链式法则

1. 链式求导规则

(1)复合函数的中间变量均为一元函数的情形.复合结构图如图 1-11-11 所示.

设 $z=f(u,v)$,$u=\varphi(t)$,$v=\psi(t)$,则 $z=f[\varphi(t),\psi(t)]$,且 $\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}$.

建议写箭头

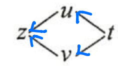


图 1-11-11

(2)复合函数的中间变量均为多元函数的情形.复合结构图如图 1-11-12 所示.

设 $z=f(u,v)$,$u=\varphi(x,y)$,$v=\psi(x,y)$,则 $z=f[\varphi(x,y),\psi(x,y)]$,且

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

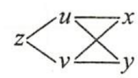


图 1-11-12

(3)复合函数的中间变量既有一元函数,又有多元函数的情形.复合结构图如图 1-11-13所示.

设 $z=f(u,v)$,$u=\varphi(x,y)$,$v=\psi(y)$,则 $z=f[\varphi(x,y),\psi(y)]$,且

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{dv}{dy}.$$

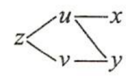
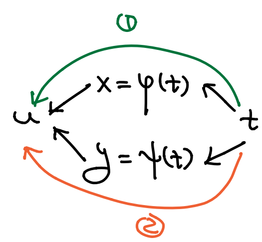


图 1-11-13

例1: $u=x^y$. 令 $x=\varphi(t)$, $y=\psi(t)$. 求 $\frac{du}{dt}$.



不能折头

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\varphi'(t)+\frac{\partial u}{\partial y}\psi'(t) \quad ①+②$$

$$=y\cdot x^{y-1}\varphi'(t)+x^y\ln x\,\psi'(t)$$

$$=x^y\left(\frac{y}{x}\varphi'+\ln x\,\psi'\right)$$

$$=(\varphi(t))^{\psi(t)}\left(\frac{\psi(t)}{\varphi(t)}\varphi'(t)+[\ln\psi(t)]\psi'(t)\right)$$

⚠ 在使用链式法则时,要求 $f(x,y)$ 在 (x_0,y_0) 可微,否则法则可能失效.

E.g. $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$\Rightarrow f_x(x,y)=\begin{cases}\frac{2xy^3}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$f_y(x,y)=\begin{cases}\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

证不可微: $\Delta f=f(\Delta x,\Delta y)-f(0,0)=\frac{\Delta x^2\cdot\Delta y}{\Delta x^2+\Delta y^2}$ 不是高阶小量 $o(r)$.

现在令 $x=y=t$, 则 $f(t,t)=\frac{t}{2}\Rightarrow\frac{df}{dt}=\frac{1}{2}$.

如果用链式法则，就有

$$\frac{df}{dt} = f_x x_t' + f_y y_t' = f_x + f_y$$

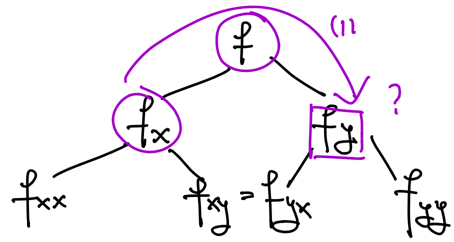
当$(x,y)=(0,0)$时 $\frac{df}{dt}=0$，问题出在$f(x,y)$在$(0,0)$不可微.

与$\frac{1}{2}$矛盾.

例2：设$f(x,y)$有连续偏导数且$f(x,x^2)=1$.

(1) 若$f_x(x,x^2)=x$，求$f_y(x,x^2)$.

(2) 若$f_y(x,y)=x^2+2y$，求$f(x,y)$.



(1) 对$f(x,x^2)=1$两边求导得

不是$f_x(x,y)$
而是$f_x(x,x^2)$

$$f_x (x)' + f_y (x^2)' = f_x + 2x f_y = 0$$

而 $f_x(x,x^2)=x$，故 $x+2xf_y(x,x^2)=0$

$$\Rightarrow f_y(x,x^2) = -\frac{1}{2} \quad \boxed{(x\neq 0)}$$

由连续性知 $f_y(x,x^2)=-\frac{1}{2}$ $(x=0)$.

(2) 将f_y的原函数视作f的一部分去还原f.

令 $F(x,y)=f(x,y)-\underbrace{(x^2y+y^2)}_{f_y\text{的积分}}$

$\Rightarrow F_y = f_y-(x^2+2y)=0$ ← 构造F的目标是完成这.

$\Rightarrow$ F只与x有关，设$F=\varphi(x)$

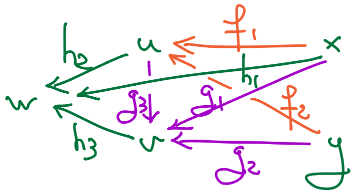
$\Rightarrow f(x,y)=x^2y+y^2+\varphi(x)$.

利用$f(x,x^2)=1$，得 $\varphi(x)=1-2x^4$

$\Rightarrow f(x,y)=x^2y+y^2+1-2x^4$.

例3：$u=f(x,y)$，$v=g(x,y,w)$，$w=h(x,u,v)$. 求$\frac{\partial w}{\partial x}$，$\frac{\partial w}{\partial y}$

先画图



每个字母只出现一遍.

$$\frac{\partial w}{\partial x}=h_1+h_2f_1+h_3(g_1+g_3f_1)$$

$$=h_1+h_2\frac{\partial f}{\partial x}+h_3\left(g_1+g_3\frac{\partial f}{\partial x}\right)$$

只有一层才能写成这种形式.

而 e.g. $h_1 \neq \frac{\partial h}{\partial x}$

$$\frac{\partial w}{\partial y}=h_2f_2+h_3(g_2+f_2g_3)$$

$$=h_2\frac{\partial f}{\partial y}+h_3\left(g_2+g_3\frac{\partial f}{\partial y}\right).$$

例4: **例 1.11.3** 设 $z=f(e^x\sin y,x^2+y^2)$，其中 f 具有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

先关于x，后关于y

解

$$\frac{\partial z}{\partial x}=e^x\sin y f_1'+2xf_2',$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}''e^{2x}\sin y\cos y+2e^x(y\sin y+x\cos y)f_{12}''+4xyf_{22}''+f_1'e^x\cos y.$$

例 1.11.4 设 $z=x^3f\left(xy,\frac{y}{x}\right)$，$f$ 具有连续的二阶偏导数，求 $\frac{\partial z}{\partial y}$，$\frac{\partial^2 z}{\partial y^2}$ 及 $\frac{\partial^2 z}{\partial x\partial y}$.

先x后y

解

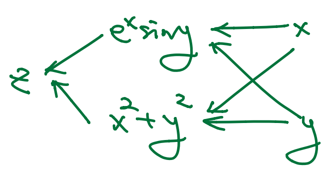
$$\frac{\partial z}{\partial y}=x^4f_1'+x^2f_2',$$

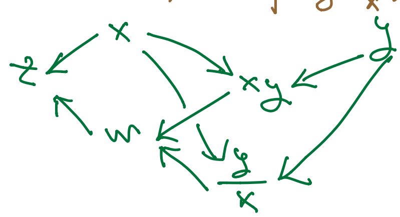
$$\frac{\partial^2 z}{\partial y^2}=x^4\left(xf_{11}''+\frac{1}{x}f_{12}''\right)+x^2\left(xf_{21}''+\frac{1}{x}f_{22}''\right)=x^5f_{11}''+2x^3f_{12}''+xf_{22}'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=4x^3f_1'+x^4\left(yf_{11}''-\frac{y}{x^2}f_{12}''\right)+2xf_2'+x^2\left(yf_{21}''-\frac{y}{x^2}f_{22}''\right)$$

$$=4x^3f_1'+2xf_2'+x^4yf_{11}''-yf_{22}''.$$

$z=x^3\cdot w$，$w=f(xy,\frac{y}{x})$

① 

② 

例5：设 $a,b\neq 0$，f 有二阶连续偏导数，且

$$a^2\frac{\partial^2 f}{\partial x^2}+b^2\frac{\partial^2 f}{\partial y^2}=0 \qquad ①$$

$$f(ax,bx)=ax \qquad ②$$

$$f_x(ax,bx)=bx^2 \qquad ③$$

求 $f_{xx}(ax,bx)$，$f_{xy}(ax,bx)$，$f_{yy}(ax,bx)$.

对②两边求导得

$$af_x(ax,bx)+bf_y(ax,bx)=a$$

对①③两边求导得

$$af_{xx}(ax,bx)+bf_{xy}(ax,bx)=2bx \quad ④$$

$$a^2f_{xx}(ax,bx)+2abf_{xy}(ax,bx)+b^2f_{yy}(ax,bx)$$

结合①得 $f_{xy}(ax,bx)=0$，将之代入④得

$$f_{xx}(ax,bx)=\frac{2b}{a}x,$$

再将之代入①得 $f_{yy}(ax,bx)=-\frac{2a}{b}x.$