derived world. We shall not go into this.

Let $\mathsf{H} \subset \mathsf{G}$ be a subgroup and S be a Hamiltonian H-space. Denote $L := S \times_{\mathfrak{h}^*} \mathfrak{g}^*$. Then L embeds into $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ as the fiber over $0 \subset \mathsf{H}\backslash\mathsf{G}$. Note that the restriction of the symplectic form on $\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$ to L is equal to the pullback of the symplectic form on S to L. It follows (by dimension counting) that

$$\iota_S\colon (\text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S))^\circ \longleftarrow L \longrightarrow S$$

is a Lagrangian correspondence. This Lagrangian correspondence is H-equivariant and compatible with the moment map in the sense that the following two compositions are equal:

$$L \longrightarrow \text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S) \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{h}^* \quad \text{and} \quad L \longrightarrow S \longrightarrow \mathfrak{h}^*.$$

Now let M be a Hamiltonian G-space with an H-equivariant Lagrangian correspondence

$$\alpha\colon M^\circ \longleftarrow L \longrightarrow S$$

compatible with moment maps in the above sense. Then there is a natural G-equivariant Lagrangian correspondence

$$\beta\colon M^\circ \longleftarrow L \times^{\mathsf{H}} \mathsf{G} \longrightarrow \text{h-ind}_{\mathsf{H}}^{\mathsf{G}}(S)$$

compatible with moment maps such that $\alpha = \beta \circ \iota_S$.

3. Structure theory of relative Langlands duality

We still work over $\mathbb{F}$, an algebraically closed field of characteristic 0.

3.1. **Hyperspherical Hamiltonian G-spaces.**

We need first to describe the coisotropic condition before defining the hyperspherical G-varieties.

3.1.1. *Coisotropicity and multiplicity-freeness.*

Definition 3.1 (Coisotropicity, cf. [BZSV, §3.5.1])**.**

(1) A symplectic G-variety (M, ω) is *coisotropic* if the field $\mathbb{F}(M)^{\mathsf{G}}$ of G-invariant rational functions on M is commutative with respect to the Poisson bracket.

(2) A G-action on the symplectic G-variety M is *coisotropic* if there exists an open dense subset $U \subset M$ with $\mathsf{G}u$ being a coisotropic subvariety for every $u \in U$.

Lemma 3.2. *A homogeneous Hamiltonian G-variety (M, ω) is coisotropic if and only if it has a coisotropic G-action.*

Proof. We use the following fact: One can check by definition of the Poisson bracket that Definition 3.1 (1) is equivalent to

$$(T_xM)^{\perp,\omega} := \{v \in T_xM : \omega(v, T_x(\mathsf{G}x)) = 0\} \subset T_xM, \quad \forall x \in M,$$

where the left-hand side is namely the orthogonal subspace of T_xM with respect to ω. Working over $\mathbb{R}$ without loss of generality, we define for every $z \in \mathfrak{g}$ the map $f_z\colon M \to \mathbb{R}$ via $f_z(x) = \mu(x)(z)$, where $\mu\colon M \to \mathfrak{g}^*$ is the moment map. Note that given any $f \in \mathbb{F}(M)^{\mathsf{G}}$, we have

$$\{f, f_z\} = \omega(\mathfrak{X}_f, \mathfrak{X}_{f_z}) = 0, \quad \forall z \in \mathfrak{g}.$$

Assume M carries a coisotropic G-action, and there is thus a generic orbit $\mathsf{G}\eta$ for some $\eta \in M$ being coisotropic. Therefore, for each $z \in \mathfrak{g}$ and $f_1, f_2 \in \mathbb{F}(M)^{\mathsf{G}}$, the vanishing $\{f_1, f_{1,z}\} = \{f_2, f_{2,z}\} = 0$ corresponds to $\mathfrak{X}_{f_1}, \mathfrak{X}_{f_2} \in (T_\eta \mathsf{G}\eta)^{\perp,\omega} \subset T_\eta \mathsf{G}\eta$. It follows that $\{f_1, f_2\}(u) = \omega(\mathfrak{X}_{f_1}, \mathfrak{X}_{f_2})(u) = 0$ as functions on M, where $u \in U$ runs through the open dense subset implicated in Definition 3.1 (2). Consequently, $\mathbb{F}(M)^{\mathsf{G}}$ is Poisson-commutative.

Conversely, assume M is a coisotropic G-variety. Given a regular point $x \in M$, there is an open neighborhood W of $\mathsf{G}x$ together with finitely many functions $f_1, \dots, f_k \in \mathbb{F}(M)^{\mathsf{G}}$, satisfying $\mathrm{d}f_1 \wedge \dots \wedge \mathrm{d}f_k \neq 0$ and

$$\mathsf{G}x = \{w \in W \colon f_1(w) = \dots = f_k(w) = 0\}.$$

Note that a priori $\mathfrak{X}_{f_i} \in (T_x M)^{\perp,\omega}$ for each $i = 1, \dots, k$. On the other hand, the assumption leads to $\{f_i, f_j\} = 0$, and hence $\mathfrak{X}_{f_i} \in T_x M$. Since all $\mathfrak{X}_{f_i}$ are independent in W, we see $\mathsf{G}x$ is a coisotropic subvariety as desired. □

Lemma 3.3. *Let X be a normal G-variety. Then X is spherical if and only if $\mathbb{F}(X)^{\mathsf{B}} = \mathbb{F}$, namely any B-invariant rational function on X is constant.*

Proof. If X is spherical then by definition $\mathbb{F}(X)^{\mathsf{B}}$ is a multiplicity free nontrivial G-module, and hence $\mathbb{F}(X)^{\mathsf{B}} = \mathbb{F}$. Conversely, we suppose that $\mathbb{F}(X)^{\mathsf{B}} = \mathbb{F}$. By a theorem of Rosenlicht [Gro97, Theorem 19.5] (cf. [Gan18, Theorem 2.8]), B-orbits in general position can be separated by B-invariant functions, that is, there exists a B-stable affine open subset $U \subset X$ such that for all $x, y \in U$ with $\mathsf{B}x \neq \mathsf{B}y$ there exists $f \in \mathbb{F}(U)^{\mathsf{B}}$ such that $f(x) \neq 0$ and $f(y) = 0$. On the other hand, f must be a constant; therefore, U is a single B-orbit. This completes the proof that X is spherical. □

Notice that the coisotropic condition in Definition 3.1 concerns dynamics on a symplectic manifold. On the other hand, the multiplicity-free condition in Definition 1.4 has a representation-theoretic flavor. It turns out that these two conditions are strongly related. The upcoming result can serve as a specific bridge *connecting symplectic geometry and representation theory.* The proof is adapted from some ingredients of [Gan18, §2].

Proposition 3.4. *Suppose the homogeneous Hamiltonian G-variety (M, ω) satisfies the multiplicity-free condition in Definition 1.4. Then M is coisotropic. In particular, if X is a spherical variety then T^*X is coisotropic.*

Proof. Since M is a homogeneous space, it is quasi-affine, and $\mathbb{F}(M)$ is the fractional field of coordinate ring $\mathbb{F}[M]$. Let $f = p/q \in \mathbb{F}(M)^{\mathsf{B}}$ for $p, q \in \mathbb{F}[M]$. We may assume p, q lie in $F[M]^{(\mathsf{B})}$, the subspace of B-eigenfunctions. Let $\mathbb{V}$ be the subspace of $\mathbb{F}[M]$ generated by the B-orbit of q, then $\mathbb{V}$ is finite-dimensional (see [PV94, Lemma 1.4] for the detailed reason). Since it is B-stable, Lie–Kolchin theorem (see [Spr98, Theorem 6.3.1] for example) dictates that it contains a B-eigenvector q'. Write $q' = \sum_i \xi_i (b_i.q)$ with $\xi_i \in \mathbb{F}$ and $b_i \in \mathsf{B}$. Denote $p' = \sum_i \xi_i (b_i.p)$. Then $f = b_i.f = (b_i.p)/(b_i.q)$ for all i, and hence $f = p'/q'$. It follows that $p' \in \mathbb{F}[M]^{\mathsf{B}}$ as well. Clearly, p' and q' have the same weight, hence they are proportional because $\mathbb{F}[M]$ is multiplicity-free by assumption, and f is a constant. Therefore, $\mathbb{F}(M)^{\mathsf{B}} = \mathbb{F}$ and the Poisson-commutativity of $\mathbb{F}(M)^{\mathsf{G}}$ follows directly. This completes the proof. □

Remark 3.5. One may naturally expect a converse of Proposition 3.4, which implies that if M is coisotropic then it is multiplicity-free. However, it fails to be valid in general, and the converse result appears to be a more complicated description in [BZSV, Proposition 3.6.3], using the language of distinguished polarization and Hamiltonian induction. In fact, $M = T^*X$ admits a distinguished polarization and we have the equivalence between coisotropicity and multiplicity-freeness for M.

Proposition 3.6. *Suppose X is a smooth quasi-affine G-variety. Then X is spherical if and only if T^*X is coisotropic.*

Proof. The proof is suggested by Zeyu Wang. By Proposition 3.4, assuming T^*X is coisotropic, it suffices to prove that X is spherical. Denote $\mu : T^*X \to \mathfrak{g}^*$ for the moment map. Fix $\mathsf{A} \subset \mathsf{B} \subset \mathsf{G}$ to be a maximal torus and a Borel subgroup of G. Denote $\mathbb{F}(X)^{(\mathsf{B})} \subset \mathbb{F}(X)$ to be the B-eigenfunctions. For each $f \in \mathbb{F}(X)^{(\mathsf{B})}$, denote $\chi_f \in X^*(\mathsf{A})$ to be the corresponding eigencharacter. Denote $X^*(\mathsf{A}_X) \subset X^*(\mathsf{A})$ to be the sublattice generated by χ_f for all $f \in \mathbb{F}(X)^{(\mathsf{B})}$, which defines a toric quotient $\mathsf{A} \twoheadrightarrow \mathsf{A}_X$. Let $\mathsf{P}(X)$ be the stabilizer of general B-orbits in X with its unipotent radical $\mathsf{U}(X)$. Write $\mathfrak{a}_X = \operatorname{Lie} \mathsf{A}_X$, $\mathfrak{a} = \operatorname{Lie} \mathsf{A}$, and $\mathfrak{u}(X) = \operatorname{Lie} \mathsf{U}(X)$. Define W_X to be the little Weyl group in the sense of [Kno94, §3], which is a subquotient of the Weyl group W of G. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of an invariant scalar product. Choose $f_1, \dots, f_n \in \mathbb{F}(X)^{(\mathsf{B})} \subset \mathbb{F}(X)^{\mathsf{U}}$ to be a transcendental basis of $\mathbb{F}(X)^{\mathsf{U}}/\mathbb{F}$ and consider maps

$$\begin{tikzcd}
\mathbb{A}^n \times X^\circ \arrow[r, "s"] \arrow[dr] & T^*X \arrow[r, "\mu"] \arrow[d] \arrow[dr] & \mathfrak{g}^* \arrow[dr, "\chi"] & \\
& T^*X /\!\!/ \mathsf{G} \arrow[r, "\varphi"] & \mathfrak{a}_X^* /\!\!/ W_X \arrow[r] & \mathfrak{a}^* /\!\!/ W
\end{tikzcd}$$

in which s is defined on an open subset $X^\circ \subset X$ by sending $((a_i), x)$ to $\sum_{i=1}^n a_i \cdot (df_i)_x / f_i(x)$.

Our goal is to construct an open B-orbit in X. By (a variant of) [BZSV, Proposition 5.3], the condition T^*X being coisotropic is equivalent to that the generic fiber of $\chi \circ \mu$ contains an open G-orbit. On the other hand, [Kno94, Lemma 3.4] dictates that the generic fiber of φ is connected. Therefore, the generic fiber of φ is a G-orbit. Also, $\mu \circ s$ factors through $\mathfrak{u}^\perp \subset \mathfrak{g}^*$.

Without loss of generality, we may assume that $\chi_{f_1}, \dots, \chi_{f_m}$ form a basis of $X^*(\mathsf{A}_X)_{\mathbb{Q}}$ for some $m \leqslant n$. It follows that the composition map $\mathbb{A}^m \times X \to T^*X \to \mathfrak{a}_X^* /\!\!/ W_X$ is dominant. Consequently, $\mathsf{G} \cdot s(\mathbb{A}^m \times X^\circ)$ is dense in T^*X. We claim that $\mathsf{B} \cdot s(\mathbb{A}^m \times X^\circ)$ is open in $\mu^{-1}(\mathfrak{u}^\perp)$. In fact, by our choice of $\mathsf{P}(X)$, f_i are also eigenvectors of $\mathsf{P}(X)$. Therefore, we know $\mathsf{B} \cdot s(\mathbb{A}^m \times X^\circ) = \mathsf{P}(X) \cdot s(\mathbb{A}^m \times X^\circ)$ hence $\operatorname{codim} \mathsf{B} \cdot s(\mathbb{A}^m \times X^\circ) \leqslant \dim \mathsf{U}(X)$. On the other hand, [Kno94, Theorem 2.3] implies that $\mathsf{U}(X)$ acts freely on general U-orbits of X hence we know $\operatorname{codim} \mu^{-1}(\mathfrak{u}^\perp) = \dim \mathsf{U}(X)$. Combining these, the claim is proved.

If we replace X by $X /\!\!/ \mathsf{U}$ in the result above, we see

$$\mathsf{A} \cdot s(\mathbb{A}^m \times (X^\circ /\!\!/ \mathsf{U})) \subset T^*(X /\!\!/ \mathsf{U})$$

is a open subset, which implies that $X /\!\!/ \mathsf{U}$ has an open dense A-orbit. By definition, $X /\!\!/ \mathsf{U}$ is a toric variety and therefore X is spherical. ☐

3.1.2. *Hyperspherical* G*-varieties.* As we have claimed before, the main player of the relative Langlands duality is the class of Hamiltonian G-varieties over $\mathbb{F}$, satisfying the *hyperspherical* condition as follows.

Definition 3.7. A *hyperspherical* G*-variety* is a Hamiltonian G-variety M such that:

(i) M is smooth and affine, equipped with a grading (see Definition 2.4) via a commuting $\mathbb{G}_m$-action;
(ii) M further satisfies several technical conditions as mentioned in [BZSV, §3.5.1];
(iii) M is coisotropic (rather than multiplicity-free, cf. §3.1.1).

We emphasize that condition (iii) is the punchline of Definition 3.7. The relation between coisotropicity and multiplicity-freeness in Proposition 3.4 and Remark 3.5 dictates that a hyperspherical G-variety M can be regarded as an object arising from representation theory as well as that from symplectic geometry.

3.2. **Whittaker induction.** We introduce the operation of Whittaker inductions for Hamiltonian spaces, which are roughly "twisted Hamiltonian inductions". Given the data of a subgroup $\mathsf{H} \subset \mathsf{G}$ and a homomorphism $\mathrm{SL}_2 \to \mathsf{G}$ that commutes with H, the Whittaker induction takes (graded) Hamiltonian H-spaces to (graded) Hamiltonian G-spaces. In the following, we introduce several possible motivations to consider Whittaker inductions.

- ◇ The graded structure of the Hamiltonian G-space output by Hamiltonian induction is inappropriate for the purpose of relative Langlands duality, and a modification called *shearing* is thus in need. The procedure of Whittaker induction is partially defined by the shearing.
- ◇ In usual practice, a Hamiltonian H-space M is possibly identified with certain symplectic H-representation S_M; in this case, the Whittaker induction is simply a vector bundle over $\mathsf{H}\backslash\mathsf{G}$.
- ◇ More generally, the Tannakian duality $M^\vee$ of M conjecturally assembles all the dual data and turns out to be a Whittaker induction of S_M. Further, $M^\vee$ defines the correct L-sheaf $\mathcal{L}_{M^\vee}$ as desired.

Before giving the definition of Whittaker inductions, we recall the parallel stories of Whittaker models for representations. Fix a G-invariant perfect pairing κ on $\mathfrak{g}$. Let $\gamma = \{e, h, f\} \subset \mathfrak{g}$ be the $\mathfrak{sl}_2$-triple defined by the morphism $\mathrm{SL}_2 \to \mathsf{G}$. Denote M_γ the centralizer of γ in G.

3.2.1. *Generalized Whittaker model.* Let F be a non-archimedean local field. Assume G, H are defined over F. We write $G = \mathsf{G}(F)$ and $H = \mathsf{H}(F)$ for the locally compact topological groups of F-points. Fix a nontrivial unitary character $\psi\colon F \to \mathbb{C}^\times$.

The Lie algebra $\mathfrak{g}$ decomposes with respect to the adjoint h-action

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j.$$

Note that $e \in \mathfrak{g}_2$ and $f \in \mathfrak{g}_{-2}$. Write

$$\mathfrak{u} := \bigoplus_{j>0} \mathfrak{g}_j, \quad \mathfrak{l} := \mathfrak{g}_0, \quad \overline{\mathfrak{u}} := \bigoplus_{j<0} \mathfrak{g}_j, \quad \mathfrak{p} := \bigoplus_{j\geqslant 0} \mathfrak{g}_j = \mathfrak{l} \oplus \mathfrak{u}.$$

Let P be the parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Write $\mathsf{P} = \mathsf{LU}$ for the Levi decomposition corresponding to $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Denote also

$$\mathfrak{u}_+ = \bigoplus_{j\geqslant 2} \mathfrak{g}_j.$$

It integrates to a unipotent subgroup U_+ of G. We call the $\mathfrak{sl}_2$-triple γ *even* if $\mathfrak{u} = \mathfrak{u}_+$, or equivalently all weights of $\mathfrak{g}$ are even, or equivalently the image of $-1 \in \mathrm{SL}_2$ in G is central.

Assume γ is even. Write $\kappa_f\colon \mathfrak{u}_+ \to \mathbb{G}_a$ for the additive character $u \mapsto \kappa(f, u)$, where $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ is the fixed isomorphism as before. Write $U = \mathsf{U}(F)$ and $U_+ = \mathsf{U}_+(F)$. There is a natural character

$$\begin{aligned} \chi_{\gamma,\psi}\colon U_+ &\longrightarrow \mathbb{C}^\times \\ \exp(u) &\longmapsto \psi(\kappa_f(u)) \end{aligned}$$

for all $u \in \mathfrak{u}_+(F)$.

Definition 3.8 (Whittaker model)**.** Assume γ is even. The *Whittaker representation* is the induced representation

$$W_{\gamma,\psi} := \mathrm{Ind}_{HU}^{G}(\chi_{\gamma,\psi}).$$

For π an irreducible smooth admissible representation of G, the *Whittaker model* of π is the space

$$W_{\gamma,\psi}(\pi) := \mathrm{Hom}_G(W_{\gamma,\psi}, \pi^\vee) = \mathrm{Hom}_{HU}(\pi, \chi_{\gamma,\psi}).$$

If γ is not even (i.e. $\mathfrak{g} \neq 0$), by $\mathfrak{sl}_2$-theory, there would be an isomorphism $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}$ induced by the $\mathrm{ad}(f)$-action. It induces a H-equivariant symplectic form κ_1 on $\mathfrak{g}_1$ by the formula

$$\kappa_1(x, y) = \kappa(\mathrm{ad}(f)x, y) = \kappa(f, [x, y]).$$

Consider the Heisenberg group $H_\gamma = \mathfrak{g}_1 \times F$ with multiplication $(x, 0)\cdot(y, 0) = (x+y, \frac{1}{2}\kappa_1(x, y))$. There is a group homomorphism

$$\begin{array}{rcl} U & \xrightarrow{\quad \alpha_\gamma \quad} & H_\gamma \\ \exp(v)\exp(u) & \longmapsto & (v, \kappa_f(u)) \end{array}$$

where $v \in \mathfrak{g}_1$ and $u \in \mathfrak{u}_+$, realizing H_γ as a quotient U/U', where $U' = \exp(\mathrm{Ker}(\kappa_f|_{\mathfrak{u}_+}))$.

Let ω_ψ be the unique smooth irreducible unitary representation of H_γ with central character ψ. Therefore U acts on ω_ψ via α_γ. Since κ_1 is stable under H-action, we get an action of $\widetilde{H}$ on ω_ψ, where $\widetilde{H}$ is a metaplectic cover of H. For a genuine representation ρ of $\widetilde{H}$, $\rho \otimes \omega_\psi$ descends to a representation of HU.

Definition 3.9 (Generalized Whittaker model)**.** Define the *generalized Whittaker representation* as

$$W_{\gamma,\rho,\psi} \coloneqq \mathrm{Ind}_{HU}^{G}(\rho \otimes \omega_\psi).$$

For π an irreducible smooth admissible representation of G, the *generalized Whittaker model* of π is the space

$$W_{\gamma,\rho,\psi}(\pi) \coloneqq \mathrm{Hom}_G(W_{\gamma,\rho,\psi}, \pi^\vee) = \mathrm{Hom}_{HU}(\pi, \rho \otimes \omega_\psi).$$

In the even case, $\omega_\psi = \chi_{\gamma,\psi}$, and we can take $\rho = 1$. Then we recover the Whittaker model in Definition 3.8.

3.2.2. *Construction of Whittaker inductions.*

Construction 3.10. By the discussion in the previous section, we know that the quotient vector space $\mathfrak{u}/\mathfrak{u}_+$ is endowed with an H-invariant symplectic form κ_1. We define the Hamiltonian HU-space $(\mathfrak{u}/\mathfrak{u}_+)_f$ as follows:

- The underlying symplectic space is $(\mathfrak{u}/\mathfrak{u}_+, \kappa_1)$.
- H acts on $(\mathfrak{u}/\mathfrak{u}_+)_f$ by the adjoint action.
- U acts by translation on $\mathfrak{u}/\mathfrak{u}_+ = \mathsf{U}/\mathsf{U}_+$.
- The moment map for H is as in Example 2.3 (2).
- The moment map for U is the shift-by-f map

$$\mathfrak{u}/\mathfrak{u}_+ \cong (\mathfrak{u}/\mathfrak{u}_+)^* \xrightarrow{\xi \mapsto \xi + f} \mathfrak{u}^*.$$

Now we can define the Whittaker induction.

Definition 3.11. Let S be a Hamiltonian H-space. The Whittaker induction of S via $\mathsf{H} \times \mathrm{SL}_2 \to \mathsf{G}$ is the Hamiltonian induction

$$M \coloneqq \text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(\widetilde{S}),$$

where $\widetilde{S}$ is the Hamiltonian HU-space $\widetilde{S} \coloneqq S \times (\mathfrak{u}/\mathfrak{u}_+)_f$.

Comparing with Definition 3.9, we see that the Whittaker induction corresponds precisely to the generalized Whittaker model under geometric quantization:

- The Hamiltonian H-space S corresponds to the representation ρ.
- The Hamiltonian HU-space $(\mathfrak{u}/\mathfrak{u}_+)_f$ corresponds to the Weil representation ω_ψ.
- The Hamiltonian induction corresponds to the usual induction in representation theory.

Example 3.12. Assume that S is trivial and the $\mathfrak{sl}_2$-triple is even. We see that the Whittaker induction is equal to

$$\text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(*_f)$$

where $*_f$ is the trivial Hamiltonian HU-space with moment map sending the point to $\kappa_f \in (\mathfrak{h} + \mathfrak{u})^*$. In particular, we recover the Whittaker space in Example 2.3 (3) if H is trivial and γ is a principal $\mathfrak{sl}_2$.

3.2.3. *Grading on a Whittaker induction.* We define the natural $\mathbb{G}_{gr}$-action on the Whittaker induction provided that the Hamiltonian H-space is graded. We can write the Whittaker induction as

$$M = (S \times (\mathfrak{u}/\mathfrak{u}_+)_f) \times^{\mathsf{HU}}_{(\mathfrak{h}+\mathfrak{u})^*} (\mathfrak{g}^* \times \mathsf{G}).$$

Let $\varpi\colon \mathbb{G}_m \to \mathsf{G}$ be the cocharacter $\lambda \mapsto \lambda^h$, where $h \in \mathfrak{g}$ is the element in the given $\mathfrak{sl}_2$-triple. The $\mathbb{G}_{gr}$-action on the Whittaker induction can be defined as follows:

- $\mathbb{G}_{gr}$ acts on S via the given grading.
- $\mathbb{G}_{gr}$ acts by scalar multiplication on $(\mathfrak{u}/\mathfrak{u}_+)_f$.
- $\mathbb{G}_{gr}$ acts on $\mathfrak{g}^*$ via the composition of the square character and the left coadjoint action of ϖ on $\mathfrak{g}^*$.
- $\mathbb{G}_{gr}$ acts on G via left multiplication by ϖ.

One can check that this defines a grading on M. There is also a more conceptual definition of the grading via *shearing*.

Definition 3.13 (Sheared Hamiltonian spaces)**.** Let $\varpi\colon \mathbb{G}_{gr} \to \mathrm{Aut}(\mathsf{G})$ be a homomorphism, i.e. G is a graded group. A Hamiltonian G-space M is *sheared* if the there is a $\mathbb{G}_{gr}$-action on M that is compatible with the grading on G and the moment map $\mu\colon M \to \mathfrak{g}^*$ is $\mathbb{G}_{gr}$-equivariant, where the grading on $\mathfrak{g}^*$ is given by the composition of the square character and the ϖ-action.

Example 3.14. We collect the following examples of sheared Hamiltonian spaces.

1. (1) If $\varpi\colon \mathbb{G}_{gr} \to \mathrm{Aut}(\mathsf{G})$ is trivial, then a sheared Hamiltonian G-space is equivalent to a graded Hamiltonian G-space.
2. (2) Consider $*$ as a $\mathbb{G}_a$-space. Let $\mathbb{G}_{gr}$ act on $\mathbb{G}_a$ by the square character. Define the moment map $* \mapsto 1 \in \mathfrak{g}_a^*$. The resulting Hamiltonian $\mathbb{G}_a$-space, denoted by $*_1$, is sheared.
3. (3) Let W be a symplectic vector space. Let $\mathsf{H} = W \ltimes \mathbb{G}_a$ be the Heisenberg group. Let $\mathbb{G}_{gr}$ act on W by scalar multiplication and act on $\mathbb{G}_a$ by the square character; this defines a grading on H. Suppose H acts on a symplectic space W by translation. The morphism
$$W \longrightarrow W \oplus \mathfrak{g}_a^* \cong W^* \oplus \mathfrak{g}_a^* = \mathrm{Lie}(\mathsf{H})^*, \quad x \longmapsto (x, 1)$$
defines a moment map for this action. Then W with scalar action by $\mathbb{G}_{gr}$ is a sheared Hamiltonian H-space.
4. (4) Let notations be as in Section 3.2.2. By the above example, $(\mathfrak{u}/\mathfrak{u}_+)_f$ is a sheared Hamiltonian HU-space. Here the grading on H is trivial and the grading on U is defined through the adjoint action by $(\lambda \mapsto \lambda^h)\colon \mathbb{G}_m \to \mathsf{G}$.
5. (5) Assume the grading on G is given by the right adjoint action by a cocharacter $\varpi\colon \mathbb{G}_m \to \mathsf{G}$. Then graded Hamiltonian G-spaces are equivalent to sheared Hamiltonian G-spaces via a twisting of gradings. Let M be a graded Hamiltonian G-space. We obtain a sheared Hamiltonian G-space by altering the grading with the action of ϖ on M.

Now the process of Whittaker induction can be depicted as follows:

$$\begin{tikzcd}
\boxed{\text{Graded Hamiltonian } \mathsf{H}\text{-spaces}} \arrow[r, "\times(\mathfrak{u}/\mathfrak{u}_+)_f"] \arrow[d] & \text{Sheared Hamiltonian } \mathsf{HU}\text{-spaces} \arrow[d, "\text{h-ind}^{\mathsf{G}}_{\mathsf{HU}}"] \\
\boxed{\text{Graded Hamiltonian } \mathsf{G}\text{-spaces}} & \text{Sheared Hamiltonian } \mathsf{G}\text{-spaces} \arrow[l]
\end{tikzcd}$$

where the lower horizontal arrow is the inverse operation of Example 3.14 (5).

3.2.4. *Vectorial cases.* We will see later that all the Hyperspherical variety are Whittaker induction from a symplectic H-vector space. In this case, the geometry of the Whittaker induction is simple. It can be realized as a vector bundle over $\mathsf{H}\backslash\mathsf{G}$.

Lemma 3.15 (Slodowy slices)**.** *There is an isomorphism*

$$\mathsf{U} \times (f + \mathfrak{g}_e) \xrightarrow{\sim} f + \mathfrak{u}_+^\perp$$

via the adjoint action of U *on* $f + \mathfrak{g}_e$*. Here* $\mathfrak{g}_e$ *is the centralizer of* e*.*

Proof. We sketch a proof of this lemma. It follows from $\mathfrak{sl}_2$-theory that $[\mathfrak{u}, f] \cap \mathfrak{g}_e = 0$ and the map $\mathfrak{u} \to [\mathfrak{u}, f]$ via $x \mapsto [x, f]$ is an isomorphism. A dimension counting shows that $\mathfrak{u}_+^\perp = [\mathfrak{u}, f] \oplus \mathfrak{g}_e$. Hence the action map

$$\alpha\colon \mathsf{U} \times (f + \mathfrak{g}_e) \to f + \mathfrak{u}_+^\perp$$

induces an isomorphism on the tangent space at $(1, f)$. Consider the action ρ of $\mathbb{G}_m$ on $\mathfrak{g}$ given by $\rho(\lambda)(x) = \lambda^2 \mathrm{ad}(\lambda^h)(x)$. The ρ-action stabilizes $f + \mathfrak{g}_e$ and $f + \mathfrak{u}_+^\perp$ and contracts them to f. Let $\mathbb{G}_m$ acts on U via conjugation by $\lambda \mapsto \lambda^h$. Then α is $\mathbb{G}_m$-equivariant and the $\mathbb{G}_m$-actions contracts both spaces to a point. Moreover, α induces an isomorphism on tangent spaces at contraction points. Now it follows from a geometric result that such a map is an isomorphism. □

3.2.5. *Simplifying Whittaker induction.* We note that

$$(S \times (\mathfrak{u}/\mathfrak{u}_+)_f) \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^* = \{(s, x) \in S \times \mathfrak{g}^* \colon \mu(s)|_{\mathfrak{h}} = x|_{\mathfrak{h}},\ x|_{\mathfrak{u}_+} = \kappa_f\}.$$

The latter condition is equivalent to $x \in f + \mathfrak{u}_+^\perp$. If follows that

$$(S \times (\mathfrak{u}/\mathfrak{u}_+)_f) \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^* = S \times_{\mathfrak{h}^*} (f + \mathfrak{u}_+^\perp) \cong (S \times_{\mathfrak{h}^*} (f + \mathfrak{g}_e)) \times \mathsf{U},$$

where the second isomorphism is given by the Slodowy slice and is HU-equivariant. It follows that as G-spaces, we have an isomorphism

$$\text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(S \times (\mathfrak{u}/\mathfrak{u}_+)_f) \cong (S \times_{\mathfrak{h}^*} (f + \mathfrak{g}_e)) \times^{\mathsf{H}} \mathsf{G}.$$

The morphism $f + \mathfrak{g}^e \to \mathfrak{h}^*$ is surjective with fiber isomorphic to $\mathfrak{h}^\perp \cap (f + \mathfrak{g}_e)$. After choosing a $\mathsf{H} \times \mathrm{SL}_2$-eqiuivariant splitting of $\mathfrak{g}^* \to \mathfrak{h}^*$, we obtain an isomorphism

$$S \times_{\mathfrak{h}^*} (f + \mathfrak{g}_e) \cong S \times (\mathfrak{h}^\perp \cap (f + \mathfrak{g}_e)).$$

We see that there is an isomorphism

$$\text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(\widetilde{S}) \cong V \times^{\mathsf{H}} \mathsf{G}, \quad V = S \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}_e)$$

compatible with G-actions, where the right hand side is an vector bundle over $\mathsf{H}\backslash\mathsf{G}$. Moreover, this isomorphism can be made to be compatible with $\mathbb{G}_{gr}$-actions, if we endow V with the $\mathbb{G}_{gr}$-action as follows:

- $\mathbb{G}_{gr}$ acts by scalar multiplication on S.
- $\mathbb{G}_{gr}$ acts with weight $2 + t$ on the weight t part of $\mathfrak{g}_e$.

Example 3.16. Consider the case when H and S are trivial. Assume the SL_2-triple is even. The Whittaker induction is equal to the Whittaker twisted bundle $T^*(\mathsf{U}\backslash\mathsf{G}, \Psi)$ in Example 2.8. By the above discussion, we see that there is an isomorphism

$$T^*(\mathsf{U}\backslash\mathsf{G}, \Psi) \cong \mathsf{G} \times \mathfrak{g}_e$$

of graded G-spaces.

3.3. **The main theorem on hyperspherical structures.** The upcoming main structure theorem states that *hyperspherical* G*-varieties always come from some Whittaker induction.*

Theorem 3.17 ([BZSV, Theorem 3.6.1])**.** *For a hyperspherical Hamiltonian* G*-space* M*, there exist*

- *a reductive subgroup* H *of* G*,*
- *an* SL_2*-action on* G *restricting to* H*, giving rise to a homomorphism* $\iota\colon \mathsf{H} \times \mathrm{SL}_2 \to \mathsf{G}$*, and*
- *a symplectic* H*-vector space* S*,*

such that

$$M \simeq (\text{Whittaker induction of } S \text{ along } \iota).$$

Proof sketch of Theorem 3.17. Recall that a Whittaker induction is roughly a "twisted Hamiltonian induction". The main idea of the proof in [BZSV, §3.6] is natural —

> Compare M with a Hamiltonian induction $\text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(\widetilde{S})$ with some symplectic space $\widetilde{S}$ by using the Lagrangian correspondence. To characterize the H-actions, first detect the weight decomposition of T_xM under $\mathsf{G} \times \mathbb{G}_{gr}$-actions, and then focus on the subspace of weight 1.

We sketch out the proof in a faithful guideline as follows.

Step I (Graded structure of T_xM via weight decomposition)**.** We fix a point $x \in M$ and take $M_0 = \mathsf{G}x$. Note that the image of T_xM_0 along the extended moment map $T_xM \to \mathfrak{g}^*$ is a subalgebra of $\mathfrak{g}^*$. Thus $T_xM_0 = \mathfrak{g}/\mathfrak{h}$ for some subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and $\mathfrak{g}_f = \mathrm{Stab}_{\mathfrak{g}}(f) \subset \mathfrak{g}$ for each $f \in \mathfrak{g}^*$. There is a natural filtration on T_xM:

$$0 \subset T_xM_0 \cap (T_xM_0)^{\perp} = \mathfrak{g}_f/\mathfrak{h} \subset T_xM_0 = \mathfrak{g}/\mathfrak{h} \subset T_xM_0 + (T_xM_0)^{\perp} = (\mathfrak{g}_f/\mathfrak{h})^{\perp} \subset T_xM.$$

Recall from §3.2.3 that we obtain the cocharacter $\varpi_x \colon \mathbb{G}_m \to \mathsf{G}$ via $\lambda^{-h} \mapsto \lambda$, and hence $\mathbb{G}_m$ acts on M by the pair $(\lambda^{-h}, \lambda) \in \mathsf{G} \times \mathbb{G}_{gr}$. Up to the choice of x, we define $M_+ \subset M$ as the subscheme of points that the $\mathbb{G}_m$-action contracts to x. Accordingly,

- The tangent space T_xM_+ admits a weight decomposition via the $\mathsf{G}\times\mathbb{G}_{gr}$-action, in which all direct summands are of positive weights (so that the notation M_+ is reasonable).[3]
- There is a noncanonical isomorphism $T_xM_+ \simeq M_+$ of symplectic G-schemes.

Combining these, we detect the weights of subquotients and subspaces of T_xM as follows:

Sub/Subquotient-spaces	$\mathfrak{g}_f/\mathfrak{h}$	$\mathfrak{g}/\mathfrak{g}_f$	T_xM_0	T_xM/T_xM_0	T_xM_+	$T_xM/(\mathfrak{g}_f/\mathfrak{h})^{\perp}$
Weights	$\leqslant 0$	$\{0,1\}$	$\leqslant 1$	$\geqslant 1$	$\geqslant 1$	$\geqslant 2$

Here $\mathfrak{g}/\mathfrak{g}_f = (\mathfrak{g}/\mathfrak{h})/(\mathfrak{g}_f/\mathfrak{h}) = (T_xM_0)/(\mathfrak{g}_f/\mathfrak{h})$. From the table we see

$$S \coloneqq (\mathfrak{g}_f/\mathfrak{h})^{\perp}/T_xM_0$$

must be of weight 1. If we denote by $(\mathfrak{g}/\mathfrak{g}_f)_1$ the component of weight 1 in $\mathfrak{g}/\mathfrak{g}_f$, then

$$\widetilde{S} = S \oplus (\mathfrak{g}/\mathfrak{g}_f)_1 = S \oplus (\mathfrak{u}/\mathfrak{u}_+)$$

is the whole subspace of weight 1 in T_xM.

Step II (Using T_xM to investigate M_+)**.** The goal is to prove that the following commutative diagram is Cartesian.

$$\begin{tikzcd} M_+ \arrow[r, "\Lambda", "x\longmapsto 0"'] \arrow[d, "\mu"'] & \widetilde{S} \arrow[d] \\ \mathfrak{g}^* \arrow[r] & (\mathfrak{h}+\mathfrak{u})^* \end{tikzcd}$$

Here the right vertical map is the moment map for HU acting on $\widetilde{S}$ and Λ is an $\mathsf{HU} \rtimes \mathbb{G}_m$-equivariant morphism. In particular, we have an isomorphism

$$\Upsilon \colon M_+ \xrightarrow{\sim} \widetilde{S} \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^*.$$

For this, the following ingredients are required:

[3] *Caution.* We have $T_xM/T_xM_0 \subset T_xM_+$, yet the equality possibly fails to hold. Indeed, $T_xM_+ = (T_xM/T_xM_0) \oplus (\mathfrak{u}/\mathfrak{u}_+)$. Further, notice from the table that subspaces of T_xM_0 are not exclusively of nonpositive weights.

(Λ) The starting point is the natural differential map[4] $\Lambda\colon M_+ \to T_xM_+$, along which the image of M_+ has weight 1, and hence $\Lambda(M_+) \subset \widetilde{S}$. This leads to the map $\Lambda\colon M_+ \to \widetilde{S}$, which we still denote by Λ as an abuse of notation.

(μ) On the other hand, the moment map $\mu\colon M \to \mathfrak{g}^*$ admits a canonical lifting to T_xM, denoted by $\mu^\natural : T_xM \to \mathfrak{g}^*$, just so $\mu^\natural(S) = \mathfrak{h}^* \subset \mathfrak{g}^*$. Also, $\operatorname{Ker}\mu^\natural = T_xM_0^\perp$ and $\operatorname{Im}\mu^\natural = \mathfrak{h}^\perp$. Hence there is an exact sequence of vector spaces:

$$0 \longrightarrow T_xM_0^\perp \longrightarrow T_xM \xrightarrow{\mu^\natural} \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/\mathfrak{h}^\perp \longrightarrow 0.$$

(Υ) In fact, the nature of $\mu_1 = \mu|_S\colon S \to \mathfrak{h}^*$ deduces that we can translate Υ to the level of tangent spaces, written as

$$\Upsilon^\natural\colon T_xM_+ \longrightarrow T_{\mu_1(x)}(\widetilde{S} \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^*) = \widetilde{S} \oplus (\mathfrak{h}\mathfrak{u})^\perp.$$

By construction, $\Upsilon^\natural$ exactly annihilates $(T_xM_+)_{\geqslant 2}$, the subspace of T_xM_+ of weight $\geqslant 2$.

Note also that $(T_xM_0^\perp)_{\geqslant 2} = (T_xM/T_xM_0)^{*,\geqslant 2}$. Using this, based on ($\mu$) and ($\Upsilon$) above, one can check

$$\operatorname{Ker}\Upsilon^\natural = (T_xM_+)_{\geqslant 2} = (T_xM)_{\geqslant 2} = \mathfrak{h}^\perp_{\geqslant 2} = (\mathfrak{h}+\mathfrak{u})^\perp.$$

Step III (Trivializing the Lagrangian correspondence on M_+)**.** Since Υ is an isomorphism by Step II, we have a Lagrangian correspondence

$$M^\circ \longleftarrow M_+ \longrightarrow \widetilde{S}$$

in the sense of §2.4.1. Applying the Hamiltonian induction from HU to G, it further induces

$$M^\circ \xleftarrow{\varpi} M_+ \times^{\mathsf{HU}} \mathsf{G} \xrightarrow{\sim} \text{h-ind}_{\mathsf{HU}}^{\mathsf{G}}(\widetilde{S}).$$

Again, the right morphism above is an isomorphism as Υ is an isomorphism. It remains to show that ϖ on the left is a G-equivariant isomorphism. To complete this final step, we need:

- Luna's lemma [Lun73, Lemme, p.89] (deduced from Zariski's main theorem), asserting that if we can check several geometric conditions (such as affineness) then ϖ is finite.
- The fact that $\#\varpi^{-1}(m_0) = 1$ for each $m_0 \in M_0^\circ$, namely ϖ has trivial fiber on $M_0\circ$.

These imply that ϖ is a finite étale morphism of degree 1, and hence an isomorphism.

Now the proof is almost completed and the remaining ambiguity lies in the SL_2-action on G and the $\mathsf{G} \times \mathbb{G}_{gr}$-action on both sides of the claimed isomorphism in Theorem 3.17. Such a capstone argument is rather subtle just so we choose to omit it.

3.4. **The converse of Theorem 3.17.** We may expect the converse structure theorem of Whittaker inductions; that is, the Whittaker induction of S along $\iota\colon \mathsf{H} \times \mathrm{SL}_2 \to \mathsf{G}$ in Theorem 3.17 is automatically hyperspherical. For this, it suffices to check that such a Whittaker induction is coisotropic (or equivalently, multiplicity-free).

Proposition 3.18 (The structure of Whittaker inductions)**.** *Keep the same assumptions as before. Let M be the Whittaker induction of S along ι as in Theorem 3.17. Then*

(1) *M is affine.*
(2) *M satisfies the technical conditions by* [BZSV, §3.5.1] *implied in Definition 3.7, and hence is hyperspherical.*

Proof. Note that (2) is simply a paraphrase of [BZSV, Proposition 3.6.3]. For (1), M as in §3.2.3 can be written as $(S \times_{\mathfrak{h}^*} (f + \mathfrak{g}^e)) \times^{\mathsf{H}} \mathsf{G}$, where $\mathfrak{g}^e$ is the centralizer of e, considered as a subspace of $\mathfrak{g}^*$ via κ. This requires a simplification on the Whittaker induction, see §3.2.5 (cf. [GW23, §4.2] and [GG02, Lemma 2.1]) for more details. □

[4] *Caution.* This Λ does not equal the prescribed noncanonical isomorphism $T_xM_+ \simeq M_+$.

3.5. **Remarks on relative Langlands parameters and BSZV duality.** The substantials about the homomorphism $\iota\colon \mathsf{H}\times\mathrm{SL}_2 \to \mathsf{G}$ in Theorem 3.17 are as follows. We introduce both the original and the relative versions of the incarnation of Langlands parameter in various contexts.

(a) (*Langlands and Arthur*). Over a global field F, one will hopefully classify all discrete automorphic representations of $\mathsf{G}(\mathbb{A}_F)$ anyway. If so, there is an extension of the Langlands conjecture due to Arthur, in which a discrete automorphic representation π should have a parameter

$$\iota_\pi\colon \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathsf{G} = \mathsf{G}^\vee \rtimes W_\mathbb{Q},$$

where $W_\mathbb{Q}$ is the Weil group and $\mathcal{L}_F$ is the conjectural "Langlands sheaf" that contains huge amounts of geometric data yet we know very little about it.

(b) (*Sakellaridis–Venkatesh*). Using spherical varieties, [SV17] associates to a spherical variety $X = \mathsf{H}\backslash\mathsf{G}$ the explicit data about Langlands dual group $X^\vee$, consisting of a spherical parameter

$$\iota_X\colon X^\vee \times \mathrm{SL}_2 \longrightarrow \mathsf{G}^\vee$$

and a graded finite-dimensional (typically) symplectic representation $\mathtt{V}_X$ of $X^\vee$ (cf. [Gan23, §1]). The representation $\mathtt{V}_X$ of $X^\vee$ is the main ingredient allowing one to form the automorphic L-function which controls the relevant period. See [SV17] or [GW18, §1] for more details.

(c) (*Ben-Zvi–Sakellaridis–Venkatesh*). From Theorem 3.17, we see any hyperspherical G-variety can be determined from a finite-dimensional symplectic H-vector space S (seen as a symplectic H-representation), together with the homomorphism

$$\iota\colon \mathsf{H}\times\mathrm{SL}_2 \longrightarrow \mathsf{G}$$

with the condition that $\mathsf{H}\subset Z_\mathsf{G}(\iota(\mathrm{SL}_2))$ being a spherical subgroup.

Note that the data in (b) and (c) above are very similar:

$$(\iota_X\colon X^\vee\times\mathrm{SL}_2 \to \mathsf{G}, \mathtt{V}_X) \dashleftarrow\dashrightarrow (\iota\colon \mathsf{H}\times\mathrm{SL}_2\to\mathsf{G}, S).$$

Indeed, the Whittaker induction of $(\iota_X, \mathtt{V}_X)$ is the hyperspherical $\mathsf{G}^\vee$-variety $M^\vee$ over $\mathbb{C}$ with $M = T^*X$. Such a phenomenon serves as the cornerstone of [BZSV].

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