Lecture 2: Local ghost conj and overview of the proof

§1 Geometry of spectral curve / Eigencurve

<u>Recall</u> $C_{\bar r}(w,t) \in \mathcal{O}[[w,t]]$ char power series of U_p.

Define $\mathrm{Spc}(\bar r) :=$ zero locus of $C_{\bar r}(w,t)$ in $\mathcal{W} \times \mathbb{G}_m^{\mathrm{rig}}$.

<u>Cor</u> $\mathrm{Spc}(\bar r) \approx$ zero locus of $G(w,t)$ w/ multi $m(\bar r)$.

| Always assume $\bar r$ abs irred.
| $\bar r_p$ reducible, $2 \le a \le p-5$, $p \ge 11$

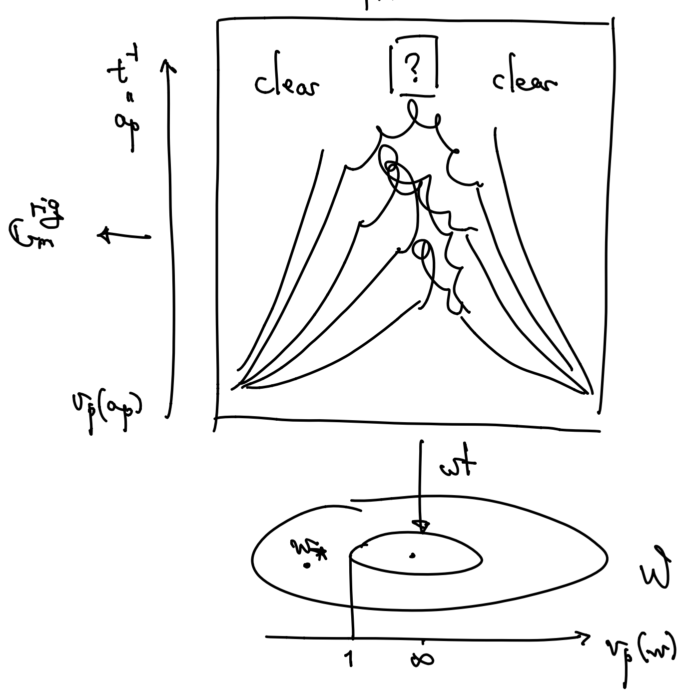
$$\mathcal{W}^{(0,1)} = \{ w \in \mathcal{W} \mid v_p(w) \in (0,1) \}$$

$$\mathrm{Spc}(\bar r)^{(0,1)} := wt^{-1}(\mathcal{W}^{(0,1)}).$$

<u>Key</u> When $w_\star \in \mathcal{W}^{(0,1)}$

$$v_p = v_p(w_\star)$$

$$v_p(g_n(w_\star)) = v_p\Big(\prod_k (w_\star - \underset{\substack{\mathrm{ss}\\ pk}}{w_k})^{m_n(k)} \Big) = v_p(w_\star) \cdot \deg g_n.$$



<u>Fact</u> $1 \le a \le p-4$, $\deg g_n - \deg g_{n-1}$ strictly increasing in n.

<u>Application F</u> (Refined spectral halo conj of Coleman, 1998)

$$\mathrm{Spc}(\bar{r})^{(0,1)} = Y_1 \sqcup Y_2 \sqcup Y_3 \sqcup \cdots$$

s.t. (1) for each point $(w_\star, a_p) \in Y_n$, $v_p(a_p) = (\deg g_n - \deg g_{n-1}) \cdot v_p(w_\star)$

possible issue with ordinary part.

(2) $\mathrm{wt}: Y_n \longrightarrow \mathcal{W}^{(0,1)}$ is finite & flat of deg $m(\bar{r})$.

<u>Remark</u> Weaker version but no constraint on $\bar{r}$: by Liu-Wan-Xiao 2017

HMF(p splits) Ren-Zhao 2022

modular symbol Diao-Yao 2023.

<u>Q</u>: Analogy b/n Hida family v.s. spectral halo?

<u>Application C</u> Write $\mathrm{Spc}(\bar{r}) = \mathrm{Spc}(\bar{r})^{\mathrm{ord}} \sqcup \mathrm{Spc}(\bar{r})^{\mathrm{nord}}$.

<u>Thm</u> $\mathrm{Spc}(\bar{r})^{\mathrm{nord}}$ has finitely many irred components.

(asked by Coleman-Mazur 1998.)

- $\lambda \in (0,1)$: A "converging" power series $F(w,t) = 1 + \sum_{n\geq 1} f_n(w) t^n \in E\langle \frac{w}{p^\lambda} \rangle [\![t]\!]$.

$\forall\, w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $v_p(w_\star) \geq \lambda$, $NP(F(w_\star, -)) = NP(G(w_\star, -))$.

stretched m times.

<u>Thm</u> If $F(w,t)$ is a power series of ghost type w/ mult m

and $F(w,t) = F_1(w,t) \cdot F_2(w,t)$

$\Rightarrow$ both $F_i(w,t)$ are of ghost type, and $m = m_1 + m_2$.

(Rigidity of ghost type power series).

§2. Local ghost conjecture

Key point 1 Don't use modular forms.

Use Betti's realization ← better integral structure.

Key point 2 (Emerton) Complete cohomology.

Assume $\bar{r}$ abs irred.

Define $\widetilde{H}_{1,\bar{r}} := \varprojlim_m H_1(X(K^p(1+p^m M_2(\mathbb{Z}_p)))(\mathbb{C}), \mathcal{O})^{\mathrm{cplx}=1}_{\bar{r}}$

Completed cohom

right $GL_2(\mathbb{Q}_p)$-action

projective $GL_2(\mathbb{Z}_p)$-mod.

Upshot This $\widetilde{H}_{1,\bar{r}}$ contains all p-adic info we need.

Note For $K_p = \mathrm{Iw}_p$ or $GL_2(\mathbb{Z}_p)$,

$$H^1(X(K^pK_p), \mathrm{Sym}^{k-2}(\mathcal{O}^{\oplus 2}))^{\mathrm{cplx}=1}_{\bar{r}} = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[K_p]}(\widetilde{H}_{1,\bar{r}}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2}).$$

as Hecke mods $\simeq$ $S_k(K^pK_p)_{\bar{r}} \circlearrowleft U_p$ $\qquad U_p$

Q: (Emerton) What's the minimum setup for ghost to work?

Def'n Let $\bar{\rho}: I_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$ be the rep'n $\begin{pmatrix} \omega_1^{a+b+1} & * \neq 0 \\ 0 & \omega_1^b \end{pmatrix}$, $1 \le a \le p-1$.

$$K_p = GL_2(\mathbb{Z}_p) \supseteq \mathrm{Iw}_p = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \supseteq \mathrm{Iw}_{p,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

$$\cup\!\shortmid \quad B^{op}(\mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

$$\cup\!\shortmid \quad \begin{pmatrix} \Delta & \\ & \mathbb{Z}_p^\times \end{pmatrix} \supseteq \begin{pmatrix} \Delta & \\ & \Delta \end{pmatrix}.$$

An <u>$\mathcal{O}[\![K_p]\!]$-projective augmented module primitive type $\bar{\rho}$</u> is

$\widetilde{H}$ = proj envelop of $\mathrm{Sym}^a \mathbb{F}^{\oplus 2} \otimes \det^b$ as a right $\mathcal{O}[\![K_p]\!]$-mod s.t. the right action extends to a cont. $GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}$-action

& (technical assumption)

$$\widetilde{H} \simeq \widetilde{H}_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![(1+p\mathbb{Z}_p)^\times]\!] \circlearrowleft (1+p\mathbb{Z}_p)^\times$$
$$\circlearrowleft$$
$$GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}(1+p\mathbb{Z}_p)^\times.$$

Rmk $\bar{\rho}$ is on Galois side $\longleftrightarrow$ corresponding Serre wt $\mathrm{Sym}^a \otimes \det^b$.

Weight discs $\longleftrightarrow$ characters of $\begin{pmatrix} \Delta & \\ & \Delta \end{pmatrix}$: $\omega^? \times \omega^?$ $\quad (\omega : \Delta \to \mathbb{Z}_p^\times)$.

"relevant character" $\varepsilon = \omega^{-s+b} \times \omega^{a+s+b}$ for some $s \in \{0, 1, \cdots, p-2\}$.

This afternoon (Lec 3)

" $S_{\widetilde{H}}^{(\varepsilon), p\text{-adic}} := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p} \chi_{univ}^{(\varepsilon)})$

$\circlearrowleft$

U_p $\quad \infty$-dim'l Banach mod$/\mathcal{O}[\![w]\!]$.

Space of abstract p-adic form.

Here $\chi_{univ}^{(\varepsilon)} : \begin{pmatrix} \Delta & \\ & \mathbb{Z}_p^\times \end{pmatrix} \longrightarrow \mathcal{O}[\![w]\!]^\times$

$$(\bar{\alpha}, \delta) \longmapsto \varepsilon(\bar{\alpha}, \bar{\delta}) \cdot (1+w)^{\log(\delta/\omega(\bar{\delta}))/p}$$

$$(1, \exp(p)) \longmapsto 1+w$$

Put $C_{\widetilde{H}}^{(\varepsilon)}(w,t) := \det(\mathbb{1} - U_p t; S_{\widetilde{H}}^{(\varepsilon), p\text{-adic}})$.

Similarly, we have ghost series $\qquad k \equiv k_\varepsilon := a+2b+2s+2 \mod p-1$

$d_k^{\mathrm{Iw}} \rightsquigarrow S_{\widetilde{H}, k}^{\mathrm{Iw}}(\omega^s) := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2} \otimes \omega^s \otimes \det)$

$\qquad i_1 \uparrow \; i_2 \uparrow \; \mathrm{pr}_1 \downarrow \; \mathrm{pr}_2 \downarrow \qquad \left(U_p(\varphi)(x) = \sum_{j=0}^{p-1} \varphi\left(x \begin{pmatrix} p & \\ pj & 1 \end{pmatrix}^{-1}\right) \right)$

$d_k^{ur} \rightsquigarrow S_{\widetilde{H}, k}^{ur}(\omega^s) := \mathrm{Hom}_{K_p}(\widetilde{H}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2} \otimes \omega^s \otimes \det)$

Define $G_{\bar{\rho}}^{(\varepsilon)}(w,t) = 1 + \sum_{n \geq 1} g_n(w) t^n \in \mathbb{Z}_p[w][\![t]\!]$ with $g_n(w) = \prod_{\substack{k \equiv a+2b+2 \\ \mathrm{mod}\, p-1}} (w - w_k)^{m_n(k)}$.

where $m_n(k) = \begin{cases} \min\{n - d_k^{ur}, d_k^{\mathrm{Iw}} - d_k^{ur} - n\} & \text{if } d_k^{ur} < n < d_k^{\mathrm{Iw}} - d_k^{ur}. \\ 0, & \text{o/w}. \end{cases}$

$$\mathrm{rk}_{\mathcal{O}}\left(\mathrm{Hom}_{K_p}\left(\tilde{H},\ \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2}\otimes\omega^{s}\otimes\det\right)\right.$$
$$=\dim_{\mathbb{F}}\left(\mathrm{Hom}_{K_p}\left(\mathrm{Proj}_{\mathbb{F}[GL_2(\mathbb{Q}_p)]}(\mathrm{Sym}^{a}\otimes\det^{b}),\ \mathrm{Sym}^{k-2}\otimes\det^{s}\right)\right)$$
$$=\#\,\mathrm{JH}_{\mathrm{Sym}^a\otimes\det^b}\left(\mathrm{Sym}^{k-2}\otimes\det^{b}\right).$$

<u>Local ghost theorem</u>

If $2\le a\le p-5$, $p\ge 11$, $\forall\, w_\star\in\mathfrak{m}_{\mathbb{C}_p}$,

$$NP\left(C_{\tilde{H}}^{(\varepsilon)}(w_\star,-)\right)=NP\left(G_{\bar{\rho}}^{(\varepsilon)}(w_\star,-)\right).$$

<u>Amazing fact</u> $\bar{\rho}$ has a comparison $\bar{\rho}'=\begin{pmatrix}\omega^{a+b+1} & 0\\ *\neq 0 & \omega^{b}\end{pmatrix}$

$\hookrightarrow$ Serre wt $\mathrm{Sym}^{p-3-a}\otimes\det^{a+b+1}$.

$G_{\bar{\rho}}^{(\varepsilon)}(w,t)=G_{\bar{\rho}'}^{(\varepsilon)}(w,t)$ except for the ordinary part

$\quad\ \ \| \qquad\qquad\quad \|$

$\ \ G^{(\varepsilon)} \qquad\quad G'^{(\varepsilon)}$ for some ε : $G^{(\varepsilon)}=1+tG'^{(\varepsilon)}$

for some other ε : $G'^{(\varepsilon)}=1+tG^{(\varepsilon)}$.

This reflects on the Galois side (when slope >0)

$\bar{r}_p$ lifts to irred rep'n V/E of $\mathrm{Gal}_{\mathbb{Q}_p}$.

But V has a lattice whose reduction is $\bar{r}_p$

$\Longleftrightarrow$ V has a lattice whose reduction is $\bar{r}_p'$. (Ribet's lemma).

For a fixed $\bar{\rho}$, $\exists!\ \varepsilon=\omega^{b}\times\omega^{a+b}$, $G_{\bar{\rho}}^{(\varepsilon)}(w,t)$ has slope 0 part.

<u>§3 Local ghost $\Rightarrow$ B-P ghost</u>

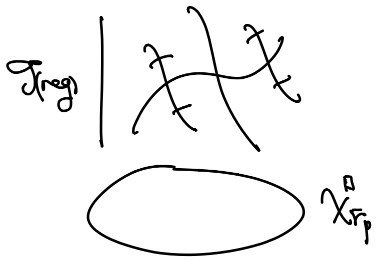
<u>Idea</u> Local ghost $\Rightarrow$ slopes on trianguline deform spaces $\Rightarrow$ B-P ghost

applying local ghost to Paškūnas module (for the first $\Rightarrow$); remove the reducible nonsplit constraint (for the second)

$\mathcal{T}:=$ rigid space of 2 conti chars $\delta_1,\delta_2:\mathbb{Q}_p^\times\longrightarrow\mathbb{C}_p^\times$.

$$\underset{\cup|}{\mathcal{T}_{reg}} = \{(\delta_1,\delta_2)\in\mathcal{T} \mid \frac{\delta_2}{\delta_1} \neq x^n,\ x^n\chi_{cycl}(x) \text{ for some } n\in\mathbb{Z}_{\geq 0}\}$$

Fix $\bar r_p \rightsquigarrow R^{\square}_{\bar r_p}$ = framed deformation ring $\mathcal{X}^{\square}_{\bar r_p} := (\mathrm{Spf}\, R^{\square}_{\bar r_p})^{rig}$



$$U^{\square,tri}_{\bar r_p} = \{(x,\delta_1,\delta_2)\in \mathcal{X}^{\square}_{\bar r_p}\times\mathcal{T}_{reg} \mid 0\to\mathcal{R}(\delta_1)\to \mathbb{D}_{rig}(V_x)\to\mathcal{R}(\delta_2)\to 0\}$$

$\mathcal{X}^{\square,tri}_{\bar r_p}$ = Zariski closure of $U^{\square,tri}_{\bar r_p}$

<u>Theorem</u> Same assumption:

For $(x,\delta_1,\delta_2)\in\mathcal{X}^{\square,tri}_{\bar r_p}$, put $w_k := \delta_1\delta_2^{-1}\chi_{cycl}^{-1}(\exp(p))-1$

$$\varepsilon = \delta_2|_\Delta \times \delta_1|_\Delta\cdot\omega^{-1}.$$

Then if $v_p(\delta_1(p)) = -v_p(\delta_2(p)) > 0$, then

$v_p(\delta_1(p))$ is a slope in $NP(G^{(\varepsilon)}_{\bar p}(w_k,-))$.

Conversely, all slopes in $NP(G^{(\varepsilon)}_{\bar p}(w_k,-))$ appears this way.

<u>Consequences</u>

<u>Application A</u> (Breuil-Buzzard-Emerton, ~2005)

(1) If r_p is a crystalline rep'n of HT wts $\{0,k-1\}$ lifting $\bar r_p$,

then $v_p(\varphi\text{-eigenval on } \mathbb{D}_{cris}(\bar r_p)) \in \begin{cases}\mathbb{Z}, & \text{if } a \text{ even}\\ \frac{1}{2}\mathbb{Z}, & \text{if } a \text{ odd.}\end{cases}$

(2) If r_p is a crystalline rep'n with wild char of conductor $p^m \geq p^2$.

φ-slope $\in \frac{1}{p^{m-2}(p-1)}\mathbb{Z}$.

Rmk Known when $v_p(-)$ small e.g. $\leq p$ by Ghate, Buzzard-Gee, Rozensztan, Berger, ...

or $wt \leq 3p$ by Breuil, Mézard (?)

(didn't have assumption on $\bar r_p$).

Application B (Gouvêa, $\lfloor \frac{k-1}{p+1} \rfloor$-conj.)

In above (1), less or φ-slope $\le \lfloor \frac{k-1}{p+1} \rfloor$.

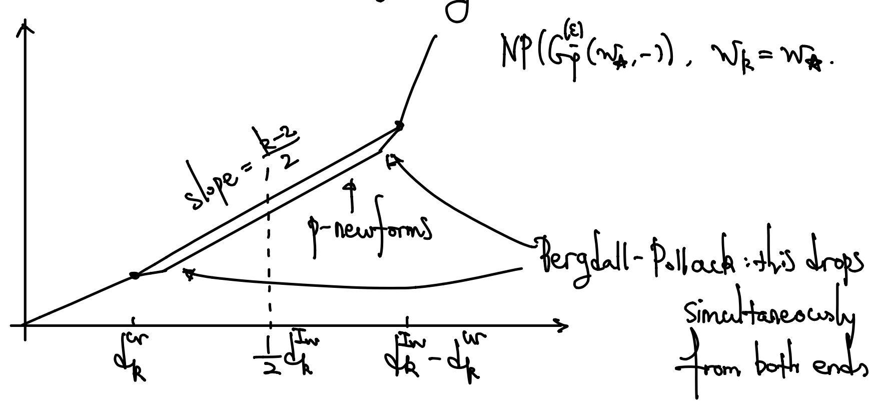
Rmk Proved by Berger-Li-Zhu for $\lfloor \frac{k-1}{p+1} \rfloor$, Bergdall-Levin for $\lfloor \frac{k-1}{p} \rfloor$.

§4 Sketch of local ghost

Step 0 Criterion for vertices of $NP(G_{\bar{p}}^{(\varepsilon)}(w,-))$.

Thm $(n, g_n(w_\star))$ is **not** a vertex iff $\exists k \equiv a+2b+2s+2 \bmod p-1$ s.t. $v_p(w_\star - w_k) \ge \cdots$ (some explicit number.)

"too close to a Steinberg wt".



Step 1 Have a matrix for U_p (power bases).

$$C_{\bar{H}}^{(\varepsilon)}(w,t) = \sum c_n(w) t^n.$$

Hope $c_n(w) \approx g_n(w)$.

Important Enough to estimate $A_k^{(n)} \cdot s + g_n(w) h_n(w)$.

Lagrange interpolation

$$c_n(w) = \sum_k g_{n,\hat{k}}(w) \cdot \left(A_{k,0}^{(n)} + A_{k,1}^{(n)}(w - w_k) + \cdots\right)$$

Will prove this for all $n \times n$ minors by induction on size.

Step 2 **Key** U_p-matrix takes a special form at $w = w_k$.

At $w = w_k$: