一阶常微分方程

(一) 恰当方程

形如 $P(x,y)dx + Q(x,y)dy = 0 \iff \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$，$Q \neq 0$.

且 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \rightsquigarrow$ 恰当方程 (exact equation)

问题：一般的 $Pdx + Qdy = 0$ 不能直接积分.

因为 $\int P\,dx$ 中 y 与 x 有关.

但是，恰当方程可以直接积.

例：1. $(3x^2-1)dx + (2x+1)dy = 0$

$P = 3x^2 - 1$，$Q = 2x+1$

$\frac{\partial P}{\partial y} = 0$，$\frac{\partial Q}{\partial x} = 2 \Rightarrow$ 不恰当.

2. $(x+2y)dx + (2x+y)dy = 0$

$\frac{\partial(x+2y)}{\partial y} = 2 = \frac{\partial(2x+y)}{\partial x} \Rightarrow$ 恰当

直接积：$\int (x+2y)dx + \int (2x+y)dy = C$

$$= \underbrace{\int x\,dx}_{\frac{x^2}{2}} + \underbrace{\int \boxed{2y\,dx}}_{2xy} + \underbrace{\int \boxed{2x\,dy}}_{2yx} + \underbrace{\int y\,dy}_{\frac{y^2}{2}}$$

全微分

$\Phi(x,y)$ 的全微分 $= \boxed{\frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy}$

原理 $P(x,y)dx + Q(x,y)dy = 0$

$\Rightarrow \int P(x,y)dx + \int Q(x,y)dy = C$

比较：希望寻找 $\Phi(x,y)$，使得 $\frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy = Pdx + Qdy$

$\rightsquigarrow$ 希望 $\frac{\partial \Phi}{\partial x} = P$，$\frac{\partial \Phi}{\partial y} = Q$

$\Rightarrow$ 必要条件 $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial x \partial y}$

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y}\right)$$

例：3. $(t^2+1)\cos u \cdot du + 2t\sin u \cdot dt = 0$

$P = (t^2+1)\cos u$，$Q = 2t\sin u$

$\frac{\partial P}{\partial t} = 2t\cos u$，$\frac{\partial Q}{\partial u} = 2t\cos u \Rightarrow$ 恰当

$$\Rightarrow \underbrace{\int t^2\cos u\, du + \int 2t\sin u \cdot dt}_{\text{全微分}\ \to\ t^2\sin u} + \underbrace{\int \cos u\, du}_{=\ \sin u} = C$$

$\Rightarrow t^2\sin u + \sin u = C$. 通解.

4. $(ye^x + 2e^x + y^2)dx + (e^x + 2xy)dy = 0$

$P = ye^x + 2e^x + y^2$，$Q = e^x + 2xy$

$\frac{\partial P}{\partial y} = e^x + 2y$，$\frac{\partial Q}{\partial x} = e^x + 2y$. $\Rightarrow$ 恰当.

$$\underbrace{\int 2e^x dx}_{2e^x} + \int (ye^x + y^2)dx + \int (e^x + 2xy)dy = 0$$

$$ye^x + y^2x + \cancel{\varphi(y)} = e^x y + xy^2 + \cancel{\psi(x)}$$

$\Rightarrow 2e^x + xy^2 + ye^x = C$. 通解.

5. $xf(x^2+y^2)\,dx + yf(x^2+y^2)\,dy = 0$ 　 f 连续可微

$$\left.\begin{aligned} P = xf(x^2+y^2) \quad &\Rightarrow \quad \frac{\partial P}{\partial y} = 2xyf' \\ Q = yf(x^2+y^2) \quad &\Rightarrow \quad \frac{\partial Q}{\partial x} = 2xyf' \end{aligned}\right\} \Rightarrow \text{恰当}.$$

$$\int \underbrace{xf(x^2+y^2)}_{\frac{\partial \Phi}{\partial x}}\,dx + \int \underbrace{yf(x^2+y^2)}_{\frac{\partial \Phi}{\partial y}}\,dy = C$$

$$\Rightarrow \Phi = \left(\int f\right)(x^2+y^2) = F(x^2+y^2)$$

$$\leadsto \frac{1}{2}\int_0^{x^2} f(t+y^2)\,dt \qquad\qquad F' = f.$$

(二) 可分离 x 和 y

如果 dx 侧：只有 x，dy 侧：只有 y $\Rightarrow$ 可直接积.

例：1. $y' = \frac{x^2}{y}$ 　 $(y \neq 0)$

$$\Rightarrow y\,dy = x^2\,dx \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C$$

2. $\frac{dy}{dx} + y^2 \sin x = 0$

① $y \neq 0 \Rightarrow \frac{dy}{y^2} + \sin x \cdot dx = 0$

$\Rightarrow -\frac{1}{y} - \cos x + C = 0$ 　 通解

② $y = 0$ 也是解. 　 特解

3. $\frac{dy}{dx} = 1 + x + y^2 + xy^2 = (1+x) + y^2(1+x) = (1+y^2)(1+x)$

$$\Rightarrow \frac{dy}{1+y^2} = (1+x)\,dx$$

$$\Rightarrow \arctan y = x + \frac{x^2}{2} + C$$

初值问题：

1. $xdx + ye^{-x}dy = 0$ ， $y(0)=1$

$xe^{x}dx + ydy = 0$

$\Rightarrow (x-1)e^{x} + \frac{y^2}{2} = C$ 通解.

$\int xe^{x}dx = \int x\,de^{x} = xe^{x} - \int e^{x}dx$

$= xe^{x} - e^{x}$

代入 $y(0)=1$

$\Rightarrow -1 + \frac{1}{2} = C = -\frac{1}{2}$

$\Rightarrow (x-1)e^{x} + \frac{y^2}{2} = -\frac{1}{2}$ 特解

2. $\frac{dy}{dx} = 1-y^2$

① $\frac{dy}{1-y^2} = dx$ ， $y \neq \pm 1$

$\int \frac{dy}{1-y^2} = \int \frac{dy}{(1-y)(1+y)} = \int -\frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy$

$= -\frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right) = -\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \int dx = x + C$

$\Rightarrow \ln\left|\frac{y-1}{y+1}\right| + 2x = C.$

② $y = \pm 1$ 也是解

本题函数积分：$\int \frac{P(x)}{Q(x)}dx$ ，P, Q 多项式.

第一步：将 Q 因式分解成
- 一些一次式之积. $\xrightarrow{\text{分开}}$ $\ln|\cdots|$
- 一些无实根的二次式.

(三) "线性" 方程

形如 $y' + p(x)y = q(x)$ ⟵ 系数是 x 的函数.

只有一种标准解法：凑积分因子：$e^{\int p(x)dx}$

$\Rightarrow e^{\int p}y' + e^{\int p}py = qe^{\int p}$

$\Rightarrow (e^{\int p}y)' = qe^{\int p}$ 可直接积分（关于 x）.

关于 x 的导数

$\Rightarrow e^{\int p(x)dx}y = \int q(x)e^{\int p(x)dx}dx + C$

$\Rightarrow y = e^{-\int p(x)dx}\cdot\int q(x)e^{\int p(x)dx}dx + C\cdot e^{-\int p(x)dx}$.

例：1. $y' + 2y = xe^{-x}$，凑 $e^{\int 2\cdot dx} = e^{2x}$

$\Rightarrow e^{2x}(y'+2y) = xe^{-x}\cdot e^{2x}$

$\Rightarrow (e^{2x}\cdot y)' = xe^{x}$

$\Rightarrow e^{2x}\cdot y = \int xe^{x}dx + C$

$\Rightarrow y = Ce^{-2x} + (x-1)e^{-x}$

2. $x\dfrac{dy}{dx} + 2y = \sin x$，$y(\pi) = \dfrac{1}{\pi}$.

$\Rightarrow y' + \dfrac{2}{x}y = \dfrac{\sin x}{x}$

$\Rightarrow (e^{\int\frac{2}{x}}y)' = \dfrac{\sin x}{x}\cdot e^{\int\frac{2}{x}dx} = x\sin x$

$\Rightarrow x^2y = \int x\sin x\,dx + C = -x\cos x + \sin x + C$

$\Rightarrow y = -\dfrac{1}{x}\cos x + \dfrac{\sin x}{x^2} + \dfrac{C}{x^2}\ (x \neq 0)$

（四）"拟线性"方程

可通过某种代换化为线性.

例: 1. $\frac{dy}{dx} = \frac{x^2+y^2}{2y}$　　关于y的二次降一次: 代 $z=y^2$.

令 $y^2=z$. $\frac{dz}{dx} = \frac{dy^2}{dx} = 2yy' = 2y\cdot\frac{x^2+y^2}{2y} = x^2+z$

$\Rightarrow \frac{dz}{dx} = z + x^2$

2. $\frac{dy}{dx} = \frac{y}{x+y^2}$　$\leftarrow$ -1次. 需要1次. 取倒数

$\Rightarrow \frac{dx}{dy} = \frac{x+y^2}{y} = \frac{1}{y}x + y$　关于x线性.

3. $3xy^2\frac{dy}{dx} + y^3 + x^3 = 0$.　　令 $z=y^3$.

$\frac{dz}{dx} = 3y^2\cdot y' = -\frac{y^3}{x} - x^2 = -\frac{z}{x} - x^2$.　关于z线性.

4. $\frac{dy}{dx} = \frac{1}{\cos y} + x\tan y$　　将z代为y的三角函数

令 $z=\sin y$,

$\frac{dz}{dx} = \frac{d\sin y}{dx} = \cos y\cdot\frac{dy}{dx} = 1 + x\sin y = 1+xz$　关于z线性.

或 $z=\cos y$,

$\frac{dz}{dx} = \frac{d\cos y}{dx} = -\sin y\cdot\frac{dy}{dx} = -\tan y - x\cdot\frac{\sin^2 y}{\cos y}$　(×).

(三) $y' = f(ax+by+c)$

例: 1. $y' = \frac{2y-x}{2x-y}$　上下齐次

令 $y = ux = u(x)\cdot x$ (u不是常数, u是关于x的函数!)

$$\Rightarrow y' = u'x + u = \frac{du}{dx}\cdot x + u = \frac{2ux - x}{2x - ux} = \frac{2u-1}{2-u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{2u-1-2u+u^2}{2-u} = \frac{u^2-1}{2-u}$$

$$\Rightarrow \frac{2-u}{u^2-1}\,du = \frac{dx}{x}$$ ← 可分离变量.

$$\Rightarrow \ln\left|\frac{1-u}{1+u}\right| - \frac{1}{2}\ln|u^2-1| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{x-y}{x+y}\right| - \frac{1}{2}\ln\left|\frac{y^2-x^2}{x^2}\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|\left(\frac{x-y}{x+y}\right)^2 \Big/ \frac{y^2-x^2}{x^2}\right| = \ln x^2 + C$$

2. $y' = \dfrac{2y - x + 5}{2x - y - 4}$ 直接令 $y = ux$ 会无效.

(先去掉常数项)

方程: $\begin{cases} 2y - x + 5 = 0 \\ 2x - y - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -2 \end{cases}$

令 $u = x - 1$, $v = y + 2$.

$$\Rightarrow y' = \frac{2v - u}{2u - v} = \frac{dv}{du}$$

练习: 将方程先处理成标准的形式.

1. $y' = \cos(x-y)$

$u = x - y$, $\dfrac{du}{dx} = 1 - y' = 1 - \cos u$

$$\Rightarrow \frac{du}{1-\cos u} = dx \quad (\cos u \neq 1)$$ ← 验证: $\cos u = 1$ 确实是解

$$\Rightarrow \frac{d(\frac{u}{2})}{\sin^2\frac{u}{2}} = dx \quad (\cos u = 1 - 2\sin^2\frac{u}{2})$$

$$\Rightarrow -\cot\frac{u}{2} = x + C$$

$$\Rightarrow \cot\frac{x-y}{2} + x + C = 0$$ 通解

2. $(3uv+v^2)\,du+(u^2+uv)\,dv=0$

$$\downarrow\frac{\partial}{\partial v}\qquad\qquad\qquad\downarrow\frac{\partial}{\partial u}$$

$$3u+2v\qquad\qquad\qquad 2u+v$$

同乘u $\Rightarrow (3u^2v+uv^2)\,du+(u^3+u^2v)\,dv=0$

$$\left.\begin{array}{l} P=3u^2v+uv^2 \Rightarrow \dfrac{\partial P}{\partial v}=3u^2+2uv \\ Q=u^3+u^2v \Rightarrow \dfrac{\partial Q}{\partial u}=3u^2+2uv. \end{array}\right\} \Rightarrow \text{恰当}$$

$$\Rightarrow \int (3u^2v+uv^2)\,du+\int(u^3+u^2v)\,dv=C$$

$$u^3v+\frac{1}{2}u^2v^2+\cancel{\varphi(v)} = u^3v+\frac{1}{2}u^2v^2+\cancel{\psi(u)}$$

$\Rightarrow u^3v+\frac{1}{2}u^2v^2=C$. 通解.

3. $\dfrac{y\,dy}{x\,dx}=\dfrac{4y^2-2x^2}{x^2+y^2+3}$

LHS $=\dfrac{dy^2}{dx^2}$. 令 $u=y^2$, $v=x^2$.

$$\Rightarrow \frac{du}{dv}=\frac{4u-2v}{u+v+3}$$

令 $\begin{cases}4u-2v=0\\ u+v+3=0\end{cases} \Rightarrow \begin{cases}u=-1\\ v=-2\end{cases}$. 令 $\begin{cases}m=u+1\\ n=v+2\end{cases}$

$\Rightarrow \dfrac{dm}{dn}=\dfrac{4m-2n}{m+n}$, 令 $m=zn$, z与n有关

$$\Rightarrow m'=z'\cdot n+z=\frac{4zn-2n}{zn+n}=\frac{4z-2}{z+1}$$

$$\Rightarrow n\cdot\frac{dz}{dn}=\frac{4z-2-z^2-z}{z+1}=\frac{-z^2+3z-2}{z+1}$$

$$d(z^2-3z+2)=(2z-3)\,dz$$

$$\Rightarrow \frac{z+1}{-(z^2-3z+2)}\,dz=\frac{1}{n}\,dn \rightsquigarrow \ln|n|+C=-\int\frac{z+1}{z^2-3z+2}\,dz$$

$$\frac{z+1}{-(z^2-3z+2)}=\frac{z+1}{-(z-2)(z-1)}\,dz$$

$$=-\frac{1}{2}\int\frac{(2z-3)+5}{z^2-3z+2}\,dz$$

$$=-\frac{1}{2}\int\frac{d(z^2-3z+2)}{z^2-3z+2}-\frac{5}{2}\int\frac{dz}{(z-1)(z-2)}$$

$$-\int\left(\frac{3}{z-2}-\frac{2}{z-1}\right)dz$$

$$=\ln\frac{(z-1)^2}{|2-z|^3}=\ln|n|+C$$

$$\begin{cases} n=v+2=x^2+2 \\ z=\dfrac{m}{n}=\dfrac{u+1}{v+2}=\dfrac{y^2+1}{x^2+2} \end{cases}$$

$$\ln(x^2+2)+C=\ln(\cdots).$$

4. $$\frac{dy}{dx}=\frac{2x^3+3xy^2-7x}{3x^2y+2y^3-8y}$$

$$\Leftrightarrow \frac{y\,dy}{x\,dx}=\frac{2x^2+3y^2-7}{3x^2+2y^2-8}.\quad 令\ u=y^2,\ v=x^2$$

$$\Rightarrow \frac{du}{dv}=\frac{2v+3u-7}{3v+2u-8}.\quad 设\ \begin{cases}2v+3u-7=0\\3v+2u-8=0\end{cases}\Rightarrow\begin{cases}u=1\\v=2\end{cases}$$

令 $m=u-1,\ n=v-2$. 则

$$\frac{dm}{dn}=\frac{2n+3m}{2m+3n}.\quad 令\ m=zn.$$

$$\Rightarrow 分离变量为\ \left(\frac{3+2z}{2-2z^2}\right)dz=\frac{dn}{n}$$

$$\Rightarrow \frac{3}{4}\ln\left|\frac{1+z}{1-z}\right|-\frac{1}{2}\ln|1-z^2|=\ln n+C$$

$$\Rightarrow (x^2-y^2-1)^5=C(x^2+y^2-3).$$

一阶微分方程的求解

- 能写成 $y'=f(x)\cdot g(y)$ ← 分离变量
- 能写成 $y'=f(ax+by+c)$
- 能写成 $y'=f\left(\frac{y}{x}\right)$
- 能写成 $\frac{1}{y}=f\left(\frac{x}{y}\right)$

拟线性.

- 能写成 $y'+p(x)y=q(x)$ ← 线性. 凑 $e^{\int p(x)dx}$
- 能写成 $y'+p(x)y=q(x)y^n$ $(n\neq 0,1)$(仅数学一) ← 拟线性