定积分

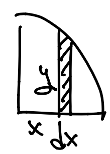
12个经典问题

例1: $I=\int_0^{\frac{\pi}{2}} \sin^n x\cos x\,dx$

$$=\int_0^{\frac{\pi}{2}}\sin^n x\,d(\sin x)=\int_0^1 t^n\,dt=\frac{t^{n+1}}{n+1}\Big|_0^1=\frac{1}{n+1}$$

例2: 求圆 $x^2+y^2\le R^2$ 的面积

$$S=4\int_0^R\sqrt{R^2-x^2}\,dx,\quad 令\ x=R\sin t$$



$y\,dx=dS$

$$=4\int_0^{\frac{\pi}{2}} R\cos t\cdot R\cos t\,dt$$

$$=4R^2\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt=4R^2\left(\frac{\pi}{4}+\frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}}\right)=\pi R^2.$$

例3: $f(x)$ 周期函数, 定义在 $\mathbb{R}$ 上. 证明: $\forall a\in\mathbb{R}$,

$$\int_a^{a+T} f(x)\,dx=\int_0^T f(x)\,dx.$$

证明: $\int_a^{a+T} f=\int_a^0 f+\int_0^T f+\int_T^{a+T} f$

对于 $\int_T^{a+T} f(x)\,dx$ 作代换 $x=t+T$

$x: T\to a+T,\quad t: 0\to a$

$$\Rightarrow \int_T^{a+T} f(x)\,dx=\int_0^a f(t+T)\,dt=\int_0^a f(t)\,dt=-\int_a^0 f(t)\,dt.$$ □

注: 若 f 为连续函数, 可对积分限求导

令 $F(a)=\int_a^{a+T} f(x)\,dx$, 去证 $F(a)=F(0)$ (或 $F(a)=$ 常数)

$\Leftarrow F'(a)=f(a+T)-f(a)=0$ (f 周期).

例4: $I=\int_0^a\sqrt{a^2+x^2}\,dx\ (a>0)$

$$\Rightarrow I = \sqrt{a^2+x^2}\cdot x\Big|_0^a - \int_0^a \frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$= \sqrt{2}a^2 - \int_0^a \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

分子要有分母的形式 again!

$$= \sqrt{2}a^2 - I + \int_0^a \frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$\Rightarrow I = \frac{\sqrt{2}}{2}a^2 + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})\Big|_0^a = \frac{a^2}{2}(\sqrt{2}+\ln(1+\sqrt{2})).$$

1245 (Wallis公式/点火公式)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$\sin^n x = \cos^n(\frac{\pi}{2}-x)$

$dx = -d(\frac{\pi}{2}-x)$, $0 \le x \le \frac{\pi}{2} \iff 0 \le \frac{\pi}{2}-x \le \frac{\pi}{2}$

分部积分降次

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}x\,d(-\cos x)$$

$$= -\cos x\cdot\sin^{n-1}x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,d(\sin^{n-1}x)$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cdot\cos^2 x\,dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n = \frac{n-1}{n}I_{n-2},\quad I_0 = \frac{\pi}{2},\ I_1 = 1.$$

$$\Rightarrow I_n = \begin{cases} \frac{(n-1)!!}{n!!}, & n\text{奇} \\ \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, & n\text{偶} \end{cases}.$$

应用：(1) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

(2) $\int_0^{\frac{\pi}{2}} \cos^2 x\sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x - \cos^4 x\,dx = I_2 - I_4$

$= \frac{\pi}{4} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{16}.$

(3) $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}.$

1246：$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\,dx$

$$\int_0^{\pi}\cos^n x\,dx = \begin{cases} 0, & n奇. \\ 2\int_0^{\frac{\pi}{2}}\cos^n x\,dx, & n偶 \end{cases}$$

例9: f 在 $[0,1]$ 上有连续导函数, $f(0)=f(1)=0$.

证明: $\left|\int_0^1 f(x)dx\right| \le \frac{1}{2}\int_0^1 |f'(x)|dx$

从左侧入手分部积分:

$$\int_0^1 f(x)dx = xf(x)\Big|_0^1 - \int_0^1 xf'(x)dx$$
$$= -\int_0^1 xf'(x)dx \quad (不够)$$

带有 $dx = d(x-\frac{1}{2})$ 的分部积分

$$\int_0^1 f(x)d(x-\tfrac{1}{2}) = (x-\tfrac{1}{2})f(x)\Big|_0^1 - \int_0^1 (x-\tfrac{1}{2})f'(x)dx$$
$$= -\int_0^1 (x-\tfrac{1}{2})f'(x)dx = \frac{1}{2}\int_0^1 f'(x)dx - \int_0^1 xf'(x)dx$$

$$\Rightarrow \left|\int_0^1 f(x)dx\right| = \left|\int_0^1 (x-\tfrac{1}{2})f'(x)dx\right|$$
$$\le \int_0^1 |x-\tfrac{1}{2}|\cdot|f'(x)|dx \le \frac{1}{2}\int_0^1 |f'(x)|dx.$$

对称性在积分计算中的应用.

在 $[a,b]$ 区间上, $x \longmapsto a+b-x$

令 $g(x) = f(a+b-x)$, $g(x)$ 与 $f(x)$ 关于 $x=\frac{a+b}{2}$ 轴对称

$$\Rightarrow g(x) = f(a+b-x) = f\left(\frac{a+b}{2} + \left(\frac{a+b}{2} - x\right)\right)$$

若 f 可积, 则

$$\int_a^b f(x)dx = \int_a^b g(x)dx = \int_a^b f(a+b-x)dx$$

特别地, 在 $[0,a]$ 上:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

⟿ 奇偶性的推论:

f在$[0,a]$上，关于$(\frac{a}{2},0)$为奇函数，

即$\forall x\in[0,a]$，$f(x)=-f(a-x)$.

则$\int_0^a f(x)dx=0$.

f在$[0,a]$上，关于$x=\frac{a}{2}$为偶函数，

即$\forall x\in[0,a]$，$f(x)=f(a-x)$，

则$\int_0^a f(x)dx=2\int_0^{a/2}f(x)dx$.

不怎么用

一个重要恒等式，$\forall f$（不一定是偶函数）

$$\int_0^a f(x)dx=\int_0^{a/2}f(x)dx+\int_0^{a/2}f(a-x)dx$$ ← 比较有用.

更一般版本: $\int_a^b f(x)dx=\int_a^{\frac{a+b}{2}}(f(x)+f(a+b-x))\,dx$.

例: $I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

希望是$[0,\pi]$上偶函数/奇函数

优先验证偶:

$$f(\pi-x)=\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}=\frac{(\pi-x)\sin x}{1+\cos^2 x}\overset{?}{=}f(x)$$

$$\Rightarrow f(x)+f(\pi-x)=\frac{\pi\sin x}{1+\cos^2 x}$$

$$\Rightarrow I=\pi\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx=-\pi\int_0^{\frac{\pi}{2}}\frac{d(\cos x)}{1+\cos^2 x}=-\pi\arctan(\cos x)\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}.$$

例: f在$[-\pi,\pi]$上偶，在$[0,\pi]$关于$\pi/2$奇，

证明: $\forall n\in\mathbb{N}_+$，$I=\int_{-\pi}^{\pi}f(x)\cdot\cos 2nx\,dx=0$

被积$f(x)\cdot\cos 2nx$在$[-\pi,\pi]$偶

$$\Rightarrow I=2\int_0^\pi f(x)\cos 2nx\,dx$$

其中$\int_0^\pi f(x)\cos 2nx\,dx=\int_0^{\frac{\pi}{2}}\Big(f(x)\cos 2nx+\underbrace{f(\pi-x)}_{=-f(x)}\underbrace{\cos 2n(\pi-x)}_{=\cos(-2nx)=\cos 2nx}\Big)dx$

$=0$

其它答案写法: 作代换$t=\pi-x$

$$\Rightarrow\int_0^\pi f(x)\cos 2nx\,dx=\int_\pi^0 f(\pi-t)\cos 2n(\pi-t)\,d(\pi-t)$$

$$= \int_0^{\pi} (-f(t)) \cos 2nt \, dt$$

$\Rightarrow I = -I \Rightarrow I = 0.$

例：$I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

直接代入 $1-x$，分母不友好，先换元：$x = \tan t$

$\Rightarrow dx = \sec^2 t \, dt$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1+\tan t) dt$ $\qquad = \frac{1-\tan t}{1+\tan t}$

$= \int_0^{\frac{\pi}{8}} \ln(1+\tan t) + \ln(1+\boxed{\tan(\frac{\pi}{4}-t)}) \, dt$

$= \int_0^{\frac{\pi}{8}} \ln(1+\tan t + 1 - \tan t) \, dt = \frac{\pi}{8} \ln 2.$

因为 $\tan(\frac{\pi}{2}-x) = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)} = \frac{\cos x}{\sin x} = \cot x.$

例：证明：$\forall \alpha \in \mathbb{R}$, $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^\alpha x} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot^\alpha x} = \frac{\pi}{4}.$

只证 $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^\alpha x} = \frac{\pi}{4}.$

左式 $= \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\tan^\alpha x} + \frac{1}{1+\cot^\alpha x}\right) dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\tan^\alpha x} + \frac{\tan^\alpha x}{1+\tan^\alpha x}\right) dx = \frac{\pi}{4}.$