Lecture 3: Abstract p-adic forms & corank thm

<u>Notations</u> $\tilde{H}$ primitive $\mathcal{O}[\![K_p]\!]$-proj augmented module of type $\bar{\rho}$.

- $\varepsilon: \Delta^2 \to \mathcal{O}^\times$, $\quad \varepsilon(\bar{\alpha}, \bar{x}) = \bar{\alpha}^{a+2b}$, $\quad \bar{\alpha} \in \Delta$.
- $\mathcal{O}[\![w]\!]^{(\varepsilon)} = \mathcal{O}[\![\Delta \times \mathbb{Z}_p^\times]\!] \otimes_{\mathcal{O}[\Delta^2], \varepsilon} \mathcal{O} \simeq \mathcal{O}[\![w]\!]$

$$w = [\![(1, \exp(p))]\!] - 1.$$

- $\chi_{univ}^{(\varepsilon)}: B^{op}(\mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \longmapsto (\mathcal{O}[\![w]\!]^{(\varepsilon)})^\times$

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \longmapsto \underbrace{[(\bar{\alpha}, \delta)] \otimes 1}_{\varepsilon(\bar{\alpha}, \bar{\delta})(1+w)^{\log(\delta/\omega(\bar{\delta}))/p}}$$

- $\mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}) := \left\{ \text{cont } f: \mathrm{Iw}_p \to \mathcal{O}[\![w]\!]^{(\varepsilon)} \;\middle|\; \begin{array}{l} f(gb) = \chi_{univ}^{(\varepsilon)}(b) f(g) \\ f\big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}(g) = f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} g\right) \end{array} \right\}$

right action convolution.

$\rightsquigarrow \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}) \simeq C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$.

$$f \longmapsto h(z) = f\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

- The action extends to $M_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}_p) : p \mid \gamma,\ p \nmid \delta,\ \det \neq 0 \right\}$.

$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = p^t d$, $\quad$ for some $d \in \mathbb{Z}_p^\times$.

$$h\big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} = [(\bar{d}/\bar{\delta}, \gamma z + \delta)]\, h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

$$= \varepsilon(\bar{d}/\bar{\delta}, \bar{\delta})(1+w)^{\log((\gamma z + \delta)/\omega(\bar{\delta}))/p}\, h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right).$$

- $S_{\tilde{H}}^{(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}))$

$\simeq \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$ $\quad$ families of p-adic forms.

$\mathrm{Iw}_p \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_p = \coprod v_j \mathrm{Iw}_p$, $\quad$ e.g. $v_j = \begin{pmatrix} p & 0 \\ pj & 1 \end{pmatrix}$.

Similarly, families of overconvergent forms

- $S_{\tilde{H}}^{\dagger, (\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \Lambda^{(\varepsilon), \leq \frac{1}{p}}\langle z \rangle)$,

$\circlearrowright \quad \circlearrowright$

$U_p \quad M_1$

where $\Lambda^{\leq\frac{1}{p}} = \mathcal{O}\langle w/p\rangle$, $\Lambda^{\geq\frac{1}{p}} = \mathcal{O}[\![w]\!]\langle p/w\rangle$, $\Lambda^{\frac{1}{p}} = \mathcal{O}\langle w/p, p/w\rangle$,

and $\Lambda^{(\varepsilon),?} = \mathcal{O}[\![w]\!]^{(\varepsilon)} \otimes_{\mathcal{O}[\![w]\!]} \Lambda^{?}$.

(U_p is compact on $S_{\tilde{H}}^{\dagger,(\varepsilon)}$.)

· $\psi : \Delta^2 \longrightarrow \mathcal{O}^\times$, $k \geq 2$,

$$S^{\dagger}_{\tilde{H},k}(\psi) := \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{O}\langle z\rangle \otimes \psi).$$

For $f \in \mathcal{O}\langle z\rangle$, $f\big|\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} := (\gamma z+\delta)^{k} \cdot f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$.

$$S^{\mathrm{Iw}}_{\tilde{H},k}(\psi) := \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{O}[z]^{k-2} \otimes \psi).$$

· $(\psi, k) \rightsquigarrow \varepsilon = \varepsilon(\psi, k) = \psi\,(1 \times w^{k})$

It turns out $S^{\mathrm{Iw}}_{\tilde{H},k}(\psi) \subset S^{\dagger}_{\tilde{H},k}(\psi) \subset S^{\dagger,(\varepsilon)}_{\tilde{H}} \otimes_{w \mapsto w_k} \mathcal{O}$.

· $\eta : \Delta \longrightarrow \mathcal{O}^\times \rightsquigarrow S^{\mathrm{ur}}_{k}(\eta) := \mathrm{Hom}_{K_p}(\tilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \eta\circ\det)$

$\circlearrowleft$

$M_2 = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in M_2(\mathbb{Z}_p) : \det \neq 0\right\}$.

s.t. $(\eta\circ\det)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \eta(d)$, $\quad p^r d = \det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $d \in \mathbb{Z}_p^\times$.

<u>Char power series of U_p</u>

$U_p \circlearrowright S^{\dagger,(\varepsilon)}_{\tilde{H}}$ compact $\rightsquigarrow$ only defined over $\Lambda^{(\varepsilon),\leq\frac{1}{p}}$.

$\rightsquigarrow$ sublattice $C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\geq\frac{1}{p}})^{\mathrm{mod}}$

$\uparrow$ "modified".

$$\widehat{\bigoplus} w^{[n/p]}\binom{p}{n}\Lambda^{(\varepsilon),\geq\frac{1}{p}} \subseteq C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\geq\frac{1}{p}}).$$

resp. $p^{[n/p]}\binom{z}{n}$ with "$\leq\frac{1}{p}$".

<u>note</u> p/w is a unit in $\Lambda^{\frac{1}{p}} \Rightarrow C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\geq\frac{1}{p}})^{\mathrm{mod}}$ & $C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\leq\frac{1}{p}})^{\mathrm{mod}}$

coincide on $\Lambda^{(\varepsilon),\frac{1}{p}}$.

They glue to a Banach subsheaf of $C(\mathbb{Z}_p, \underset{\mathcal{O}[\![w]\!]^{(\varepsilon)}}{\underset{\parallel}{\Lambda^{(p)}}})$

$$w^{[\frac{n}{p}]}\binom{z}{n}\Big|_{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in M_1} = \sum_{m\geq 0} P_{m,n}\, w^{[m/p]}\binom{z}{m}.$$

Fact If $p \mid \alpha$, $(p \mid \gamma,\ p \nmid \delta)$, then $P_{m,n} \in w^{m\lceil \frac{n}{p} \rceil} \Lambda^{\geq \frac{1}{p}}$.

Similar for $p^{\lceil n/p \rceil}\binom{z}{n}$ $\Rightarrow$ U_p is compact on this Banach subsheaf!

$\Downarrow$

Can define $C_{\tilde{H}}^{(\varepsilon)}(w,t)$ as an element of $\mathcal{O}[\![w,t]\!]$.

(Stronger than Coleman-Mazur.)

From now on, assume $b=0$ $\left(\Rightarrow \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \text{ acts trivially on } \tilde{H}.\right)$

$$\bar{T} \cdot \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \subseteq \mathcal{O}[\![\mathrm{Iw}_p]\!].$$

$\rightsquigarrow$ $\tilde{H} \simeq \bigoplus_{i=1}^{r} e_i \mathcal{O} \otimes_{\chi_i, \mathcal{O}[\![\bar{T}]\!]} \mathcal{O}[\![\mathrm{Iw}_p]\!]$, $\quad r = \mathrm{rank}_{\mathrm{Iw}_p} \tilde{H}$.

noetherian local ring

Since $\tilde{H}$ is primitive,

$$\mathrm{Proj}_{a,0} \simeq \bigoplus_{i=1}^{2} e_i \mathcal{O} \otimes_{\chi_i} \mathbb{F}(\bar{B}).$$

Fact $\chi_1 = 1 \times \omega^a$, $\chi_2 = \omega^a \times 1$. (Possibly, will exchange χ_1 & χ_2.)

Lemma We may replace e_2 by $e_2' = e_1 \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ $\left(\Rightarrow e_1 = e_2' \begin{pmatrix} 0 \\ p \end{pmatrix}\right)$.

Proof If e_1 has $\bar{T}$-char $1 \times \omega^a$, then $\bar{e_2'}$ has $\bar{T}$-char $\omega^a \times 1$. $\square$

$\rightsquigarrow$ From now on, assume $e_2 = e_1 \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

$$S_{\tilde{H}}^{\dagger,(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}\left(\tilde{H}, \Lambda^{(\varepsilon), \leq 1/p}\langle z \rangle \otimes \varepsilon\right), \quad \varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon}, \quad s_\varepsilon \in \{0, \cdots, p-2\},$$

$$\simeq e_1^*\left(\Lambda^{(\varepsilon), \leq 1/p}_{\langle z \rangle \otimes \varepsilon}\right)^{\bar{T} = 1 \times \omega^a} \oplus e_2^*\left(\Lambda^{(\varepsilon), \leq 1/p}_{\langle z \rangle \otimes \varepsilon}\right)^{\bar{T} = \omega^a \times 1}.$$

$\equiv a + s_\varepsilon \mod p-1$

Prop $\{e_1^* \cdot z^{s_\varepsilon + (p-1)i}\}_{i \geq 0} \cup \{e_2^* \cdot z^{\{a+s_\varepsilon\} + (p-1)i}\}_{i \geq 0}$

is a basis of $S_{\tilde{H}}^{\dagger,(\varepsilon)}$ as a Banach module over $\Lambda^{\leq 1/p}$.

or of $S^{\dagger}_{\tilde{H},k}(\varepsilon(1 \times \omega^{2-k}))$ as a Banach $\mathcal{O}$-mod.

Moreover, terms whose (powers in z) $\leq k-2$ form an $\mathcal{O}$-basis of $S_k^{\mathrm{Iw}}(\varepsilon(1 \times \omega^{2-k}))$.

Proof · z^j has T-char $\omega^j \times \omega^{-j}$

· $(j - s_\varepsilon,\ j + a + s_\varepsilon) \equiv (a,a)$ or $(a,0)$ $\Big\}$ (mod $p-1$). □

$\Rightarrow$ $j \equiv s_\varepsilon$ or $j \equiv \{a + s_\varepsilon\}$

$B^{(\varepsilon)}$ or $B_k^{(\varepsilon)} \circlearrowleft U_p$

Remark $U^{\dagger,(\varepsilon)}_{(k)}$ = matrix of U_p w.r.t. $B^{(\varepsilon)}_{(k)}$ $\rightsquigarrow$ $C^{(\varepsilon)}_{\tilde{H}}(w,t) = \mathrm{char}(1 - U_p \cdot ?)$

Prop (Theta map) $\underset{\varepsilon'}{\overset{\varepsilon'(\psi,\,2-k)}{\shortparallel}} = \varepsilon(\omega^{k-1} \times \omega^{1-k})$, $\psi = \varepsilon(1 \times \omega^{2-k})$, $\psi' = \varepsilon'(1 \times \omega^{k})$.

(1) $\exists$ short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & S_k^{Iw}(\psi) \arrow[r] \arrow[loop above, "U_p"] & S_k^{\dagger}(\psi) \arrow[rr, "\left(\frac{d}{dz}\right)^{k-1} \circ (\cdot)"] \arrow[loop above, "U_p"] & & S_{2-k}^{\dagger}(\psi) \arrow[loop above, "p^{k-1}U_p"]
\end{tikzcd}
\quad (\text{not surj})
$$

$$\underbrace{\qquad\qquad}_{\text{equivariant wrt } U_p\text{-action}} \left(\left(\frac{d}{dz}\right)^{k-1} \circ \varphi\right)(x) := \left(\frac{d}{dz}\right)^{k-1}(\varphi(x)).$$

$p^{k-1}U_p$-equivariant here

(2) $C^{(\varepsilon)}(w_k, t) = C^{(\varepsilon)}(w_{2-k}, p^{k-1}t) = \mathrm{char}(U_p, S_k^{Iw}(\psi))$.

(3) All finite U_p-slopes that are $< k-1$ belong to $S_k^{Iw}(\psi)$.

Proof (1) By direct computation.

(2) · $\mathrm{Ker}\left(\left(\frac{d}{dz}\right)^{k-1} \circ (-)\right)$ is spanned by $B_k^{(\varepsilon)}$

$\rightsquigarrow$ $\left(\frac{d}{dz}\right)^{k-1}(e_1^* \cdot z^j) = \underbrace{j \cdot (j-1) \cdots (j-k+2)}_{\text{"integrating is not integral"}}\, e_1^* \cdot z^{j-k}$.

· $$U_k^{\dagger,(\varepsilon)} = \begin{pmatrix} U_k^{Iw} & 0 \\ 0 & p^{k-1} D^{-1} U_{2-k}^{\dagger,(\varepsilon)} D \end{pmatrix}$$

D = diagonal matrix given by the coefficients $j(j-1)\cdots(j-k+2)$.

$\Rightarrow$ (2).

(3) U_p-slopes of $S^{\dagger}_{2-k}(\psi)$ is nonnegative

$\Rightarrow$ if U_p-slope of f is $< k-1$, then $\left(\frac{d}{dz}\right)^{k-1}(f) = 0$.

Prop (Atkin-Lehner involution). $\psi = \psi_1 \times \psi_2$, $\psi^s = \psi_2 \times \psi_1$ chars of Δ^2.

$$\varepsilon'' = \varepsilon \cdot \psi^s \cdot \psi^{-1}.$$

(1) $\exists$ natural morphism

$$\begin{aligned} AL_{(k,\psi)} : S_k^{Iw}(\psi) &\longrightarrow S_k^{Iw}(\psi^s) \\ \varphi &\longmapsto \left(x \mapsto \varphi\left(x \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}\right)\right)\Big|_{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}} \Big). \end{aligned}$$

(2) $\forall j \geq 0$, $l = 1, \dots, d_k^{Iw}(\psi^s)$, $i = 1, 2$

$$AL_{(k,\psi)}(e_i^* z^j) = p^{k-2-j} \cdot e_{3-i}^* \cdot z^{k-2-j}$$

$$\left(\Rightarrow \quad AL_{(k,\psi^s)} \circ AL_{(k,\psi)} = p^{k-2}. \right)$$

(3) When $\psi_1 \neq \psi_2$ ($\Leftrightarrow k \not\equiv k_\varepsilon \mod p-1$), we have

$$U_p \circ AL_{(k,\psi)} \circ U_p = p^{k-1} \cdot AL_{(k,\psi)}.$$

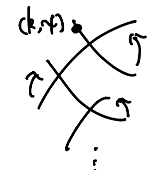
In this case, can pair U_p-slopes on $S_k^{Iw}(\psi)$ & $S_k^{Iw}(\psi^s)$

s.t. each pair adds up to $k-1$.

Rmk If $\psi_1 \neq \psi_2$, then (k,ψ), (k,ψ^s) are often on different wt disks.

Some Ping-Pong type phenomenon:

$(k,\psi) \rightsquigarrow (k',\psi') \rightsquigarrow (k,\psi)$.



From now on, assume $k \equiv k_\varepsilon$.

Then Atkin-Lehner $\rightsquigarrow$ $S_k^{Iw}(\tilde{\varepsilon}_1) \longrightarrow S_k^{Iw}(\tilde{\varepsilon}_2)$ $\quad (\varepsilon = \varepsilon_1 \times \varepsilon_2)$.

Fix $\eta : \Delta \to \mathcal{O}^\times$, $\tilde{\eta} := \eta \times \eta : \Delta^2 \to \mathcal{O}^\times$.

Def'n

$$\begin{tikzcd} S_k^{ur}(\varepsilon_1) \arrow[r, bend left=40, "i_1"] \arrow[r, bend left=15, "i_2"] & S_k^{Iw}(\tilde{\varepsilon}_1) \arrow[l, bend left=15, "pr_1"] \arrow[l, bend left=40, "pr_2"] \end{tikzcd}.$$

$i_1(\varphi) = \varphi$, $i_2(\varphi) = AL(\varphi)$

$$pr_1(\varphi)(x) = \sum_{j=0,\dots,p-1,\star} \varphi(x u_j)\big|_{u_j^{-1}}$$

$$pr_2(\varphi)(x) = \sum_{j=0,\dots,p-1,\star} \varphi\left(x \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix} u_j\right)\Big|_{u_j^{-1}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}.$$

$$\quad\quad = pr_1(AL(\varphi))(x).$$

Prop $U_p(\varphi) = i_2(pr_1(\varphi)) - AL(\varphi), \quad \varphi \in S_k^{Iw}(\widetilde{\varepsilon})$

$$= AL(pr_1(\varphi)) - AL(\varphi).$$

Put U_k^{Iw} = matrix of U_p w.r.t. $B_k^{(\varepsilon)}$

L_k^{cl} = matrix of AL w.r.t. $B_k^{(\varepsilon)}$.

Prop (1) L_k^{cl} is anti-diag with entries $p^{\deg e_1^{(\varepsilon)}}, p^{\deg e_2^{(\varepsilon)}}, \cdots$

(where $e_1^{(\varepsilon)}, e_2^{(\varepsilon)}, \cdots$ is the power basis indexed by exponents of z)

from upper right to lower left.

(2) $U_k^{Iw} = -L_k^{cl} + \square$, where $\square$ has $rk \le d_k^{ur}(\varepsilon_1)$.

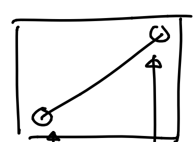
size $(d_k^{Iw} \times d_k^{Iw})$

Proof (2) $\square = U_k^{Iw} + L_k^{cl}$ is the matrix corresponds to $\varphi \mapsto L_2(pr_1(\varphi))$. $\square$

Consider $d_k^{new}(\varepsilon_1) = d_k^{Iw}(\varepsilon_1) - 2d_k^{ur}(\varepsilon_1)$.

Cor The multiplicities of $\pm p^{(k-2)/2}$ as eigenvals of U_p on $S_k^{Iw}(\widetilde{\varepsilon}_1)$

are at least $\frac{1}{2} d_k^{new}(\varepsilon_1)$ each.



Proof $U_k^{Iw} \pm p^{\frac{k-2}{2}} I = -\underbrace{(L_k^{cl} \pm p^{\frac{k-2}{2}} I)}_{\text{rank } \frac{1}{2} d_k^{Iw}} + \underset{\text{rank} \le d_k^{ur}}{\square}$

L_k^{cl} has eigenvals $\pm p^{\frac{k-2}{2}}$ each with multiplicity $\frac{1}{2} d_k^{Iw}$. two entries for eigenvals.

$\Rightarrow$ rank $U_k^{Iw} \pm p^{\frac{k-2}{2}} I \le d_k^{ur} + \frac{1}{2} d_k^{Iw}$

$\Rightarrow$ Corank $U_k^{Iw} \pm p^{\frac{k-2}{2}} I \ge \frac{1}{2} d_k^{Iw} - d_k^{ur} = \frac{1}{2} d_k^{new}$. $\square$

Cor (Weak corank thm)

Write $U^{\dagger,(\varepsilon)}(\underline{n}) \in M_n(\mathcal{O}\langle \frac{w}{p} \rangle)$ for the upper-left $n \times n$-submatrix of $U^{\dagger,(\varepsilon)}$.

Then $p^{-\deg g_n^{(\varepsilon)}} g_n^{(\varepsilon)}(w) \mid \det(U^{\dagger,(\varepsilon)}(\underline{n})) \in \mathcal{O}\langle \frac{w}{p} \rangle$.

~~Proof~~ Need to show: $\forall\ k \equiv k_\varepsilon \bmod p-1$ s.t. $m_n(k) > 0$,

$$\left(\frac{w}{p} - \frac{w_k}{p}\right)^{m_n(k)} \Big| \det\left(U^\dagger(\underline{n})\right).$$

(note: coeffs $\in \mathcal{O}\langle\frac{w}{p}\rangle \leadsto$ need to devide each ghost factor by p).

$\Leftarrow$ (eval $U^\dagger(\underline{n})$ at $w = w_k$) $= U_k^\dagger(\underline{n})$ has corank $\geq m_n(k)$.

Indeed, for $L_k^{cl}(\underline{n}) :=$ upper left $n\times n$-submat of L_k^{cl}.

Prop (1)(2) above $\Rightarrow$ $\operatorname{rank}(U_k^\dagger(\underline{n})) \leq d_k^{ur} + \operatorname{rank} L_k^{cl}(\underline{n})$

$$= \begin{cases} d_k^{ur}, & n \leq \frac{1}{2} d_k^{Iw}, \\ d_k^{ur} + 2\left(n - \frac{1}{2} d_k^{Iw}\right), & n \geq \frac{1}{2} d_k^{Iw}. \end{cases}$$

$$\Rightarrow \operatorname{corank} U_k^\dagger(\underline{n}) \geq \begin{cases} n - d_k^{ur} & \text{if } n \leq \frac{1}{2} d_k^{Iw}. \\ d_k^{Iw} - d_k^{ur} - n & \text{if } n \geq \frac{1}{2} d_k^{Iw}. \end{cases}$$

$\Rightarrow$ $\operatorname{corank} U_k^\dagger(\underline{n}) \geq m_n(k)$. $\square$