差分方程

差分定义：离散版微分

若 $f(x)=y$ 只在 $x_0, x_1, \cdots$ 上定义.

x_k 变到 x_{k+1} 时，$y=f(x)$ 的变化量 $\Delta y_k = f(x_{k+1}) - f(x_k)$

$\Rightarrow \Delta y_k = y_{k+1} - y_k$，$y_k = f(x_k)$ 一阶差分.

完整版本 $\Delta y = \min\limits_{\Delta k>0} \dfrac{y_{k+\Delta k} - y_k}{\Delta k} = y_{k+1} - y_k \quad (\Delta k = 1)$，

微分：$y' = \boxed{\lim\limits_{\Delta x \to 0}} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

本质是让Δx无穷小（以直代曲误差无穷小）.

二阶差分：$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$.

n阶差分：$\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$.

差分运算法则.

(1) $\Delta C = 0$ （C常数）

(2) $\Delta(C y_k) = C \Delta y_k$

(3) $\Delta(C_1 y_k \pm C_2 z_k) = C_1 \Delta y_k \pm C_2 \Delta z_k$

(4) $\Delta(y_k \cdot z_k) = y_{k+1} \cdot \Delta z_k + z_k \cdot \Delta y_k = y_k \cdot \Delta z_k + z_{k+1} \cdot \Delta y_k$!

(5) $\Delta\left(\dfrac{y_k}{z_k}\right) = \dfrac{z_k \Delta y_k - y_k \Delta z_k}{z_k \cdot z_{k+1}} = \dfrac{z_{k+1} \Delta y_k - y_{k+1} \Delta z_k}{z_k \cdot z_{k+1}}$.

例：$\Delta^4 y_x - 6 y_{x+2} + 4 y_{x+1} - y_x + 1 = 0$

$\Delta^4 y_x = \Delta^3 y_{x+1} - \Delta^3 y_x$

$= \Delta^2 y_{x+2} - \Delta^2 y_{x+1} - \Delta^2 y_{x+1} + \Delta^2 y_x$

$= \Delta y_{x+3} - \Delta y_{x+2} - \Delta y_{x+2} + \Delta y_{x+1} - \Delta y_{x+2} + \Delta y_{x+1} + \Delta y_{x+1} - \Delta y_x$

$$= \Delta y_{x+3} - 3\Delta y_{x+2} + 3\Delta y_{x+1} - \Delta y_x$$
$$= y_{x+4} - y_{x+3} - 3(y_{x+3} - y_{x+2}) + 3(y_{x+2} - y_{x+1}) - (y_{x+1} - y_x)$$
$$= y_{x+4} - 4y_{x+3} + 6y_{x+2} - 4y_{x+1} + y_x$$

$$\Rightarrow \quad \Delta^4 y_x - 6y_{x+2} + 4y_{x+1} - y_{x+1}$$
$$= y_{x+4} - 4y_{x+3} + 1 = 0$$

作变换 $u = x+3 \rightsquigarrow y_{u+1} - 4y_u + 1 = 0.$

结论：高阶一定可以化为低阶.

差分方程通解结构.

如果 $y_x^{(1)}, y_x^{(2)}, \cdots, y_x^{(n)}$ 是n阶齐次常系数差分方程

$$a_n \Delta^n y_x + a_{n-1} \Delta^{n-1} y_x + \cdots + a_0 y_x = 0$$
$$\Leftrightarrow \quad b_n y_{x+n} + b_{n-1} y_{x+(n-1)} + \cdots + b_1 y_{x+1} + b_0 y_x = 0$$

那么通解为

$$Y_x = C_1 y_x^{(1)} + C_2 y_x^{(2)} + \cdots + C_n y_x^{(n)}.$$

齐次差分方程通解.

① $y_{x+1} + m y_x = 0$

通解 $y_c(x) = C \cdot (-m)^x$.

验证：$y_c(x+1) = (-m) \cdot C \cdot (-m)^x = -m y_c(x)$.

② $y_{x+1} + m y_x = f(x)$

找特解，通解 = 特解 + 齐次通解

(i) $f(x)$ 是 n次多项式 $P_n(x)$

$m \neq -1$: $y_x^* = Q_n(x)$ （系数不是 P_n 的系数）

（n次项不会被消掉）

$m=-1$：n次项会被消掉

$$y_x^* = x \cdot Q_n(x) \quad (\text{提一次})$$

原因：设 $Q_n(x) = B_n x^n + B_{n-1} x^{n-1} + \cdots + B_0$

$$Q_n(x+1) = B_n(x+1)^n + B_{n-1}(x+1)^{n-1} + \cdots + B_0$$

$$= B_n x^n + C_{n-1} x^{n-1} + \cdots + C_0 \quad (C_{n-1} \neq B_{n-1})$$

$$\Rightarrow Q_n(x+1) - Q_n(x) = (C_{n-1} - B_{n-1}) x^{n-1} + \cdots + C_0 - B_0$$

是一个 $n-1$ 次多项式

若 $m=-1$，$y_{x+1} + m y_x = y_{x+1} - y_x = n-1$ 次多项式 $\neq P_n(x)$.

(所以左式必须补一次).

(ii) $f(x) = d^x \cdot P_n(x)$，$d \neq 0$ 为常数

$m+d \neq 0$，$y_x^* = d^x \cdot Q_n(x)$.

检验：$y_{x+1} = d^{x+1} Q_n(x+1)$

$$\Rightarrow y_{x+1} + m y_x = d \cdot d^x Q_n(x+1) + m d^x Q_n(x) = d^x \cdot P_n(x)$$

$$\Rightarrow d \cdot Q_n(x+1) + m \cdot Q_n(x) = P_n(x).$$

(若 $d+m=0$，n次项被消).

$m+d=0$，$y_x^* = x \cdot d^x \cdot Q_n(x)$ (补一次)

(iii) $f(x) = b_1 \cos\omega x + b_2 \sin\omega x$ ($\omega \neq 0$，b_1, b_2 不同时 $=0$)

$$D = \begin{vmatrix} m+\cos\omega & \sin\omega \\ -\sin\omega & m+\cos\omega \end{vmatrix} \neq 0,$$

$$y_x^* = \alpha \cos\omega x + \beta \sin\omega x, \quad \alpha, \beta \text{ 待定}.$$

$D=0$，$y_x^* = x(\alpha \cdot \cos\omega x + \beta \sin\omega x)$.

差分方程的例子

Hansen-Samuelson 国民收入分析模型

Y_t - 国民收入，C_t - 消费，I_t - 投资，G_t - 政府支出总额

作为最终差分方程的未知量 $\longrightarrow$ $Y_t = C_t + I_t + G_0$ $\longleftarrow$ 方便起见，$G_t = G_0$ 为常数

$I_t = \beta(C_t - C_{t-1})$, $\beta > 0$.

$Y(0) = Y_0$, $Y_1 = Y_1$

$C_t = \alpha Y_{t-1}$, $0 < \alpha < 1$

$$\Rightarrow Y_t = \alpha Y_{t-1} + \beta(C_t - C_{t-1}) + G_0$$

$$= \alpha Y_{t-1} + \beta(\alpha Y_{t-1} - \alpha Y_{t-2}) + G_0$$

$$\Rightarrow Y_t - (1+\beta)\alpha Y_{t-1} + \alpha\beta Y_{t-2} - G_0 = 0.$$

二阶常系数非齐次差分方程.