换元法和分部积分法

换元法1：凑微分法

目的：$\int f(u(x))u'(x)\,dx = F(u(x))+C$

其中F是f的原函数. （原因：链式法则）

$u'(x)dx = d(u(x)) \Rightarrow \int f(u(x))\,du(x)$

例：幂代换

设$F'(u)=f(u)$，$\alpha\neq 0$

$$\int f(x^\alpha)x^{\alpha-1}\,dx = \frac{1}{\alpha}\int f(x^\alpha)(\alpha x^{\alpha-1})\,dx = \frac{1}{\alpha}F(x^\alpha)+C$$

$\alpha=-1$时：

$$\int f\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx = -F\left(\frac{1}{x}\right)+C$$

另：$\int f(\ln x)\frac{dx}{x} = \int f(\ln x)\,d(\ln x) = F(\ln x)+C$.

习题：1. 求$\int \frac{1}{x^2}\sin\frac{1}{x}\,dx = \cos\frac{1}{x}+C$

2. 求$\int \frac{dx}{x\sqrt{x^2+1}}$　$\frac{1}{x^2}$在哪？

$$= \int \frac{x\,dx}{x^2\sqrt{x^2+1}} = \int \frac{-x}{\sqrt{x^2+1}}\,d\left(\frac{1}{x}\right) = \int \frac{-1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}\,d\left(\frac{1}{x}\right)$$

$\frac{dx}{x^2} = -d\left(\frac{1}{x}\right)$

$$= -\ln\left|\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right|+C$$

3. 求$\int \frac{dx}{x(1+x^n)} = \int \frac{x^{n-1}}{x^n(1+x^n)}\,dx = \frac{1}{n}\int \frac{dx^n}{x^n(1+x^n)}$

$$= \frac{1}{n}\int \frac{du}{u(1+u)} = \frac{1}{n}\left(\int\left(\frac{1}{u}-\frac{1}{1+u}\right)du\right) = \frac{1}{n}\ln\left|\frac{x^n}{1+x^n}\right|+C$$

4. 求$\int \frac{dx}{x\ln x}$

$u=\ln x$，原式$=\ln|\ln x|+C$.

与三角代换的关系.

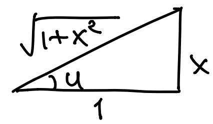
例: $\int (x^2+1)^{-3/2}\,dx$

• 利用 $(\arctan x)' = (1+x^2)^{-1}$

取 $u=\arctan x$ $(x=\tan u)$ $\Rightarrow du = \frac{dx}{1+x^2}$

$\Rightarrow I = \int \frac{dx}{(x^2+1)\sqrt{x^2+1}} = \int \frac{du}{\sqrt{\tan^2 u+1}} = \int \cos u\,du = \sin u + C$

$= \sin(\arctan x)+C = \frac{x}{\sqrt{1+x^2}}+C.$



• 凑微分法

$$I = \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)^{3/2}} \underset{\uparrow\ 令u=\frac{1}{x}}{=} \int \frac{u^3\cdot \overbrace{\frac{du}{-u^2}}^{=dx}}{(1+u^2)^{3/2}} = -\int \frac{u\,du}{(1+u^2)^{3/2}}$$

$$= -\frac{1}{2}\int\frac{d(u^2+1)}{(1+u^2)^{3/2}} = -\frac{1}{2}\int \frac{dv}{v^{3/2}} \qquad du^2 = 2u\,du = d(u^2+1)$$

$$= \frac{1}{\sqrt{v}}+C = \frac{x}{\sqrt{1+x^2}}+C.$$

三角凑微分

(1) $\int f(\sin x)\cos x\,dx = \int f(\sin x)\,d\sin x$

(2) $\int f(\cos x)\sin x\,dx = -\int f(\cos x)\,d\cos x$

(3) $\int f(\tan x)\frac{dx}{\cos^2 x} = \int f(\tan x)\sec^2 x\,dx = \int f(\tan x)\,d\tan x.$

(4) $\int f(\cot x)\frac{dx}{\sin^2 x} = \int f(\cot x)\csc^2 x\,dx = -\int f(\cot x)\,d\cot x.$

进阶 $\int f(\tan x)\,dx = \int f(\tan x)\cdot\cos^2 x\cdot \overbrace{\boxed{\frac{1}{\cos^2 x}dx}}^{d\tan x}$

$= \int \frac{f(\tan x)}{1+\tan^2 x}\,d\tan x.$

$f(u) \rightsquigarrow \frac{f(u)}{1+u^2}$

e.g. $I=\int \tan x\,dx$

解一: $I = -\int\frac{d\cos x}{\cos x} = -\ln|\cos x| + C = \ln|\sec x|+C$

解二：$I=\int \frac{\tan x}{1+\tan^2 x}\, d\tan x = \int \frac{u}{1+u^2}\, du$

$= \frac{1}{2}\int \frac{d(1+u^2)}{1+u^2} = \frac{1}{2}\ln|1+u^2| + C$

$= -\frac{1}{2}\ln|\cos^2 x| + C = -\ln|\cos x| + C$

应用：$\sin^n x$ 和 $\cos^n x$ 的积分。

例1：$I=\int \sin^3 x\, dx$

两种主要方法：

① 分部积分 $I=\int \sin x\cdot(1-\cos^2 x)\, dx = \int (\cos^2 x - 1)\, d\cos x$

$= \frac{1}{3}\cos^3 x - \cos x + C.$

② 倍角公式 $\sin 3x = 3\sin x - 4\sin^3 x.$

$(\sin^3 x = \frac{1}{2} 2\sin^2 x\cdot \sin x = \frac{1}{2}(1-\cos 2x)\sin x$

$= \frac{1}{2}\sin x - \frac{1}{4}\cdot \underbrace{2\cos 2x\cdot \sin x}_{\text{积化和差}}$

$\sin(\alpha+\beta)-\sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

$= \frac{1}{2}\sin x - \frac{1}{4}(\sin 3x - \sin x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x)$

$\Rightarrow I = \frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$

例2：$I=\int \frac{dx}{\sin^3 x\cdot \cos x}$ ← 此类问题一定可以化为有理分式，求不定积分。

• 凑微分，化为有理分式

$I=\int \frac{\cos x\, dx}{\sin^3 x \cos^2 x} = \int \frac{d\sin x}{\sin^3 x(1-\sin^2 x)}$

$= \int \frac{du}{u^3(1-u^2)} = \frac{1}{2}\int \frac{2u\, du}{u^4(1-u^2)} = \frac{1}{2}\int \frac{dv}{v^2(1-v)}$ $(v=u^2)$

待定系数法：$\frac{1}{v^2(1-v)} = \frac{Av+B}{v^2} + \frac{C}{1-v} = \frac{A}{v} + \frac{B}{v^2} + \frac{C}{1-v}$

$\Rightarrow \frac{1}{1-v} = Av + B + C\cdot \frac{v^2}{1-v}$

$\cdot$ 令 $v \to 0 \Rightarrow B = 1$ $\quad\quad$ $\cdot$ 令 $v \to +\infty \Rightarrow A = C = 1$

$\cdot$ 令 $v \to 1 \Rightarrow C = 1$

$$\Rightarrow I = \frac{1}{2}\int\left(\frac{1}{v} + \frac{1}{v^2} + \frac{1}{1-v}\right) dv$$

$$= \frac{1}{2}\ln\left|\frac{v}{1-v}\right| - \frac{1}{2v} + C = \cdots$$

$\cdot$ 另解: $I = \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos x} dx = \int \frac{dx}{\sin x \cos x} + \int \frac{\cos x}{\sin^3 x} dx$

$$= \int \frac{dx}{\tan x \cdot \cos^2 x} + \int \frac{d\sin x}{\sin^3 x}$$

$$= \int \frac{d\tan x}{\tan x} + \int \frac{d\sin x}{\sin^3 x} = \ln|\tan x| - \frac{1}{2\sin^2 x} + C$$

注: 还可以

$$I = \int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^3 x \cos x} dx = \int \frac{\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x}{\sin^3 x \cos x} dx$$

$$= \int \tan x \, dx + \int \frac{2}{\tan x} dx + \int \frac{1}{\tan x} \cdot \left(\frac{1}{\sin^2 x} - 1\right) dx$$

$$= \int \tan x \, dx + \int \frac{1}{\tan x} dx + \int \cot x \cdot \boxed{\frac{dx}{\sin^2 x}} = -d\cot x$$

$$= -\ln|\cos x| + \ln|\sin x| - \frac{1}{2}\cot^2 x + C.$$

总结: 若分子上只有 dx, 则可以乘某些东西化为 $du = (\cdots)dx$.

特别部分: $\sin x$、$\cos x$ 二次式积分.

$$I = \int \frac{dx}{A\cos^2 x + 2B\cos x \sin x + C\sin^2 x} \quad (A \neq 0)$$

首先转化为有理分式

$$I = \int \frac{\frac{1}{\sin^2 x} dx}{A\cot^2 x + 2B\cot x + C} = -\int \frac{d\cot x}{A\cot^2 x + 2B\cot x + C}$$

$$= -\int \frac{dt}{At^2 + 2Bt + C} = -\frac{1}{A}\int \frac{dt}{\left(t + \frac{B}{A}\right)^2 + \frac{1}{A^2}(AC - B^2)}$$

令 $t_0 = \frac{B}{A}$, $\beta = \frac{1}{A^2}(AC - B^2)$.

① $AC-B^2>0$: $I=-\frac{1}{A}\cdot\frac{1}{\sqrt{\beta}}\arctan\frac{\cot x+t_0}{\sqrt{\beta}}+C$

② $AC-B^2=0$: $I=\frac{1}{A}\cdot\frac{1}{\cot x+t_0}+C$

③ $AC-B^2<0$: $I=-\frac{1}{A}\cdot\frac{1}{2\sqrt{-\beta}}\ln\left|\frac{\cot x+t_0-\sqrt{-\beta}}{\cot x+t_0+\sqrt{-\beta}}\right|+C$

换元法2：代入法 (换元法1的反向用法)

目的： $\int f(x)dx=\int f(x(t))\,x'(t)\,dt$

$(dx=d(x(t))=x'(t)dt)$

例： $I=\int\frac{dx}{\sqrt{x^2+a^2}}$ 令 $x=a\tan t$

$$\Rightarrow I=\int\frac{dx}{a\sqrt{\tan^2t+1}}=\frac{1}{a}\int\frac{\frac{a\,dt}{\cos^2t}}{\frac{1}{\cos t}}=\int\frac{dt}{\cos t}$$

$dx=\frac{a}{\cos^2t}dt$, $1+\tan^2t=\frac{1}{\cos^2t}$

查积分表

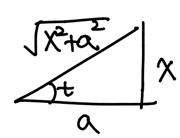
?

$\Rightarrow I=\ln\left|\tan\left(\frac{t}{2}+\frac{\pi}{4}\right)\right|+C$. 恢复为含x的式子.

$\tan\frac{t}{2}=\frac{1-\cos t}{\sin t}$ (恢复要用式子)

$\tan\left(\frac{t}{2}+\frac{\pi}{4}\right)=\frac{1-\cos\left(t+\frac{\pi}{2}\right)}{\sin\left(t+\frac{\pi}{2}\right)}=\frac{1+\sin t}{\cos t}$

$\rightsquigarrow I=\ln\left|\frac{1+\sin t}{\cos t}\right|+C$ $(\tan t=\frac{x}{a})$



$=\ln\left|\frac{1+x/\sqrt{x^2+a^2}}{a/\sqrt{x^2+a^2}}\right|+C$

$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right|+C$

补充： $\int\frac{dt}{\cos t}=\int\frac{\cos t\,dt}{\cos^2t}=\int\frac{d\sin t}{1-\sin^2t}$

$=-\frac{1}{2}\ln\left|\frac{1-\sin t}{1+\sin t}\right|+C$ $=\frac{1-\sin^2t}{(1+\sin t)^2}=\frac{\cos^2t}{(1+\sin t)^2}$

$=\ln\left|\frac{1+\sin t}{\cos t}\right|+C$

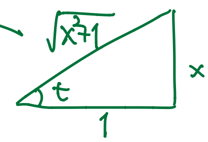
例：$I=\int\dfrac{dx}{x\sqrt{x^2+1}}$ （之前：同乘 x）

令 $x=\tan t$（去根号）

$$\Rightarrow I=\int\frac{\frac{1}{\cos^2 t}\cdot dt}{\tan t\cdot\frac{1}{\cos t}}=\int\frac{dt}{\sin t}$$ ← 比较上方"补充".

$$=\int\frac{\sin t\,dt}{1-\cos^2 t}=-\int\frac{d\cos t}{1-\cos^2 t}=\frac{1}{2}\ln\left|\frac{1-\cos t}{1+\cos t}\right|+C$$

$$=\ln\left|\frac{\sin t}{1+\cos t}\right|+C$$



$$=\ln\left|\frac{x/\sqrt{x^2+1}}{1+1/\sqrt{x^2+1}}\right|+C$$

$$=\ln\left|\frac{x}{\sqrt{x^2+1}+1}\right|+C.$$

例：$I=\int\dfrac{dx}{x(1+x^n)}$ （之前：上下同乘 x^{n-1}）

令 $x^n=t$（即 $x=t^{1/n}$）. 这样有 $dt=nx^{n-1}dx$

$$\Rightarrow\quad I=\int\frac{\frac{1}{n}t^{1/n-1}dt}{t^{1/n}(1+t)}=\frac{1}{n}\int\frac{dt}{t(1+t)}$$

$$=\frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt=\frac{1}{n}\ln\left|\frac{t}{1+t}\right|+C$$

$$=\frac{1}{n}\ln\left|\frac{x^n}{x^n+1}\right|+C$$

例：$I=\int\dfrac{dx}{(x^2+1)^{3/2}}$.

令 $x=\tan t$，则 $I=\int\cos t\,dt=\sin t+C=\dfrac{x}{\sqrt{x^2+1}}+C$.

例：$I=\int\sin^3 x\,dx$,

令 $t=\sin x\Rightarrow I=\int t^3\dfrac{dt}{\sqrt{1-t^2}}=\dfrac{1}{2}\int\dfrac{t^2}{\sqrt{1-t^2}}d(t^2)$. 令 $v=t^2$

$$\Rightarrow I=\frac{1}{2}\int\frac{v\,dv}{\sqrt{1-v}}=\frac{1}{2}\int\frac{1-(1-v)}{\sqrt{1-v}}dv=-\frac{1}{2}\int\left(\frac{1}{\sqrt{1-v}}-\sqrt{1-v}\right)d(1-v)$$

$(1-v)^{-1/2}$

$$=-(1-v)^{1/2}+\frac{1}{2}\cdot\frac{2}{3}(1-v)^{3/2}+C$$

其中 $1-v=1-t^2=1-\sin^2 x=\cos^2 x$

$\Rightarrow I=-\cos x+\frac{1}{3}\cos^3 x+C.$

分部积分法.

公式 $\int u\,dv = uv-\int v\,du.$ (没有C).

例: 有些积分不分部无法处理

$I=\int x\cdot e^x dx$

尝试: 换元法. 令 $e^x=t.$ $dx=d(\ln t)=\frac{1}{t}dt$

$\Rightarrow I=\int t\cdot \ln t\cdot \frac{1}{t}dt=\int \ln t\, dt$

($\ln t$ 的原函数: $(t\ln t)'=\ln t+1 \Rightarrow (t\ln t-t)'=\ln t$)

无法绕开分部.

另法: $x\,dx=d(\frac{x^2}{2})$

$I=\int e^x d(\frac{x^2}{2})=\frac{x^2}{2}e^x-\underbrace{\int \frac{x^2}{2}d(e^x)}_{\text{无法处理}}$

$\rightsquigarrow \int x^2 e^x dx$, 次数反而升高.

正解: $I=\int x\,d(e^x)=xe^x-\int e^x dx$

$=xe^x-e^x+C=(x-1)e^x+C.$

Upshot 分部积分可以提高或降低某项次数来求积分.

用分部积分求 $\ln t$:

$I=\int \ln t\cdot dt=t\ln t-\int t\,d(\ln t)$

$=t\ln t-\int 1\cdot dt=t\ln t-t.$

例: $I=\int x\sin x\,dx$

$=\int x\,d(-\cos x)=-x\cos x-\int(-\cos x)dx$

$=-x\cos x+\sin x+C.$

例：$I=\int x\ln^2 x\,dx$ ← 核心：用upshot
将$\ln x$次数 $2\to1\to0$.

扔到d后面的项 → 次数在升.

另一项 → 次数在降.

$$I=\frac{1}{2}\int \ln^2 x\,d(x^2)=\frac{1}{2}x^2\cdot\ln^2 x-\int\frac{x^2}{2}d(\ln^2 x)$$

$$=\frac{x^2}{2}\ln^2 x-\int x\ln x\,dx \quad \text{1次}$$

$$=\frac{x^2}{2}\ln^2 x-\left(\int \ln x\,d\frac{x^2}{2}\right)$$

$$=\frac{x^2}{2}\ln^2 x-\left(\frac{x^2}{2}\cdot\ln x-\int\frac{x^2}{2}d(\ln x)\right)$$

$$=\frac{x^2}{2}\ln^2 x-\frac{x^2}{2}\ln x+\int\frac{x^2}{2}\cdot\frac{1}{x}dx$$

$$\Rightarrow I=\frac{x^2}{2}\left(\ln^2 x-\ln x+\frac{1}{2}\right)+C.$$

有时看似无法直接下手（只给一个函数）

例：$I=\int \arctan x\,dx$

$$=x\arctan x-\int x\,d(\arctan x)$$

$$=x\arctan x-\int\frac{x}{1+x^2}dx$$

$$=x\arctan x-\int\frac{1}{1+x^2}d\left(\frac{x^2}{2}\right)$$

$$=x\arctan x-\frac{1}{2}\ln(1+x^2)+C$$

最麻烦的一种：循环现象.

例：$I=\int e^{ax}\cdot\sin bx\,dx$ 不断计算，回到自己，解方程.

$$=\frac{1}{a}\int\sin bx\cdot d(e^{ax})$$

$$=\frac{1}{a}e^{ax}\sin bx-\frac{1}{a}\int e^{ax}d\sin bx$$

$$=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\boxed{\int e^{ax}\cos bx\,dx}$$

同理，$J=\int e^{ax}\cos bx\,dx=\frac{1}{a}e^{ax}\cos bx+\frac{b}{a}\int e^{ax}\sin bx\,dx$

$$\Rightarrow\ I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}\boxed{\int e^{ax}\sin bx\,dx}$$

$$\Rightarrow\ \left(1+\frac{b^2}{a^2}\right)I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx$$

$$\Rightarrow\ I=\frac{a^2}{a^2+b^2}e^{ax}\left(\frac{1}{a}\sin bx-\frac{b}{a^2}\cos bx\right)+C$$

$$=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C.$$

同理 $J=\dfrac{e^{ax}}{a^2+b^2}(a\cos x+b\sin x)+C.$

$(\cot x)'=-\dfrac{1}{\sin^2 x}$

$(\tan x)'=\dfrac{1}{\cos^2 x}.$

习题: 1. $I=\displaystyle\int\frac{\ln\cos x}{\sin^2 x}\,dx$

$$=-\cot x\ln\cos x+\int\cot x\,d(\ln\cos x)$$

$$=-\cot x\ln\cos x+\int(-1)\cdot dx$$

$$=-\cot x\ln\cos x-x+C.$$

2. $I=\displaystyle\int\sqrt{a^2-x^2}\,dx\quad(a>0)$

$$=x\sqrt{a^2-x^2}-\int x\,d\sqrt{a^2-x^2}=x\sqrt{a^2-x^2}-\int x\cdot\frac{-2x}{2\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$

原则: 使分子也出现 a^2-x^2

$$=x\sqrt{a^2-x^2}+\int\frac{a^2}{\sqrt{a^2-x^2}}\,dx-\underbrace{\boxed{\int\sqrt{a^2-x^2}\,dx}}_{I}$$

$$\int\frac{a^2\,d\left(\frac{x}{a}\right)}{\sqrt{1-\frac{x^2}{a^2}}}=a^2\arcsin\frac{x}{a}.\qquad a\,dx=a^2\,d\left(\frac{x}{a}\right)$$

$$\Rightarrow\ I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$$

例: $I_n=\displaystyle\int\sin^n x\,dx$ 降次

$$=\int\sin^{n-1}x\,d(-\cos x)=-\cos x\sin^{n-1}x+\int\cos x\,d(\sin^{n-1}x)$$

$$=-\cos x\sin^{n-1}x+\int\cos x\,(n-1)\cdot\sin^{n-2}x\cdot\cos x\,dx$$

$$=-\cos x\sin^{n-1}x+(n-1)\int\underbrace{\cos^2 x}_{1-\sin^2 x}\sin^{n-2}x\,dx$$

$$\Rightarrow I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \left(1-\frac{1}{n}\right)I_{n-2}.$$

例：$I_n = \displaystyle\int \frac{dx}{\sin^n x}$　　原则：要使dx前有东西.

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^n x}\,dx = I_{n-2} + \int \frac{\cos^2 x}{\sin^n x}\,dx$$

$$= I_{n-2} + \int \cos x \, d\left(\frac{\sin^{1-n}x}{1-n}\right)$$

$$= I_{n-2} + \cos x \cdot \frac{\sin^{1-n}x}{1-n} - \int \frac{\sin^{1-n}x}{1-n}\,d(\cos x)$$

$$= I_{n-2} + \cos x\,\frac{\sin^{1-n}x}{1-n} + \frac{1}{1-n}\underbrace{\boxed{\int \sin^{2-n}x\,dx}}_{I_{n-2}}$$

$$\Rightarrow I_n = \frac{1}{1-n}\sin^{1-n}x\cos x + \frac{2-n}{1-n}I_{n-2}.$$

脑筋急转弯

用凑微分法

$$\int 2\sin x\cos x\,dx = \int 2\sin x\,d\sin x = \sin^2 x + C \quad \leftarrow C_1$$

$$\int 2\sin x\cos x\,dx = -\int 2\cos x\,d\cos x = -\cos^2 x + C \quad \leftarrow C_2$$

$$\Rightarrow \sin^2 x + \underset{\substack{\uparrow\\ C_1}}{C} = -\cos^2 x + \underset{\substack{\uparrow\\ C_2}}{C} \Rightarrow \sin^2 x + \cos^2 x = 0. \quad (\times)$$