Lecture 10: Proof of local ghost conjecture (I)

Fix $\bar{\rho} = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix} : I_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$,

$\varpi \in \mathcal{O}, \mathbb{F}$, $\mathcal{O} \subseteq E/\mathbb{Q}_p$

$\tilde{H}$ = primitive $\mathcal{O}[\![K_p]\!]$-proj augmented module of type $\bar{\rho}$

$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially on $\tilde{H}$. $(1 \leq a \leq p-4.)$

$\varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^\times \longrightarrow \mathcal{O}^\times$ a relevant char

$(s_\varepsilon \in \{0, \cdots, p-2\})$.

$\circlearrowright S_{p\text{-adic}}^{(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$

$U_p \circlearrowright S^{\dagger, (\varepsilon)} := \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{O}\langle \frac{w}{p} \rangle^{(\varepsilon)} \langle z \rangle)$.

$C^{(\varepsilon)}(w, t) = \sum_{n \geq 0} c_n^{(\varepsilon)}(w) t^n \in \mathcal{O}[\![w, t]\!]$.

Ghost series $G^{(\varepsilon)}(w, t) = \sum_{n \geq 0} g_n^{(\varepsilon)}(w) \cdot t^n \in \mathbb{Z}_p[w][\![t]\!] \subseteq \mathcal{O}[\![w, t]\!]$.

Our goal in the last 3 lectures: to prove

<u>Thm</u> When $p \geq 11$, $2 \leq a \leq p-5$, we have

$$NP(C^{(\varepsilon)}(w_*, -)) = NP(G^{(\varepsilon)}(w_*, -)), \quad w_* \in \mathfrak{m}_{\mathbb{C}_p}.$$

<u>Warm up</u> <u>Prop 4.1</u> Fix ε.

If (1) $g_1^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times \iff \varepsilon = 1 \times \omega^a$

(2) For $k \geq 2$, let $d_{\varepsilon,k} = d_k^{\mathrm{Iw}}(\varepsilon \cdot (1 \times \omega^{2-k}))$.

Then $(d_{\varepsilon,k}, v_p(c_{d_{\varepsilon,k}}^{(\varepsilon)}(w_k)))$ (resp. $(d_{\varepsilon,k}, v_p(g_{d_{\varepsilon,k}}^{(\varepsilon)}(w_k)))$)

is a vertex of $NP(C^{(\varepsilon)}(w_k, -))$ (resp. $NP(G^{(\varepsilon)}(w_k, -))$).

Proof (1) We write $\varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^2 \longrightarrow \mathcal{O}^\times$

Assume first $s_\varepsilon = 0$. We consider $k = 2 + s_\varepsilon + \{a + s_\varepsilon\}$.

By dim formula, $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = 2$, $d_k^{\mathrm{ur}}(\varepsilon_1) = 0$.

$\Rightarrow$ U_p-slopes on $S_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ is $\frac{k-2}{2} = \frac{s_\varepsilon + \{a+s_\varepsilon\}}{2} > 0$.

$\Rightarrow$ $v_p(C_1^{(\varepsilon)}(w_k)) > 0$

$\Rightarrow$ $C_1^{(\varepsilon)}(w)$ is not a unit.

Now we assume $s_\varepsilon = 0$, $\varepsilon = 1 \times \omega^a$, $\varepsilon' = \omega^a \times 1$. $\psi = \omega^a \times \omega^a$.

By dimention, for $k = 1 + p - a$ on $\mathcal{W}^{(\varepsilon')}$,

we have $d_k^{\mathrm{Iw}}(\psi) = 2$ and $d_k^{\mathrm{ur}}(\omega^a) = 0$.

$\Rightarrow$ U_p-slopes on $S_k^{\mathrm{Iw}}(\psi)$ is $\frac{p+1-a-2}{2} > 1$.

$\Rightarrow$ $v_p(C_1^{(\varepsilon')}(w_k)) \geq \frac{p+1-a-2}{2} > 1$.

$C_1^{(\varepsilon')}(w) \in \mathcal{O}[\![w]\!] \Rightarrow v_p(C_1^{(\varepsilon')}(w_2)) \geq 1$.

By dim formula, $d_2^{\mathrm{Iw}}(\varepsilon) = d_2^{\mathrm{Iw}}(\varepsilon') = 1$.

By classicality the U_p-slope on $S_2^{\mathrm{Iw}}(\varepsilon')$ is 1.

By Atkin-Lehner $S_2^{\mathrm{Iw}}(\varepsilon) \longleftrightarrow S_2^{\mathrm{Iw}}(\varepsilon')$

the U_p-slope on $S_2^{\mathrm{Iw}}(\varepsilon)$ is 0 $\Rightarrow v_p(C_1^{(\varepsilon)}(w_2)) = 0$.

$\Rightarrow C_1^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times$.

The 1st slope of $NP(C^{(\varepsilon)}(w_\star, -))$ is 0 $\Leftrightarrow$ $\varepsilon = 1 \times \omega^a$.

$w_\star \in \mathfrak{m}_{\mathbb{C}_p}$

(2) $\psi = \varepsilon(1 \times \omega^{2-k}) = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon+2-k}$

$\psi^s = \omega^{a+s_\varepsilon+2-k} \times \omega^{-s_\varepsilon}$

$\psi' = \psi \cdot (\omega^{k-1} \times \omega^{k-1}) = \omega^{-s_\varepsilon+k-1} \times \omega^{a+s_\varepsilon+1}$.

By Atkin-Lehner:

$$AL_{(k,\psi)} : S_k^{\mathrm{Iw}}(\psi) \longrightarrow S_k^{\mathrm{Iw}}(\psi^s).$$

The $(d_{\varepsilon,k})$th slope of $S_k^{Iw}(\psi)$ is $\leq k-1$.

and equality holds $\iff a+s_\varepsilon+2-k \equiv 0 \mod p-1$ ①

By Theta map:

$$0 \to S_k^{Iw}(\psi) \to S_k^{\dagger}(\psi) \xrightarrow{\theta} S_{2-k}^{\dagger}(\psi')$$

the $(d_{\varepsilon,k}+1)$st slope of $S_k^{\dagger}(\psi) \geq k-1$

and equality holds $\iff -s_\varepsilon+k-1 \equiv 0 \mod p-1$ ②

But ① & ② cannot simultaneously happen

(by assumption on a). □

Def (Lagrangian interpolation formula)

Let $f(w) \in \mathcal{O}\langle \frac{w}{p} \rangle$ (later: $f(w) = C_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]$).

and $g(w) = (w-x_1)^{m_1} \cdots (w-x_s)^{m_s} \in \mathbb{Z}_p[w]$ (later: $g(w) = g_n(w)$.)

$x_i \in p\mathbb{Z}_p$, $m_1, \dots, m_s \in \mathbb{Z}_{>0}$.

Then we write $f(w)$ uniquely as

$$f(w) = \sum_{i=1}^{s} \Big(\underbrace{A_i(w)}_{E[w]^{<m_i}} \frac{g(w)}{(w-x_i)^{m_i}} \Big) + \underbrace{h(w)}_{E\langle \frac{w}{p} \rangle} \cdot g(w)$$

Lagrange interpolation of $f(w)$ along $g(w)$.

s.t. $f(w) \equiv A_j(w) \frac{g(w)}{(x-x_j)^{m_j}} \mod (w-x_j)^{m_j}$ in $E[\![w-x_j]\!]$, $\forall j = 1, \dots, s$.

Fix n & ε. Consider the Lagrange interpolation of $C_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]$

along $g_n^{(\varepsilon)}(w) \in \mathbb{Z}_p[w]$.

$$(\star) \quad C_n^{(\varepsilon)}(w) = \sum_{\substack{k \equiv k_\varepsilon (p-1) \\ m_n(k) \neq 0}} \left(A_k^{(n,\varepsilon)}(w) \cdot g_{n,\hat{k}}(w) \right) + h_n^{(\varepsilon)}(w)\, g_n^{(\varepsilon)}(w)$$

$$A_k^{(n,\varepsilon)}(w) = \sum_{i=0}^{m_n(k)-1} A_{k,i}^{(n,\varepsilon)} (w-w_k)^i \in E[w].$$

Prop 4.4 The local ghost conj is true if the following holds:

$\forall n, \varepsilon$ and every ghost zero w_k of $g_n^{(\varepsilon)}(w)$,

we have $v_p(A_{k,i}^{(n,\varepsilon)}) \geq \Delta^{(\varepsilon)}_{k,\frac{1}{2}d_k^{new}-i} - \Delta'^{(\varepsilon)}_{k,\frac{1}{2}d_k^{new}-m_n(k)}$.

for all $i = 0, 1, \cdots, m_n(k)-1$.

Recall $\Delta'^{(\varepsilon)}_{k,l} = v_p(g^{(\varepsilon)}_{\frac{1}{2}d_k^{Iw}+l,\hat{k}}(w_k)) - \frac{k-2}{2}\cdot l$.

$|l| \leq \frac{1}{2}d_k^{new}$, $\underline{\Delta}_k^{(\varepsilon)}$ lower convex hull of $(l, \Delta'_{k,l})$.

Proof It suffices to prove $\forall w_* \in \mathfrak{m}_{\mathbb{C}_p}$,

Claim 1 The pt $(n, v_p(C_n^{(\varepsilon)}(w_*)))$ lies on or above $NP(G^{(\varepsilon)}(w_*, -))$.

Claim 2 If $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*, -))$

then $v_p(g_n^{(\varepsilon)}(w_*)) = v_p(C_n^{(\varepsilon)}(w_*))$.

Lemma $A = A_{k,i}^{(n,\varepsilon)}$. The pt $(n, v_p(A(w-w_*)^i g_{n,\hat{k}}(w_*)))$

lies on or above $NP(G^{(\varepsilon)}(w_*, -))$ and it lies strictly on this NP

if $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*, -))$.

Pf of Claim 1 We know $A_k^{(n,\varepsilon)} \in \mathcal{O}[w] \Rightarrow h_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]$.

Pf of Claim 2 Can show $h_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times$.

We take a wt $k \equiv k_\varepsilon \bmod p-1$ s.t. $d_k^{Iw}(\varepsilon(1 \times \omega^{2-k})) = n$

$s_{\varepsilon''} = \{k-2-a-s_\varepsilon\}$.

Then the pt $(n, v_p(g_n^{(\varepsilon)}(w_k)))$ (resp. $(n, v_p(g_n^{(\varepsilon'')}(w_k)))$)

is a vertex of $NP(G^{(\varepsilon)}(w_*, -))$ (resp. $NP(G^{(\varepsilon'')}(w_*, -))$).

Lem $\Rightarrow$ $v_p(C_n^{(\varepsilon)}(w_k)) \geq v_p(g_n^{(\varepsilon)}(w_k))$, equality holds iff $h_n^{(\varepsilon)}(w_k) \in \mathcal{O}^\times$

and similar result for ε''. $\qquad\Downarrow$ $h_n(w) \in \mathcal{O}[\![w]\!]^\times$.

$$v_p(C_n^{(\varepsilon)}(w_k)) + v_p(C_n^{(\varepsilon)}(w_k)) = (k-1)n$$

$$= v_p(g_n^{(\varepsilon)}(w_k)) + v_p(g_n^{(\varepsilon'')}(w_k)).$$

<u>Rmk</u> (1) intuition of Prop 4.4.

When w_* is not close to any ghost zero w_k of $g_n(w)$.

$\Rightarrow$ $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*, -))$.

Because $v_p(A_{k,i}^{(n,\varepsilon)})$ is big $\Rightarrow$ $v_p(C_n^{(\varepsilon)}(w_*)) = v_p(g_n^{(\varepsilon)}(w_*))$.

When w_* is close to some w_k

$\Rightarrow$ $v_p(C_n^{(\varepsilon)}(w_*))$ is large.

(2) In $(*)$, let $w = w_k$ for some ghost zero w_k of $g_n(w)$.

$\Rightarrow$ $A_{k,0}^{(n,\varepsilon)} = C_n^{(\varepsilon)}(w_k) / g_{n,\hat{k}}^{(\varepsilon)}(w_k)$

$\rightsquigarrow$ the equality in Prop 4.4 becomes $\quad (n = \frac{1}{2} d_k^{\mathrm{Iw}})$.

$$v_p(C_n^{(\varepsilon)}(w_k)) \geq v_p(\underbrace{g_{d_k^{\mathrm{ur}}}(w_k)}_{\text{sum of } d_k^{\mathrm{ur}}\ U_p\text{-slopes}}) + (n - d_k^{\mathrm{ur}}) \cdot \frac{k-2}{2}.$$

$S^{\dagger,(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathcal{O}\langle \frac{w}{p}\rangle^{(\varepsilon)}\langle z\rangle)$.

We have a power basis $\quad \{e_1^{(\varepsilon)}, e_2^{(\varepsilon)}, \cdots\}$

$\qquad \| $

$B^{(\varepsilon)} = \{e_1^* \cdot z^i \cdot e_2^* \cdot z^j : i \equiv s_\varepsilon \bmod p-1,\ j \equiv a + s_\varepsilon \bmod p-1\}$

$U^\dagger = U^{\dagger,(\varepsilon)} \in M_\infty(\mathcal{O}\langle \frac{w}{p}\rangle)$ matrix of the U_p-operator on $S^\dagger$

w.r.t. the power basis $B^{(\varepsilon)}$.

Take $\underline{\zeta} = \{\zeta_1 < \zeta_2 < \cdots < \zeta_n\}$, $\underline{\xi} = \{\xi_1 < \cdots < \xi_n\}$.

(1) $U^\dagger(\underline{\zeta}, \underline{\xi})$ = $n \times n$ submatrix of $U^\dagger$

with row indices in $\underline{\zeta}$ & column indices in $\underline{\xi}$.

(2) $\deg(\underline{\zeta}) = \sum_{i=1}^{n} \deg e_{\zeta_i}$. $\det(U^\dagger(\underline{\zeta}, \underline{\xi})) \in \mathcal{O}\langle \frac{w}{p}\rangle$.

Fix ε and n. Consider the Lagrange interpolation

of $\det(U^\dagger(\underline{\zeta}, \underline{\xi}))$ along $g_n^{(\varepsilon)}(w) \in \mathbb{Z}_p[w]$.

$$\det\left(U^{\dagger}(\underline{\zeta}, \underline{\xi})\right) = \sum_{\substack{k \equiv k_\varepsilon \ (p-1) \\ m_n(k) \neq 0}} \left(A_k^{(\underline{\zeta} \times \underline{\xi})}(w) \cdot g_{n,\hat{k}}(w) \right) + h_{\underline{\zeta} \times \underline{\xi}}(w) \cdot g_n(w)$$

$$= \sum_{i=0}^{m_n(k)-1} A_{k,i}^{(\underline{\zeta} \times \underline{\xi})} (w - w_k)^i \in E[w].$$

<u>Thm 5.2</u> Assume $p \geq 11$ and $2 \leq a \leq p-5$.

For $\forall\, \underline{\zeta}, \underline{\xi}$ of size n, w_k zero of $g_n(w)$,

we have

$$v_p\left(A_{k,i}^{(\underline{\zeta} \times \underline{\xi})}\right) \geq \Delta_{k, \frac{1}{2} d_k^{\mathrm{new}} - i} - \Delta'_{k, \frac{1}{2} d_k^{\mathrm{new}} - m_n(k)} + \frac{1}{2}\left(\deg \underline{\zeta} - \deg \underline{\xi}\right).$$