Lecture 4: Basic properties of ghost series

Fix $p \geq 5$. $E/\mathbb{Q}_p$ finite ext'n, $\varpi \in \mathcal{O} \subseteq E \rightsquigarrow \mathbb{F} = \mathcal{O}/(\varpi)$.

Fix a residual rep'n $\bar{\rho} : \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix} \cdot I_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$ with $1 \leq a \leq p-4$.

Let $\widetilde{H}$ be a primitive $\mathcal{O}[\![K_p]\!]$-proj augmented mod

of type $\bar{\rho}$ on which $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially.

$\rightsquigarrow \widetilde{H}$ $\mathcal{O}[\![K_p]\!] = \mathcal{O}[\![GL_2(\mathbb{Z}_p)]\!]$-mod

where $GL_2(\mathbb{Z}_p)$-action extends to $GL_2(\mathbb{Q}_p)$.

Fix a character $\varepsilon = \varepsilon_1 \times \varepsilon_2 = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^2 \rightarrow \mathcal{O}^\times$

$\Delta \subseteq \mathbb{Z}_p^\times$ torsion subgrp, $\omega : \Delta \rightarrow \mathcal{O}^\times$ Teichmuller char.

Relevant to $\bar{\rho}$: $s_\varepsilon \in \{0, \ldots, p-2\}$.

for $k \geq 2$, $\psi : \Delta^2 \rightarrow \mathcal{O}^\times$

$\rightsquigarrow$ we define $S_k^{\mathrm{Iw}}(\psi) = \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \psi) \leftrightarrow S_k(\Gamma \cap \Gamma_0(p), \psi)$.

and $d_k^{\mathrm{Iw}}(\psi) = \mathrm{rank}_{\mathcal{O}} S_k^{\mathrm{Iw}}(\psi)$.

$\rightsquigarrow \mathcal{O}[z]^{\leq k-2}$ subspace of $\mathcal{O}\langle z \rangle$ consisting of polynomials of deg $\leq k-2$.

$$f\Big|\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = (\gamma z + \delta)^{k-2} f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in K_p, \ f \in \mathcal{O}[z]^{\leq k-2}.$$

For $\eta : \Delta \rightarrow \mathcal{O}^\times$ we define

$S_k^{\mathrm{ur}}(\eta) = \mathrm{Hom}_{K_p}(\widetilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \eta \circ \det) \longleftrightarrow S_k(\Gamma, \eta)$, $\Gamma = \Gamma_0(N)$, $(N, p) = 1$.

$d_k^{\mathrm{ur}}(\eta) = \mathrm{rank}_{\mathcal{O}} S_k^{\mathrm{ur}}(\eta)$.

<u>Goal</u> To give formulas to compute $d_k^{\mathrm{Iw}}(\psi)$ & $d_k^{\mathrm{ur}}(\eta)$

in terms of k, ψ, and η (with a, p).

For $\eta : \Delta \rightarrow \mathcal{O}^\times$ we set $\tilde{\eta} : \Delta^2 \rightarrow \mathcal{O}^\times$, $\tilde{\eta} = \eta \times \eta$

if $S_k^{Iw}(\psi)$ belongs to $\mathcal{W}^{(\varepsilon)}$

$\Leftrightarrow$ $(1\times\omega^{k-2})\cdot\psi = \varepsilon \Rightarrow \psi = \varepsilon(1\times\omega^{2-k})$.

<u>Computation</u> $d_k^{Iw}(\psi) = d_k^{Iw}(\varepsilon(1\times\omega^{2-k}))$.

$\Delta^2 \subseteq Iw_p$, $Iw_{p,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \subseteq K_p$.

Recall $\tilde{H}$ is a free $\mathcal{O}[[Iw_{p,1}]]$-mod of rk 2.

We can choose a basis $\{e_1, e_2\}$

s.t. Δ^2 acts on e_1 (resp. e_2) via the character $1\times\omega^a$ (resp. $\omega^a\times 1$).

<u>Rmk</u> Can further assume $e_1\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = e_2$,

$\mathcal{O}[z]^{\le k-2}\otimes\psi$ has a basis $\{1, z, \cdots, z^{k-2}\}$

Δ^2 acts on z^i via the character $\omega^{i-s_\varepsilon}\times\omega^{a+s_\varepsilon-i}$

$\Rightarrow$ the $\mathcal{O}$-module $S_k^{Iw}(\psi)$ has an $\mathcal{O}$-basis

$\{e_1^*\cdot z^i \mid i\equiv s_\varepsilon \bmod p-1,\ 0\le i\le k-2\} \cup \{e_2^*\cdot z^j \mid j\equiv a+s_\varepsilon \bmod p-1,\ 0\le j\le k-2\}$.

$\leadsto$ $(e_1^*\cdot z^i : e_1\longmapsto z^i,\ e_2\longmapsto 0) \in \mathrm{Hom}_{Iw_p}(\tilde{H}, \mathcal{O}[z]^{\le k-2}\otimes\psi)$.

<u>Prop 4.1</u> We have

$$d_k^{Iw}(\psi) = d_k^{Iw}(\varepsilon\cdot(1\times\omega^{2-k})) = \left\lfloor\frac{k-2-s_\varepsilon}{p-1}\right\rfloor + \left\lfloor\frac{k-2-\{a+s_\varepsilon\}}{p-1}\right\rfloor + 2.$$

$\forall m\in\mathbb{Z}$, $\{m\}$ is the unique integer $\in\{0,\cdots,p-2\}$ s.t. $m\equiv\{m\} \bmod p-1$.

In particular, $d_{k+(p-1)}^{Iw}(\varepsilon(1\times\omega^{2-k})) - d_k^{Iw}(\varepsilon(1\times\omega^{2-k})) = 2$.

$d_k^{Iw}(\varepsilon(1\times\omega^{2-k})) = \frac{2k}{p-1} + O(1)$.

- When $\psi = \varepsilon(1\times\omega^{2-k}) = \varepsilon_1\times\varepsilon_2\omega^{2-k} = \varepsilon_1\times\varepsilon_1 = \tilde{\varepsilon}_1$,

  we may have $S_k^{ur}(\varepsilon_1) \subseteq S_k^{Iw}(\psi)$.

- Recall we define $k_\varepsilon = 2+\{a+2s_\varepsilon\}$.

  we consider $k\equiv k_\varepsilon \bmod (p-1) \Rightarrow \psi = \varepsilon\cdot(1\times\omega^{2-k}) = \tilde{\varepsilon}_1$

The number $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ $(d_k^{\mathrm{ur}}(\varepsilon_1))$ is used in the def'n of ghost series.

- $k = k_\varepsilon + (p-1)k_0$

$\rightsquigarrow$ Define $\delta_\varepsilon = \left\lfloor \frac{s_\varepsilon + \{a + s_\varepsilon\}}{p-1} \right\rfloor = \begin{cases} 0, & s_\varepsilon + \{a + s_\varepsilon\} < p-1 \\ 1, & s_\varepsilon + \{a + s_\varepsilon\} \geq p-1. \end{cases}$

<u>Cor</u> For the above k,

$$d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = 2k + 2 - 2\delta_\varepsilon.$$

In particular, $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$, $d_k^{\mathrm{new}} = d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - 2d_k^{\mathrm{ur}}(\varepsilon_1)$ are even.

$(k \mapsto k + (p-1),\ d_k^{\mathrm{Iw}}(\tilde{\varepsilon}) \longrightarrow d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) + 2)$

Computation of $d_k^{\mathrm{ur}}(\tilde{\varepsilon}_1)$:

$k \equiv k_\varepsilon \mod p-1$, $k_\varepsilon = 2 + \{a + 2s_\varepsilon\}$

For two integers $a \geq 0$, b,

use $\sigma_{a,b}$:= right rep of $GL_2(\mathbb{F}_p)$, say $\mathrm{Sym}^a \mathbb{F}^{\oplus 2} \otimes \det^b$.

When $0 \leq a \leq p-1$, $\sigma_{a,b}$ irred, we let

$\mathrm{proj}_{a,b}$ = proj envelop of $\sigma_{a,b}$ in $\mathrm{Rep}_{\mathbb{F}[GL_2(\mathbb{F}_p)]}$.

$\tilde{H} = \mathcal{O}[\![K_p]\!]$-module,

$\rightsquigarrow$ $\tilde{H}/(\varpi, \underbrace{I_{1+pM_2(\mathbb{Z}_p)}}_{K_1''}) \cong \mathrm{Proj}_{a,0}$ as $\mathbb{F}[GL_2(\mathbb{F}_p)]$-mod.

For $k \equiv k_\varepsilon \mod p-1$,

$$\begin{aligned} d_k^{\mathrm{ur}}(\varepsilon_1) &= \mathrm{rank}_{\mathcal{O}} \mathrm{Hom}_{K_p}(\tilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \varepsilon_1 \circ \det) \\ &= \mathrm{rank}_{\mathbb{F}} (\mathrm{Proj}_{a,0}, \sigma_{k-2} \otimes \det^{s_\varepsilon}) \\ &= \mathrm{rank}_{\mathbb{F}} (\mathrm{Proj}_{a,s_\varepsilon}, \sigma_{k-2}) = \mathrm{Multi}_{\sigma_{a,s_\varepsilon}}(\sigma_{k-2,0}). \end{aligned}$$

We define $t_1 < t_2$ as follows:

(i) When $a + s_\varepsilon < p-1$, set $t_1 = s_\varepsilon + \delta_\varepsilon$, $t_2 = a + s_\varepsilon + \delta_\varepsilon + 2$,

(ii) When $a+\delta_\varepsilon \geq p-1$, set $t_1 = \{a+\delta_\varepsilon\} + \delta_\varepsilon + 1$, $t_2 = \delta_\varepsilon + \delta_\xi + 1$.

<u>Prop</u> $k = k_\varepsilon + (p-1)k_0$. We have

$$d_k^{ur}(\tilde{\varepsilon}_1) = \left\lfloor \frac{k_0 - t_1}{p+1} \right\rfloor + \left\lfloor \frac{k_0 - t_2}{p+1} \right\rfloor + 2 . \quad (k \mapsto d_k^{ur}).$$

In particular, we have

$$d^{ur}_{k+(p-1)(p+1)}(\tilde{\varepsilon}_1) - d^{ur}_k(\tilde{\varepsilon}_1) = 2$$

and $d_k^{ur}(\varepsilon_1) = \frac{2k}{p^2-1} + O(1)$, $d_k^{new}(\varepsilon_1) = \frac{2k}{p+1} + O(1)$.

§ Application of dimension formulas

(1) Refined spectral halo of eigencurves.

Recall $k \in \mathbb{Z}$, $w_k = \exp((k-2)p) - 1 \in \mathfrak{m}_{\mathbb{C}_p}$.

For $k \equiv k_\varepsilon \bmod p-1$,

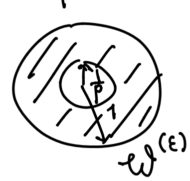
we define $\{m_n(k)\}_{n\geq 1}$ as a seq of integers:

$$\underbrace{0, \cdots, 0}_{d_k^{ur}(\varepsilon_1)}, 1, 2, 3, \cdots, \tfrac{1}{2} d_k^{new}, \tfrac{1}{2} d_k^{new} - 1, \cdots, 3, 2, 1, 0, \cdots$$

Take $g_n(w) = \prod_{\substack{k \equiv k_\varepsilon \\ k \geq 2}} (w - w_k)^{m_n(k)} \in \mathbb{Z}_p[w]$.

Define $G^{(\varepsilon)}(w,t) = G(w,t) = 1 + \sum_{n \geq 1} g_n(w) \cdot t^n \in \mathcal{O}[\![w,t]\!]$.

<u>Goal</u> Compute $NP(G^{(\varepsilon)}(w_\star, -))$ when $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$ lies in the halo range



i.e. $v_p(w_\star) \in (0,1)$.

For such $w_\star$ we have $v_p(w_\star - w_k) = v_p(w_\star)$

$\Rightarrow v_p(g_n(w_\star)) = \deg(g_n(w)) \cdot v_p(w_\star)$.

We define $\{d_n\}_{n\geq 1} = \{ d \equiv \delta_\varepsilon \text{ or } a+\delta_\varepsilon \bmod p-1,\ d \geq 0\}$.

$\lambda_n = d_n - \lfloor \frac{d_n}{p} \rfloor \longleftarrow$ Hodge bound.

We define the Hodge polygon whose nth slope $= \lambda_n v_p(w_\star)$, $n \geq 1$.

[LWX] $\Rightarrow$ $NP(C^{(\varepsilon)}(w_\star, -))$ lies on or above this ghost Hodge polygon.

essentially gives an estimation of the matrix for the operator

$$(-)\Big|\begin{pmatrix} pa & b \\ pc & d \end{pmatrix} \text{ on } C(\mathbb{Z}_p, \mathcal{O}[[w]]\langle \tfrac{p}{w} \rangle) \text{ w.r.t. } \left\{ \binom{z}{n} \,\Big|\, n \geq 0 \right\}.$$

<u>Prop</u> If $a + s_\varepsilon < p-1$ for $n \geq 0$,

$$\deg g_{n+1}(w) - \deg g_n(w) - \lambda_{n+1} = \begin{cases} 1, & n - 2s_\varepsilon \equiv 1, 3, \cdots, 2a+1 \mod 2p \\ -1, & n - 2s_\varepsilon \equiv 2, 4, \cdots, 2a+2 \mod 2p \\ 0, & \text{otherwise.} \end{cases}$$

If $a + s_\varepsilon \geq p-1$,

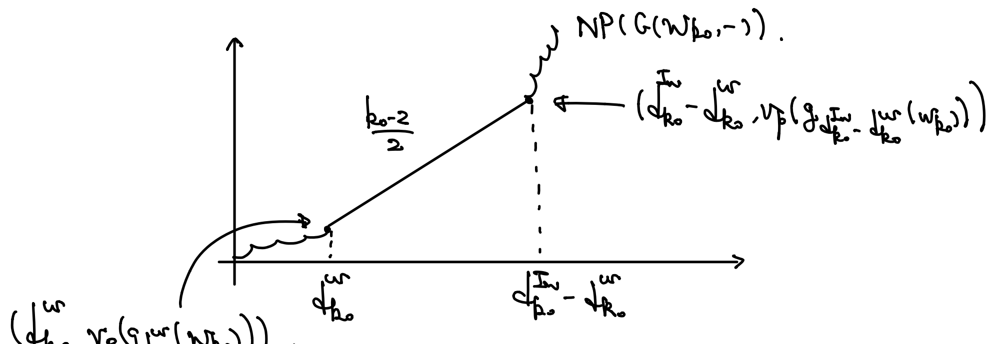
$$\deg g_{n+1}(w) - \deg g_n(w) - \lambda_{n+1} = \begin{cases} 1, & n - 2s_\varepsilon \equiv 2, 4, \cdots, 2a+2 \mod 2p \\ -1, & n - 2s_\varepsilon \equiv 3, 5, \cdots, 2a+3 \mod 2p. \\ 0, & \text{otherwise.} \end{cases}$$

<u>Rmk</u> If we set $a_n = \deg g_n(w) - \deg g_{n-1}(w)$, $n \geq 1$,

then (1) $\{a_n\}_{n\geq1}$ is strictly increasing

(2) $a_{n+2p} - a_n = (p-1)^2$, $n \geq 1$.

$\Rightarrow \{a_n\}_{n\geq1}$ is a disjoint union of $2p$ arithmetic progressions.



Application (Ghost duality)

For $k \equiv k_\varepsilon \bmod p-1$, for each $l = 0, \dots, \frac{1}{2} d_{k_0}^{new} - 1$,

we have

$$v_p\left(g_{d_{k_0}^{Iw} - d_{k_0}^{ur} - l, \hat{k}_0}(w_{k_0})\right) - v_p\left(g_{d_{k_0}^{ur} + l, \hat{k}_0}(w_{k_0})\right) = \frac{k_0 - 2}{2}\left(d_{k_0}^{new} - 2l\right).$$

Here $\displaystyle g_{n, \hat{k}_0}(w) := \prod_{\substack{k \equiv k_\varepsilon (p-1) \\ k \neq k_0}} (w - w_k)^{m_n(k)}.$

$$\left(g_n(w) = \prod_{\substack{k \equiv k_\varepsilon (p-1) \\ k \geq 2}} (w - w_k)^{m_n(k)}. \right)$$

When $l = 0$, the slope of the line segment

connecting $\left(d_{k_0}^{ur}, v_p\left(g_{d_{k_0}^{ur}}(w_{k_0})\right)\right)$ and

$\left(d_{k_0}^{Iw} - d_{k_0}^{ur}, v_p\left(g_{d_{k_0}^{Iw} - d_{k_0}^{ur}}(w_{k_0})\right)\right)$ is $\frac{k_0 - 2}{2}$.

Question What is the meaning of ghost duality for $l \geq 1$?