Lecture 12: Completing the proof of local ghost conjecture

Fix $\underline{\zeta} = \{\zeta_1 < \cdots < \zeta_n\}$, $\underline{\xi} = \{\xi_1 < \cdots < \xi_n\}$.

$$\det U^{\dagger}(\underline{\zeta} \times \underline{\xi}) = \sum_{m_n(k) \neq 0} \underbrace{A_k^{(\underline{\zeta}\times\underline{\xi})}(w)}\, g_{n,\hat{k}}(w) + h_{\underline{\zeta}\times\underline{\xi}}(w)\, g_n(w).$$

$$A_k^{(\underline{\zeta}\times\underline{\xi})}(w) = \sum_{i=0}^{m_n(k)-1} A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}(w - w_k)^i \in E[w], \quad h_{\underline{\zeta}\times\underline{\xi}} \in E\langle \tfrac{w}{p} \rangle.$$

Goal Prop 5.4 Assume that ghost zero w_k of $g_n(w)$,

we have $v_p(A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \Delta'_{k,\frac{1}{2}d_k^{new}-m_n(k)} + \frac{1}{2}(\deg(\underline{\zeta}) - \deg(\underline{\xi}))$ ①

for $i = 0, \ldots, m_n(k)-1$.

Then (1) $h_{\underline{\zeta}\times\underline{\xi}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})}$

(2) for every ghost zero w_{k_0} of $g_n(w)$, if we expand

$$\det(U^{\dagger}(\underline{\zeta}\times\underline{\xi}))/g_{n,\hat{k}_0}(w) = \sum_{i\geq 0} a_{k_0,i}^{(\underline{\zeta}\times\underline{\xi})}(w - w_{k_0})^i$$

then $v_p(a_{k_0,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq \frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \frac{1}{2}(d_{k_0}^{new} - i)^2 - (\frac{1}{2}d_{k_0}^{new} - m_n(k_0))^2$

$+ \Delta_{k_0,\frac{1}{2}d_k^{new}-i} - \Delta'_{k_0,\frac{1}{2}d_k^{new}-i}$

for $i = m_n(k_0), \ldots, \frac{1}{2}d_{k_0}^{new}$.

Let $p \geq 11$, $2 \leq a \leq p-5$.

Can show (1) $\Rightarrow$ (2) by a direct computation.

In the rest of the lecture, we will focus on (1).

We have $\Delta_{k,l'} - \Delta'_{k,l} \geq l' - l$, $\forall\, l' > l \geq 0$.

$\overset{①}{\Longrightarrow}$ $v_p(A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq m_n(k) - i + \frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})$

$\Rightarrow$ $A_k^{(\underline{\zeta}\times\underline{\xi})}(w) \cdot g_{n,\hat{k}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \deg g_n}$.

To show $h_{\underline{\zeta}\times\underline{\xi}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})}\mathcal{O}\langle \frac{w}{p} \rangle$,

it suffices to show $\det U^{\dagger}(\underline{\zeta}\times\underline{\xi}) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \deg g_n(w)}\mathcal{O}\langle \frac{w}{p} \rangle$.

Modified Mahler basis

Recall that on $\mathcal{C}^0(\mathbb{Z}_p; \mathcal{O}[\![w]\!]^{(\varepsilon)})$, we have a right action of

$$M_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}_p) \,\middle|\, \alpha\delta - \beta\gamma \neq 0,\ p \nmid \delta \right\}$$

$$\rightsquigarrow\ h\Big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}(z) = \varepsilon(\bar{\alpha}/\bar{\delta}, \bar{\delta})\,(1+w)^{\frac{1}{p}\log \frac{\gamma z+\delta}{\omega(\bar{\delta})}} \cdot h\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$$

$$(\alpha\delta - \beta\gamma = p^m \cdot d,\ d \in \mathbb{Z}_p^\times).$$

[LWX] Let $P = (P_{m,n})_{m,n\geq 0} \in M_\infty(\mathcal{O}[\![w]\!])$ be the matrix of the operator $\cdot\Big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}$ on $\mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$ w.r.t. Mahler basis $\left\{\binom{z}{n} : n \geq 0\right\}$.

Then $P_{m,n} \in (p,w)^{\max\{m-n,0\}}\mathcal{O}[\![w]\!] \subseteq p^{\max\{m-n,0\}}\mathcal{O}\langle \frac{w}{p}\rangle$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_1$

and $\in (p,w)^{\max\{m-\lfloor\frac{n}{p}\rfloor,0\}}\mathcal{O}[\![w]\!] \subseteq p^{\max\{m-\lfloor\frac{n}{p}\rfloor,0\}}\mathcal{O}\langle\frac{w}{p}\rangle$ with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

In our application, the elements $\binom{z}{n}$ is not eigenvectors under the action of $\bar{T} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$.

On $\mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$,

Consider $f(z) = f_1(z) = \frac{1}{p}(z^p - z) \longleftrightarrow \binom{z}{p}$

$$f_{i+1}(z) = f \circ f_i(z) = \frac{1}{p}(f_i(z)^p - f_i(z)), \quad i \geq 1.$$

$\forall n \geq 0$, write $n = n_0 + pn_1 + \cdots$ with $n_i \in \{0, \ldots, p-1\}$ and we define

$$\mathrm{m}_n(z) = z^{n_0} f_1(z)^{n_1} f_2(z)^{n_2} \cdots \in \mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}).$$

<u>Lemma</u> (1) $\forall n \geq 0$, deg of each monomial term in $\mathrm{m}_n(z)$ is convergent to n mod $p-1$.

In particular, $\bar{T}$ acts on $\mathrm{m}_n(z)$ via the character $\omega^n \times \omega^{-n}$.

(2) $\{\mathrm{m}_n(z) : n \geq 0\}$ is an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_p; \mathcal{O}[\![w]\!]^{(\varepsilon)})$ and is called the modified Mahler basis.

(3) If $P = (P_{m,n})_{m,n\geq 0}$ denote the matrix of $\cdot|_{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}$ on $C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$
w.r.t. the modified Mahler basis,
then we have the same estimation as before.

It admits an orthonormal basis $\mathbb{C} = \mathbb{C}^{(\varepsilon)}$

$$= \{ e_1^* \cdot m_i(z),\ e_2^* \cdot m_j(z) : i \equiv s_\varepsilon \bmod p-1,\ j \equiv a + s_\varepsilon \bmod p-1 \}$$

$$\varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon},\ s_\varepsilon \in \{0, \cdots, p-2\}.$$

Define $\deg(e_i^* \cdot m_j(z)) = \deg(m_j(z))$.

Write $\mathbb{C}^{(\varepsilon)} = \{ f_1^{(\varepsilon)}, f_2^{(\varepsilon)}, \cdots \}$ with increasing degrees.

If $e_n^{(\varepsilon)} = e_i^* \cdot z^j$, then $f_n^{(\varepsilon)} = e_i^* \cdot m_j(z)$ for $i = 1, 2$.

Define two matrices of the U_p-operator:

$U^{\dagger,(\varepsilon)} := (U^{\dagger,(\varepsilon)}_{e_m, e_n})_{m,n\geq 0}$ for $U_p : S^\dagger \to S^\dagger$ w.r.t. $\mathbb{B}^{(\varepsilon)}$ (power basis)

$U_{\mathbb{C}} = U_{\mathbb{C}}^{(\varepsilon)} = (U^{(\varepsilon)}_{\mathbb{C}, f_m, f_n})_{m,n\geq 0}$ for $U_p : S^{(\varepsilon)}_{p\text{-adic}} \to S^{(\varepsilon)}_{p\text{-adic}}$

w.r.t. $\mathbb{C}^{(\varepsilon)} \to$ modified Mahler basis.

<u>Prop 3.18</u> We have $U^{(\varepsilon)}_{\mathbb{C}, f_m, f_n} \in p^{\deg f_m^{(\varepsilon)} - \lfloor \frac{\deg f_n^{(\varepsilon)}}{p} \rfloor} \cdot \mathcal{O}\langle \frac{w}{p} \rangle \in \mathcal{O}[\![w]\!]$.

<u>Rmk</u> On $E[z]^{\deg \leq k-2}$ we have two basis

$\{1, \cdots, z^{k-2}\}$ (corank thm)

$\{m_0(z), \cdots, m_{k-2}(z)\}$ (halo bound).

Let $Y = (Y_{m,n})_{m,n\geq 0} \in M_\infty(\mathbb{Q}_p)$ be the change of basis matrix b/w $\{m_n(z)\}$ & $\{z^n\}$.

$\leadsto\quad m_n(z) = \sum_{m\geq 0} Y_{m,n} z^m$.

Define $Y = (Y_{e_m, f_n})_{m,n\geq 0}$ be the change of basis matrix from $\mathbb{C}$ to $\mathbb{B}$.

Then $Y_{e_m, f_n} = Y_{\deg e_m, \deg f_n}$.

Lemma $Y \in M_\infty(\mathbb{Q}_p)$ is upper-triangular with diagonal entries $Y_{n,n} \in (n!)^{-1} \cdot \mathbb{Z}_p^\times$.

$Y_{m,n} = 0$ unless $m \equiv n \mod p-1$.

Moreover, for $m<n$,

$$v_p(Y_{m,n}) \geq -v_p(m!) + \lfloor \tfrac{m}{p} \rfloor - \lfloor \tfrac{n}{p} \rfloor - \lfloor \tfrac{m-n}{p^2-p} \rfloor.$$

$$v_p((Y^{-1})_{m,n}) \geq v_p(n!) + \lfloor \tfrac{m}{p} \rfloor - \lfloor \tfrac{n}{p} \rfloor - \lfloor \tfrac{m-n}{p^2-p} \rfloor.$$

$\rightsquigarrow$ $U_p^\dagger = Y \cdot U_C \cdot Y^{-1}$.

Notation For $m, n \geq 0$, we write $m = m_0 + pm_1 + \cdots$

and $n = n_0 + pn_1 + \cdots$ with $m_i, n_i \in \{0, \ldots, p-1\}$.

We define $D(m,n) = \#\{i : n_{i+1} > m_i\}$.

Example $n = (p-1) + (p-1)p + \cdots + (p-1)p^k$, $k \geq 1$.

$m = n+1 = p^{k+1} \Rightarrow D(m,n) = k$.

Prop Let $P = (P_{m,n})_{m,n \geq 0}$ be the matrix of $\cdot|\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}^{\det \neq 0}$ on $C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$

w.r.t. $\{m_n(z) : n \geq 0\}$.

Then $P_{m,n} \in p^{D(m,n) + m - \lfloor \frac{n}{p} \rfloor} \cdot \mathcal{O}\langle \frac{w}{p} \rangle$.

(note $v_p\left(\binom{m}{m - \lfloor \frac{n}{p} \rfloor}\right) \geq D(m,n)$.)

Notation Take $\underline{\eta} = \{\eta_1 < \cdots < \eta_n\}$, $\underline{\lambda} = \{\lambda_1 < \cdots < \lambda_n\}$.

For each λ_i, write $\deg e_{\lambda_i} = \lambda_{i,0} + p \cdot \lambda_{i,1} + \cdots$.

$\forall j \geq 0$, define $D^{(\varepsilon)}_{=0}(\underline{\lambda}, j) = \#\{i \mid \lambda_{i,j} = 0\}$.

We define $D^{(\varepsilon)}_{=0}(\underline{\eta}, j+1)$ similarly.

$\rightsquigarrow$ Also define

$$D(\underline{\lambda}, \underline{\eta}) = \sum_{j \geq 0} \left(\max\{D_{=0}(\underline{\lambda}, j) - D_{=0}(\underline{\eta}, j+1), 0\}\right)$$

tuple version of $D(\lambda, \eta) = \#\{i : \lambda_i < \eta_{i+1}\}$.

Cor 7.2? $v_p(\det(U_{\mathbf{c}}^{(\varepsilon)}(\underline{\lambda}\times\underline{\eta}))) \geq D(\underline{\lambda},\underline{\eta}) + \underbrace{\sum_{i=1}^{n}\left(\deg e_{\lambda_i} - \left\lfloor \frac{\deg e_{\eta_i}}{p}\right\rfloor\right)}_{\text{LWX halo bound}}.$

Lemma. Let $\underline{n}=\{1,\dots,n\}$. Then

(1) $D_{=0}(\underline{n},j) \leq D_{=0}(\underline{n},j+1)$, $\forall j\geq 0$.

(2) Write $\deg e_n = n_0 + pn_1 + \cdots$. If $n_{j+1}=p-1$ or $n_j=n_{j+1}=0$ then $D_{=0}(\underline{n},j) = D_{=0}(\underline{n},j+1)$.

In particular, $D(\underline{n},\underline{n})=0$.

Let $\tilde{D}_{=0}(\underline{n},j)=\{m \mid m\leq \deg e_n,\ m_j=0,\ m\equiv s_\varepsilon \text{ or } a+s_\varepsilon \bmod p-1\}$.

$\Rightarrow$ $D_{=0}(\underline{n},j) = \#\tilde{D}_{=0}(\underline{n},j)$,

$\tilde{D}_{=0}(\underline{n},j) \longrightarrow \tilde{D}_{=0}(\underline{n},j+1)$.

$m=\sum_{i\geq 0} p^i m_i \xrightarrow[\text{digits}]{\text{switching } j\text{th } \& \ (j+1)\text{st}} m' = m_0 + pm_1 + \cdots + p^{j-1}m_{j-1} + p^{j+1}m_{j+1} + \cdots$.

From (*), to prove $v_p(U^{\dagger}(\underline{\xi}\times\underline{\zeta})) \geq \cdots$

it suffices to show

$$v_p(\det(U_{\mathbf{c}}(\underline{\lambda}\times\underline{\eta}))) \geq \deg g_n(w) + \frac{1}{2}(\deg\underline{\lambda}-\deg\underline{\eta}) + \sum_{i=1}^{n} v_p(\deg e_{\lambda_i}!) - v_p(\deg e_{\eta_i}!) \qquad (**)$$

Consider the special case

$\underline{\lambda}=\{1,2,\dots,n-1,n+1\}$, $\underline{\eta}=\underline{n}$.

$\delta = \deg g_n(w) - \sum_{i=1}^{n}\left(\deg e_i - \left\lfloor\frac{\deg e_i}{p}\right\rfloor\right) \in \{0,1\}$

& $\delta=1$ only when $\deg e_{i+1}-\deg e_i = p-1-a$.

Take $r=\max\{v_p(i): \deg e_n+1 \leq i \leq \deg e_{n+1}\}$.

In case (**) becomes

$$v_p(\det(U_{\mathbf{c}}(\underline{\lambda}\times\underline{\eta}))) \geq \deg g_n(w) + \frac{1}{2}(\deg e_{n+1}-\deg e_n) + r.$$

By the refined halo bound.

$$\mathrm{LHS} \geq D(\underline{\lambda}, \underline{n}) + \sum_{i=1}^{n} \left(\deg e_{\lambda_i} - \left\lfloor \frac{\deg e_i}{p} \right\rfloor\right).$$

$\rightsquigarrow$ Enough to show

$$D(\underline{\lambda}, \underline{n}) + \frac{1}{2}(\underbrace{\deg e_{n+1} - \deg e_n}_{\in \{a,\, p-1-a\}}) \geq \delta + r.$$

We use $2 \leq a \leq p-5$

$$\Rightarrow \frac{1}{2}(\deg e_{n+1} - \deg e_n) \geq \delta + 1.$$

Key : $D(\underline{\lambda}, \underline{n}) \geq r - 1.$