Lecture 1: Introduction to ghost conj and some Corollaries.

§1 Question of slopes

- Fix a prime number $p \geq 5$,
  $E/\mathbb{Q}_p$ fin ext'n (coeff field).
  $E \supseteq \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) = \mathbb{F}$. Assume $\sqrt{p} \in E$.
- Fix integer $N \geq 4$, $p \nmid N$.
- $S_k(p, N; \psi) :=$ space of cusp modular forms of level pN & wt k.
  with Nebentypus char $\psi: (\mathbb{Z}/p^m N\mathbb{Z})^\times \to (\mathbb{Z}/p^m\mathbb{Z})^\times \to E^\times$.
  (Want $m \geq 1$ even, if ψ = trivial; m minimal.)
- Fix $\bar{r}$:= absolute irred residue rep'n, $\bar{r}: \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{F})$

$\rightsquigarrow$ $S_k(p^m N; \psi)_{\bar{r}}$ = "localization of $S_k(p^m N; \psi)$ at the Hecke ideal corresp to $\bar{r}$."

$\circlearrowleft$ U_p-operator

i.e. $\mathbb{T} = \mathrm{Im}(\mathcal{O}[T_\ell; \ell \nmid pN] \to \mathrm{End}_{\mathcal{O}}(S_k))$
$\cup$
$m_{\bar{r}} = (\varpi, T_\ell - \widetilde{tr(\bar{r}(\mathrm{Frob}_\ell))}; \ell \nmid pN)$.

where $U_p(f) = \sum_{n \geq 1} a_{pn} q^n$, $a_1 = 1$, $f = \sum_{n \geq 1} a_n q^n$.

<u>Question</u> For an eigenform f of U_p (i.e. $U_p(f) = a_p \cdot f$),
what's the possible value of $\underbrace{v_p(a_p)}_{\text{slopes of } f}$?

* Why do we care this?

① Deligne $\begin{cases} \text{either } \psi = \mathbb{1} \text{ and } f \text{ "}p\text{-new"} \Rightarrow a_p = \pm p^{\frac{k-2}{2}} \\ \text{o/w } |a_p|_\infty = p^{\frac{k-1}{2}}. \end{cases}$

($\ell \neq p$. $v_\ell(a_p) = 0$.).

③ From the point of view of (p-adic) local Langlands corresp:

$$\begin{array}{ccc} f \bmod p & \longleftrightarrow & \bar{r} : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}) \\ \Big\updownarrow \text{lift} & & \Big\updownarrow \text{lift} \\ f \text{ wt } k, \cdots & \longleftrightarrow & r_f : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathcal{O}(f)) \\ \text{w/ } U_p \text{ eigenval } a_p & & \text{s.t. } r_f|_{\mathrm{Gal}_{\mathbb{Q}_p}} \text{ is semistabelline} \\ & & \& \ \mathbb{D}_{pst}(r_{f,p}) \circlearrowleft \varphi \text{ has a } \varphi\text{-eigenval } a_p. \end{array}$$

all possible slopes of f $\longleftrightarrow$ all possible φ-slopes for semistabelline lifts of $\bar{r}_p$.

More precisely, $\mathbb{D}_{pst}(r_p^{univ}) \circlearrowleft \varphi$ (w/ def ring $R_{\bar{r}_p}^{\square,(0,k-1),\psi}$).

<u>Side goal</u> Information of φ-slopes

$\Rightarrow$ geometry of $(\mathrm{Spf}\, R_{\bar{r}_p}^{\square,(0,k-1),\psi})^{rig}$.

<u>§2 Newton polygon</u>

<u>Def'n</u> For a polynomial / power series $f(t) = 1 + c_1 t + c_2 t^2 + \cdots \in E[\![t]\!]$,

we define its <u>Newton polygon</u> to be the convex hull of $(n, v_p(c_n))$, denoted by $NP(f)$.

Then (with some growth condition)

$$\{v_p(\text{zeros of } f(t))\} = -\{\text{slopes of } NP(f)\}.$$

- Now, linear operator $T = U_p \circlearrowleft V$ v.s. $/E$ (V will be $S_k^{\dagger}$).

$\rightsquigarrow$ characteristic power series of T

$$C(t) := \det(1 - T\cdot t\, ;\, V) \quad (\text{if } \dim V = \infty, \text{ need } T \text{ to be compact.})$$

Thm $\{v_p(\text{eigenvals of } T)\} = -\{v_p(\text{zeros of } C(t))\} = \{\text{slopes of } NP(C)\}$.

__Related tool__ If for some basis of V, the matrix (T_{ij}) of T

satisfies $v_p(T_{ij}) \geq \lambda_i$, $\forall i,j$, $\lambda_1 \leq \lambda_2 \leq \cdots$, $\rightsquigarrow \begin{pmatrix} p^{\lambda_1} & p^{\lambda_1} & \cdots \\ p^{\lambda_2} & p^{\lambda_2} & \cdots \\ & \cdots & \end{pmatrix}$.

$\Rightarrow$ $C(t) = 1 + c_1 t + c_2 t^2 + \cdots$

$v_p(c_1) \geq \lambda_1$, $v_p(c_2) \geq \lambda_1 + \lambda_2$, $\cdots$.

$\Rightarrow$ $NP(C)$ lies above the polygon w/ slopes $\lambda_1, \lambda_2, \cdots$.

"__Cor__" $C_n \approx \det(\text{upper-left } n \times n \text{ minor})$.

__§3 p-adic weights__

__Slogan__ Even if we care about only one wt k,

it still helps to vary k p-adically.

$\Delta := (\mathbb{Z}/p\mathbb{Z})^\times$, Teichmüller char $\omega: \Delta \longrightarrow \mu_{p-1} \subseteq \mathbb{Z}_p^\times$

$\mathbb{Z}_p^\times \simeq \Delta \times (1+p\mathbb{Z}_p)^\times$.

__p-adic wts__ = conti char of $\mathbb{Z}_p^\times$ weight k, nebentypus $\psi: (\mathbb{Z}/p^m\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$

$\rightsquigarrow \chi_{(k,\psi)}: \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$

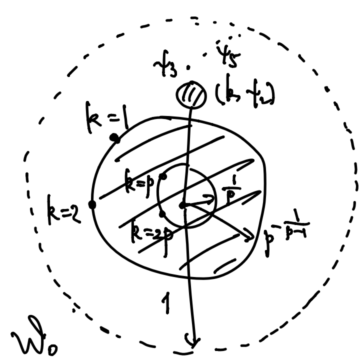
$a \longmapsto a^k \cdot \psi(a \bmod p^m)$

For an open disc $\mathcal{W}_0$ (w/ radius $=1$)

$\mathcal{W} = \mathrm{Hom}_{\text{cont gp}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$

$= \underbrace{\mathrm{Hom}_{gp}(\Delta, \mathbb{C}_p^\times)}_{p-1 \text{ copies}} \times \mathrm{Hom}((1+p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$

like $\omega^0, \omega^1, \cdots, \omega^{p-2}$ $\qquad \exp(p) \mapsto 1+w$.



$\mathcal{W}_0$

Each char $\chi: \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ corresponds to $(\chi|_\Delta,\ w_\chi := \chi(\exp(p)) - 1)$

E.g. $\chi = \chi_{(k,\mathbb{1})}$, $w_\chi = \chi_{(k,\mathbb{1})}(\exp(p)) - 1 = \exp(kp) - 1$.

$\rightsquigarrow v_p(w_\chi) = 1 + v_p(k) \ge 1$.

Consider $\psi_m: (\mathbb{Z}/p^m\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$, $\chi = \chi_{(k,\psi)}$,

$$1+p \longmapsto \zeta_{p^{m-1}}.$$

$$w_\chi = \exp(pk)\cdot \underset{\zeta_{p^{m-1}}}{\underset{\parallel}{\psi(\exp(p))}} - 1 = \underbrace{\zeta_{p^{m-1}} - 1}_{\text{main term}} + p(k)\,\zeta_{p^{m-1}} + \cdots$$

$$\Rightarrow v_p(w_\chi) = v_p(\zeta_{p^{m-1}} - 1) = \frac{1}{(p-1)p^{m-2}}.$$

<u>Major distinction</u> $v_p(w) \in (0,1)$ v.s. $v_p(w) \ge 1$

rim of the wt space (halo region) — center of weight space.

<u>Notation</u> $\omega_1: I_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$ first fundamental char

$\omega_2: I_{\mathbb{Q}_p} = I_{\mathbb{Q}_{p^2}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_{p^2}(\sqrt[p^2-1]{p})/\mathbb{Q}_{p^2}) \simeq \mathbb{F}_{p^2}^\times$.

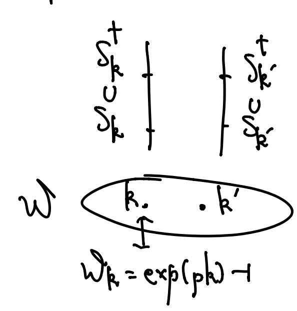
<u>Started with</u> $\bar{r}: \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F})$

$$(\det \bar{r})|_{I_{\mathbb{Q}_p}} = \omega_1^c, \quad c \in \{0, 1, \cdots, p-2\}.$$

Only consider $S_k(pN; \omega^{k-1-c})_{\bar{r}}$ (others are zero)

$$\cap\!\!\mid \quad S_k^\dagger(pN; \omega^{k-1-c})_{\bar{r}} := \lim_{M\to\infty} S_{k+(p-1)p^M}(pN, \omega^{k-1-c})_{\bar{r}}.$$

dim ~ linear in k up to $O(1)$.



<u>Theorem</u> (Coleman, Coleman-Mazur)

$$S_k(pN; \omega^{k-1-c})_{\bar{r}} = S_k^\dagger(pN; \omega^{k-1-c})_{\bar{r}}^{U_p\text{-slope} \le k-1}.$$

$$\circlearrowleft U_p.$$

$\exists !$ a char power series $C_{\bar{r}}(w,t) \in \mathcal{O}[\![w,t]\!]$ (compact)

integral !

s.t. $C_{\bar{r}}(w=w_k, t) = \det(1-U_p t \,;\, S_k^{\dagger}(pN; \omega^{k-1-c})_{\bar{r}})$.

<u>Question of slopes</u> For any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, what is $NP(C_{\bar{r}}(w_\star,-))$?

§4 Main Theorem

<u>Serre</u> $\bar{r}_p : \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$ has two kinds:

(1) Reducible
$$\bar{r}_p = \begin{pmatrix} \omega_1^{a+1} \cdot \mathrm{unr}(\bar{\alpha}) & * \\ 0 & 1 \end{pmatrix} \otimes \omega_1^b \mathrm{unr}(\bar{\beta})$$

w/ $a, b \in \{0, \dots, p-2\}$, $\bar{\alpha}, \bar{\beta} \in \mathbb{F}^\times$.

Say $\bar{r}_p$ is generic if $1 \le a \le p-4$.

In this case, $\dim H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \omega_1^{a+1}\mathrm{unr}(\bar{\alpha})) = 1$

$\Rightarrow$ up to isom, two possibilities $\begin{cases} * = 0 \\ * \neq 0 . \end{cases}$

(2) Irreducible (will not talk about)
$$\bar{r}_p = (\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{a+1}) \otimes \omega_1^b \mathrm{unr}(\bar{\beta}).$$

<u>Assume</u> $\bar{r}|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is reducible + generic, $\bar{r}$ abs irred.

<u>Fact</u> $\exists$ multiplicity $m(\bar{r}) \in \mathbb{Z}_{\ge 1}$, s.t.
$$\dim S_k(pN; \omega^{k-1-c})_{\bar{r}} = \frac{2k}{p-1} m(\bar{r}) + O(1).$$

<u>Theorem</u> (ghost conjecture of Bergdall-Pollack, Liu-Truong-Xiao-Zhao).

Assume $p \ge 11$, and $2 \le a \le p-5$.

(Ghost series).

There's an explicitly comb def'd $G(w,t) = 1 + \sum_{n \ge 1} g_n(w) t^n \in \mathbb{Z}_p[w][\![t]\!]$.

s.t. $\forall w_k \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(G_{\bar{r}}(w_k, -)) = NP(G(w_k, -))$

(except for slope 0 part)

stretched in both x-y directions $m(\bar{r})$ times.

Rmk (1) $p=2$, $N=1$: raised by Buzzard-Calegari

... Loeffler, Lisa Clay,

... Buzzard's algorithm of slopes.

$\rightsquigarrow$ Bergdall-Pollack: more conceptual explanation of $G(w,t)$.

(2) $a=1$, $p=4$, or $p=7$? Need technical issue.

(3) irreducible case, maybe not hopeless.

<u>Working def'n of $G(w,t)$</u>

$$\bar{r}|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & \mathbb{1} \end{pmatrix} \otimes \omega_1^{b}, \qquad c = 2b+a+1 \bmod p-1.$$

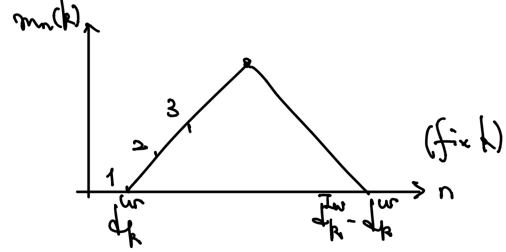
Put for $k \equiv a+2b+2 \bmod p-1$.

- $\underbrace{S_k(p^N;\mathbb{1})_{\bar{F}}}_{m(\bar{r})\cdot d_k^{Iw}} = \underbrace{S_k(N)_{\bar{F}}^{\oplus 2}}_{m(\bar{r})\cdot d_k^{ur}} \oplus S_k(pN)_{\bar{F}}^{p\text{-new}}$

- $g_n(w) = \prod_{\substack{k \equiv a+2b+2 \\ \bmod p-1}} (w - \underset{\overset{\shortparallel}{\exp(pk)-1}}{w_k})^{m_n(k)}$.

- $m_n(k) = \begin{cases} \min\{n - d_k^{ur},\ d_k^{Iw} - d_k^{ur} - n\} & \text{if } d_k^{ur} < n < d_k^{Iw} - d_k^{ur}. \\ 0, & \text{o/w} \end{cases}$

$\rightsquigarrow$



§5 Applications

All assume $\bar{r}$ irred, $\bar{r}_p$ red, $p \geq 11$, $2 \leq a \leq p-5$.

Application D (Gouvêa–Mazur Conj, 1992)

Let $n \in \mathbb{N}$. For weights $k_1, k_2 > 2n+2$

s.t. $k_1 \equiv k_2 \equiv a + 2b + 2 \mod p-1$

and $v_p(k_1 - k_2) \geq n+5$.

$\Rightarrow$ Slope seq of $S_{k_1}(pN)_{\bar{r}}$ and $S_{k_2}(pN)_{\bar{r}}$ agree up to slope n.

(originally conj'd for all $\bar{r}$, $n+5 \to n$; but $\exists$ counterexample).

- Daqing Wan showed for $n+5 \rightsquigarrow An^2 + Bn + C$.
- Combining thm with Rufei Ren.

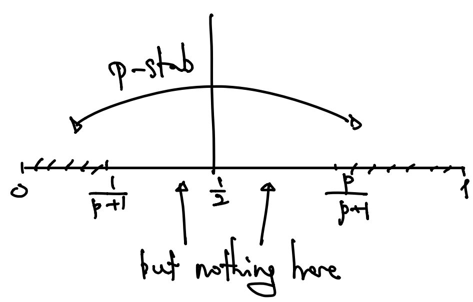
Application E (Gouvêa's slope distribution, 2001)

For each $k \equiv a + 2b + 2 \mod p-1$, write U_p-slopes on $S_k(Np)$ as

$\alpha_1(k), \dots, \alpha_d(k) \in [0, k-1]$ (with multiplicity).

$\mu_k :=$ probability measure of $\left\{ \frac{\alpha_1(k)}{k-1}, \dots, \frac{\alpha_d(k)}{k-1} \right\}$ on $[0,1]$.

$$\Rightarrow \lim_{\substack{k \to +\infty \\ k \equiv \cdots}} \mu_k = \frac{1}{p+1}\left(\delta_{[0, \frac{1}{p+1}]} + \delta_{[\frac{p}{p+1}, 1]} + \frac{p-1}{p+1} \delta_{\frac{1}{2}} \right)$$



$$S_k(pN) = \underbrace{S_k(N)^{\oplus 2}}_{p\text{-stabilization}} \oplus \boxed{\underbrace{S_k(pN)^{p\text{-new}}}_{U_p\text{-eigenval} = \pm p^{\frac{k-2}{2}}}} \leftarrow \dim = \frac{p-1}{p+1} \text{ of total dim.}$$