Lecture 11: Proof of ghost conjecture (II) — cofactor expansion

Setup $p>11$, $2\le a\le p-5$, $\quad E\supseteq \mathcal{O}\twoheadrightarrow \mathcal{O}/(\varpi)=\mathbb{F}$.

$$\bar\rho=\begin{pmatrix}\omega_1^{a+1} & * \\ 0 & 1\end{pmatrix}: I_{\mathbb{Q}_p}\longrightarrow GL_2(\mathbb{F})$$

- $K_p=GL_2(\mathbb{Z}_p)\supseteq Iw_p=\begin{pmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p\\ p\mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}\xrightarrow{\omega}\Delta=\begin{pmatrix}\mathbb{F}_p^\times & \\ & \mathbb{F}_p^\times\end{pmatrix}\supseteq\begin{pmatrix}1+p\mathbb{Z}_p & \mathbb{Z}_p\\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p\end{pmatrix}=Iw_{p,1}$.

$\widetilde{H}$ = primitive K_p-augmented module of type $\bar\rho$:

i.e. $\widetilde{H}=\mathrm{Proj}^{\mathrm{right}}_{\mathcal{O}[[K_p]]}(\mathrm{Sym}^a\,\mathbb{F}^{\oplus 2})\ \circlearrowleft$ extends the K_p-action to $GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}$.

+ centralizer, and $\cdots$

- As $\mathcal{O}[[Iw_p]]$-mods, $\widetilde{H}=e_1\mathcal{O}\otimes_{1\times\omega^a,\,\mathcal{O}[\Delta]}\mathcal{O}[[Iw_p]]\oplus e_2\mathcal{O}\otimes_{\omega^a\times 1,\,\mathcal{O}[\Delta]}\mathcal{O}[[Iw_p]]$.

Here, we choose basis e_1,e_2, s.t. $e_1\begin{pmatrix}0&1\\p&0\end{pmatrix}=e_2$.

- $\varepsilon:\Delta=(\mathbb{F}_p^\times)^2\longrightarrow\mathcal{O}^\times$. $\varepsilon=\omega^{-s_\varepsilon}\times\omega^{a+s_\varepsilon}$.

$$S^{(\varepsilon),\,p\text{-adic}}_{\widetilde{H}}:=\mathrm{Hom}_{Iw_p}\Big(\widetilde{H},\underbrace{\mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{GL_2(\mathbb{Z}_p)}\chi^{(\varepsilon)}_{\mathrm{univ}}}_{=\ \mathcal{C}(\mathbb{Z}_p,\,\mathcal{O}[[w]])}\Big)$$

$\circlearrowleft$ U_p

$\chi^{(\varepsilon)}_{\mathrm{univ}}:\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times\longrightarrow\mathcal{O}[[w]]^\times$

$(\alpha,\omega(\bar d))\longmapsto\varepsilon(\bar\alpha,\bar d)$

$(1,\exp(p))\longmapsto 1+w$

(ε parametrizes wt disc)

$$=e_1^*\,\mathcal{C}(\mathbb{Z}_p,\mathcal{O}[[w]])^{\deg_z\equiv s_\varepsilon \bmod p-1}\ \oplus\ e_2^*\,\underbrace{\mathcal{C}(\mathbb{Z}_p,\mathcal{O}[[w]])}_{=\ \widehat{\bigoplus}_{n\ge 0}\mathcal{O}[[w]]\binom{z}{n}}{}^{\deg_z\equiv a+s_\varepsilon \bmod p-1}.$$

"power basis": $e_1^*z^{s_\varepsilon},\ e_1^*z^{s_\varepsilon+p-1},\ e_1^*z^{s_\varepsilon+2(p-1)},\ \cdots$.

$e_2^*z^{\{a+s_\varepsilon\}},\ e_2^*z^{\{a+s_\varepsilon\}+(p-1)},\ \cdots$ $\qquad$ degree of basis (on z).

$\{A\}\equiv A \bmod p-1$, with $\{A\}\in\{0,\dots,p-2\}$.

Rename these by $e_1^{(\varepsilon)},e_2^{(\varepsilon)},\cdots$, ordered by deg

$\rightsquigarrow$ $U_p^\dagger:=U_p$-action on this basis.

Define $C_{\bar{H}}^{(\varepsilon)}(w,t) := \det(I - U^{\dagger} t) = \sum C_n^{(\varepsilon)}(w) t^n \in \mathbb{Q}[\![w,t]\!]$.

* Ghost series $G_{\bar{p}}^{(\varepsilon)}(w,t) = \sum g_n^{(\varepsilon)}(w) t^n$

where $g_n^{(\varepsilon)}(w) = \prod (w - w_k)^{m_n^{(\varepsilon)}(k)}$

$$m_n^{(\varepsilon)}(k) = \begin{cases} \min\{n - d_k^{ur},\ d_k^{Iw} - d_k^{ur} - n\} & \text{if } d_k^{ur} \le n \le d_k^{Iw} - d_k^{ur} \\ 0, & \text{o/w.} \end{cases}$$

<u>Thm</u> (Local ghost) For any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$

$$NP(C_{\bar{H}}^{(\varepsilon)}(w_\star, -)) = NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -)) \quad \text{(Omit } (\varepsilon) \text{ from this notation.)}$$

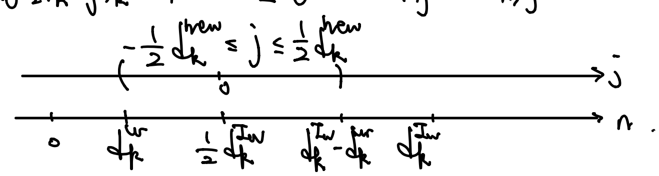
<u>Step I: Lagrange interpolation</u>

Write
$$C_n(w) = \sum_{k \equiv a+2s+2 \bmod p-1} \underbrace{g_{n,\hat{k}}(w)}_{= g_n(w)/(w-w_k)^{m_n(k)}} \left(A_{k,0}^{(n)} + A_{k,1}^{(n)}(w - w_k) + \cdots + A_{k,m_n(k)-1}^{(n)}(w-w_k)^{m_n(k)-1}\right) + h_n(w) g_n(w)$$

<u>Last time</u> To prove local ghost conj, it suffices to prove

$$v_p(A_{k,i}^{(n)}) \ge \Delta_{k,\frac{1}{2}d_k^{new} - i} - \Delta'_{k,\frac{1}{2}d_k^{new} - m_n(k)} \quad \text{for all } i = 0, 1, \dots, m_n(k) - 1.$$

Here $\Delta'_{k,j} = v_p\left(g_{\frac{1}{2}d_k^{Iw} + j, \hat{k}}(w_k)\right) - \frac{k-2}{2} j$, $\quad \Delta'_{k,j} := \Delta'_{k,-j}$.



$\{(j, \Delta_{k,j})\}$ is the convex hull of $(j, \Delta'_{k,j})$.

We prove a stronger statement.

$$\underline{\zeta} = (\zeta_1 < \cdots < \zeta_n), \quad \underline{\xi} = (\xi_1 < \cdots < \xi_n).$$

Apply the same Lagrange interpolation to $\det U^{\dagger}(\underline{\zeta} \times \underline{\xi})$. Same as conjugating by $\begin{pmatrix} p^{\frac{1}{2}\deg e_1} & & \\ & \ddots & \\ & & p^{\frac{1}{2}\deg e_{d_k^{Iw}}} \end{pmatrix}$.

$\rightsquigarrow A_{k,i}^{(\underline{\zeta},\underline{\xi})} \in E$.

<u>Need to show</u>

$$(\bigstar) \quad v_p(A_{k,i}^{(\underline{\zeta},\underline{\xi})}) \ge \Delta_{k,\frac{1}{2}d_k^{new} - i} - \Delta'_{k,\frac{1}{2}d_k^{new} - m_n(k)} + \frac{1}{2}\big(\underbrace{\deg(\underline{\zeta})}_{= \text{total deg of } e_{\zeta_i}\text{'s}} - \deg(\underline{\xi})\big)$$

by induction on size n.

<u>Step II</u> Tells about how to understand this.

<u>Step III</u> Cofactor expansion

<u>Key Input</u> When $k \equiv a+2s+2 \mod p-1$.

$$\begin{tikzcd} T_p \circlearrowright S_k^{ur}(\omega^s) \arrow[r, bend left, "i_1"] \arrow[r, shift left, "i_2"] & S_k^{I_w}(\omega^s \times \omega^s) \arrow[l, shift left, "pr_1"] \arrow[l, bend left, "pr_2"] \circlearrowleft U_p,\ AL \end{tikzcd} \qquad e_1\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = e_2$$

We have $U_p = \underbrace{i_1 \circ T_p \circ pr_2}_{\text{rank} \le d_k^{ur}} - \underset{\substack{\uparrow \\ \text{easy}}}{AL}$ $\qquad (AL(e_i) = p^{\deg e_i} \cdot e)$.

anti-diagonal,

Contributions $= p^{k-2}$

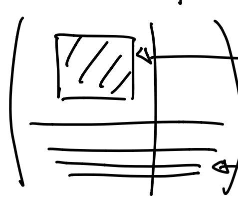
$$U^\dagger|_{w=w_k} = \left(\begin{array}{c|c} \text{rank} \le d_k^{ur} & 0 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} \begin{matrix} 0 & p^{\deg e_1} \\ p^{\deg e_{d_k^{I_w}}} & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

(idea: $k-2 \approx \deg e_{d_k^{I_w}}$)

Naive bound: ith row lies in $p^{\deg e_i}\mathcal{O}\langle \frac{w}{p} \rangle$.

Write

$$\underset{M_{\infty\times\infty}(\mathcal{O}\langle \frac{w}{p} \rangle)}{\overset{U^\dagger}{\rotatebox{90}{$\in$}}} = \left(\begin{array}{c|c} \begin{matrix} & & -p^{\deg e_1} \\ & -p^{\deg e_2} & \\ \cdot^{\cdot^{\cdot}} & & \end{matrix} & U^\dagger|_{w=w_k} \\ \hline 0 & U^\dagger|_{w=w_k} \end{array}\right) + T_k$$

Here $T_k =$ 

Can do elementary row operation so that at least $d_k^{I_w} - d_k^{ur}$ rows are div by $w - w_k$

all div by $w - w_k$.

<u>Precise version</u> $U^\dagger(\underline{\zeta} \times \underline{\xi}) = L_k(\underline{\zeta} \times \underline{\xi}) + T_k(\underline{\zeta} \times \underline{\xi})$.

For simplicity, $\underline{\zeta}, \underline{\xi} \subseteq \{1, \dots, d_k^{I_w}\}$, $d_k^{I_w} \le n \le \frac{1}{2} d_k^{I_w}$.

$$r_{\underline{\zeta}\times\underline{\xi}} := \#\{\eta \in \underline{\xi} \mid \eta^{op} \in \underline{\zeta}\}, \quad \text{rank } L_k = r_{\underline{\zeta}\times\underline{\xi}}.$$

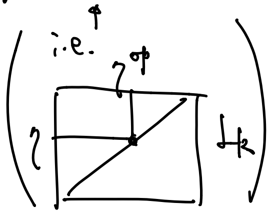
$$\text{Corank } U^\dagger(\underline{\zeta}, \underline{\xi})|_{w=w_k} \ge \underbrace{n - d_k^{ur}}_{m_n(k)} - r_{\underline{\zeta}\times\underline{\xi}}.$$

If $r_{\underline{\zeta}\times\underline{\xi}} = 0$, i.e. $L_k(\underline{\zeta}\times\underline{\xi}) = 0$ so that $\det U^\dagger(\underline{\zeta}\times\underline{\xi})$ is divisible by $(w-w_k)^{m_n(k)}$

$\Rightarrow$ All $A_{k,i}^{(\underline{\zeta}\times\underline{\xi})} = 0$. (nothing to prove.)

If $r_{\underline{\xi}\times\xi} = 1$, in this case, $\det U^{\dagger}(\underline{\xi}\times\xi)$ is div by $(w-w_k)^{m_n(k)-1}$

i.e.



$\Rightarrow$ So suffices to look at $A^{(\underline{\xi},\xi)}_{k,m_n(k)-1}$.

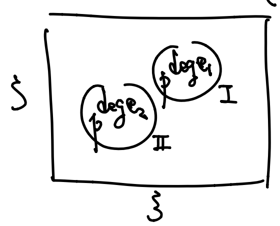
Write $U^{\dagger}(\underline{\xi}\times\xi) = T_k(\underline{\xi}\times\xi) + L_k(\underline{\xi}\times\xi)$.

$$\Rightarrow \det U^{\dagger}(\underline{\xi}\times\xi) = \underbrace{\det T_k(\underline{\xi}\times\xi)}_{\text{div by } (w-w_k)^{m_n(k)}} \pm p^{\deg e_\eta}\cdot \underbrace{\det U^{\dagger}(\underline{\xi}\setminus\eta,\ \xi\setminus\eta^{op})}_{\text{use ``}A^{(\underline{\xi}\setminus\eta,\ \xi\setminus\eta^{op})}_{k,m_n(k)-1}\text{''}}.$$

$m_{n-1}(k)$.

If $r_{\underline{\xi}\times\xi} = 2$, say $\eta_1,\eta_2\in\underline{\xi}$, $\eta_1^{op},\eta_2^{op}\in\xi$:

Write $T_k(\underline{\xi}\times\xi,\eta_i) = U^{\dagger}(\underline{\xi}\times\xi) + p^{\deg e_{\eta_i}}$ at (η_i,η_i^{op})-entry



$$\begin{cases} \det U^{\dagger}(\underline{\xi}\times\xi) \text{ div by } (w-w_k)^{m_n(k)-2} \\ \det T_k(\underline{\xi}\times\xi,\eta_i) \text{ div by } (w-w_k)^{m_n(k)-1} \\ \det T_k(\underline{\xi}\times\xi) \text{ div by } (w-w_k)^{m_n(k)}. \end{cases}$$

$\rightsquigarrow$ need to consider $A^{(\underline{\xi},\xi)}_{k,m_n(k)-1}$, $A^{(\underline{\xi},\xi)}_{k,m_n(k)-2}$.

items \ choices	neither	only $p^{\deg e_{\eta_1}}$	only $p^{\deg e_{\eta_2}}$	both
$\det U^{\dagger}(\underline{\xi}\times\xi)$	✓	✓	✓	✓
$\det T_k(\underline{\xi}\times\xi,\eta_1)$	✓		✓	
$\det T_k(\underline{\xi}\times\xi,\eta_2)$	✓	✓		
$\det T_k$	✓			
$p^{\deg e_{\eta_1}}\cdot\det U^{\dagger}(\underline{\xi}\setminus\eta_1,\ \xi\setminus\eta_1)$		✓		✓
$p^{\deg e_{\eta_2}}\cdot\det U^{\dagger}(\underline{\xi}\setminus\eta_2,\ \xi\setminus\eta_2)$			✓	✓
$p^{\deg e_{\eta_1}+\deg e_{\eta_2}}\det U^{\dagger}(\underline{\xi}\setminus\{\eta_1,\eta_2\},\ \xi\setminus\{\eta_1,\eta_2\})$				✓

$$\det U^{\dagger}(\underline{\xi}\times\xi) = \underbrace{\det(T_k(\underline{\xi}\times\xi,\eta_1))}_{\text{div by } (w-w_k)^{m_n(k)-1}} + p^{\deg e_{\eta_1}}\det U^{\dagger}(\underline{\xi}\setminus\eta_1,\ \xi\setminus\eta_1^{op})$$

$\rightsquigarrow A^{(\underline{\xi}\times\xi)}_{k,m_n(k)-2}(w-w_k)^{m_n(k)-2}$.

* note also do a cofactor expansion of $\det U^{\dagger}(\underline{\xi}\setminus\eta_1,\ \xi\setminus\eta_1^{op})$

$$\det U^{\dagger}(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op}) = \underbrace{\det T_k(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op})}_{\text{div by } (w-w_k)^{m_n(k)-1}} + p^{\deg e_{\eta_2}} \det U^{\dagger}(\underline{\xi}\backslash\eta, \underline{\xi}\backslash\eta^{op})$$

$$\underbrace{\det U^{\dagger}(\underline{\xi}\times\underline{\xi})}_{\text{div by } (w-w_k)^{m_n(k)-2}} = \underbrace{\det T_k}_{\text{div by } (w-w_k)^{m_n(k)}} + \sum_{i=1}^{2} p^{\deg e_{\eta_i}} \det U^{\dagger}(\underline{\xi}\backslash\eta_i, \underline{\xi}\backslash\eta_i^{op}) - p^{\deg e_{\eta_1}+\deg e_{\eta_2}} \det U^{\dagger}(\underline{\xi}\backslash\eta, \underline{\xi}\backslash\eta^{op})$$

<u>Recall</u> $\det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \sum_{k'} g_{n,\hat{k}}(w)\left(A_{k',0}^{(\underline{\xi}\times\underline{\xi})} + \cdots\right) + h^{(\underline{\xi}\times\underline{\xi})}(w)\, g_n(w).$

Put $p^{\frac{1}{2}(\deg(\underline{\xi})-\deg(\underline{\xi}))} \cdot \dfrac{\det U^{\dagger}(\underline{\xi}\times\underline{\xi})}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)} = \sum B_{k,i}^{(\underline{\xi}\times\underline{\xi})}(w-w_k)^i$ in $E[\![w-w_k]\!]$.

<u>NTS</u>: $v_p\left(B_{k,i}^{(\underline{\xi}\times\underline{\xi})}\right) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \frac{k-2}{2}\left(d_k^{Iw}-n\right), \quad i=0,\dots,m_n(k)-1.$

<u>More generally</u>

$$p^{\frac{1}{2}(\deg\underline{\xi}-\deg\underline{\xi})} \cdot \frac{\sum_{\substack{\text{minors of } L_k \\ \text{of size } \ell}} \det(\text{minor}^{\ell\times\ell}) \cdot \det(U^{\dagger}(\text{ample}))}{g_{n-\ell,\hat{k}}(w)/g_{n-\ell,\hat{k}}(w_k)}$$

$$=: \sum B_{k,i}^{(\underline{\xi}\times\underline{\xi};\,\ell)}(w-w_k)^i.$$

By induction hypothesis (Step II)

$$v_p\left(B_{k,i}^{(\underline{\xi}\times\underline{\xi};\,\ell)}\right) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \frac{k-2}{2}\left(\frac{1}{2}d_k^{Iw}-n\right)$$

when $i \geq m_{n-\ell}(k)$ and $i \leq m_n(k)-1$, $\ell \neq 0$.

When $m = m_n(k)$,

$$\det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \det(T_k(\underline{\xi}\times\underline{\xi}, \eta_1)) + p^{\deg e_{\eta_1}} \det U^{\dagger}(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op})$$

$$\Rightarrow \left(g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)\right) \cdot B_{k,m-2}^{(\underline{\xi},\underline{\xi})}(w-w_k)^{m-2} = \frac{1}{2}\cdot\left(g_{n-1,\hat{k}}(w)/g_{n-1,\hat{k}}(w_k)\right)\cdot B_{k,m-2}^{(\underline{\xi},\underline{\xi};\,1)}(w-w_k)^{m-2}$$

$$= \left(g_{n-2,\hat{k}}(w)/g_{n-2,\hat{k}}(w_k)\right)\cdot B_{k,m-2}^{(\underline{\xi},\underline{\xi};\,2)}(w-w_k)^{m-2}.$$

$$\overset{\text{mod } w-w_k}{\Longrightarrow} B_{k,m-2}^{(\underline{\xi},\underline{\xi})} = \frac{1}{2}B_{k,m-2}^{(\underline{\xi},\underline{\xi};\,1)} = B_{k,m-2}^{(\underline{\xi},\underline{\xi};\,2)}.$$

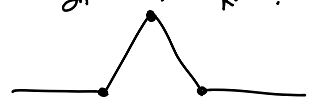
$$B_{k,m-2} + B_{k,m-1}(w-w_k) \equiv \overset{\eta(w)}{\overset{\shortparallel}{\boxed{\frac{g_{n-1,\hat{k}}(w)/g_{n-1,\hat{k}}(w_k)}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)}}}}\cdot\left(B_{k,m-2}^{(1)} + B_{k,m-1}^{(1)}(w-w_k)\right)$$

$$+ \underset{\eta'(w)}{\underset{\shortparallel}{\boxed{\frac{g_{n-2,\hat{k}}(w)/g_{n-2,\hat{k}}(w_k)}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)}}}}\cdot\left(B_{k,m-2}^{(2)} + B_{k,m-1}^{(2)}(w-w_k)\right)$$

$$\text{mod } (w-w_k)^2$$

<u>Key observation</u> $(g_{n-1}/g_n)^2 \approx g_{n-2}/g_n$

$g_n = \prod (w-w_k)^{m_n(k)}$.



$\eta(w) = 1 + \eta_1 \cdot (w-w_k) + \cdots$

Compare coeff of $w-w_k$

$$\Rightarrow B_{k,m-1} \approx \underbrace{B^{(1)}_{k,m-1} - B^{(1)}_{k,m-1}}_{=0} + \underbrace{B^{(1)}_{k,m-2}\cdot \eta_1 - 2\eta_1 \cdot B^{(2)}_{k,m-2}}_{=0}.$$