Lecture 5: Further properties of ghost NP.

<u>Question</u> Fix $\varepsilon : \Delta^2 \to \mathcal{O}^\times$, a positive int n, and $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $v_p(w_\star) \geq 1$.

How to determine when $(n, v_p(g_n^{(\varepsilon)}(w_\star)))$ is a vertex of $NP(G^{(\varepsilon)}(w_\star, -))$ or not?

<u>Later</u> To prove $NP(G(w_\star, -)) = NP(C(w_\star, -))$,

it suffices to prove two claims:

<u>Claim 1</u> $\forall n \geq 1$, $(n, v_p(C_n(w_\star)))$ lies on or above $NP(G(w_\star, -))$.

<u>Claim 2</u> If $(n, v_p(g_n(w_\star)))$ is a vertex of $NP(G^{(\varepsilon)}(w_\star, -))$

then $v_p(g_n(w_\star)) = v_p(C_n(w_\star))$.

If $w_\star = w_k$ for some $k \equiv k_\varepsilon \bmod p-1$, and

n lies in the Steinberg range of w_k, i.e. $n \in \left(\frac{1}{2} d_k^{Iw} - \frac{1}{2} d_k^{ur}, d_k^{Iw} - d_k^{ur}\right)$

then $(n, v_p(g_n(w_\star)))$ is not a vertex.

<u>Intuition</u> (1) If $w_\star$ is "very closed" to w_k and n lies in some subinterval of $(d_k^{ur}, d_k^{Iw} - d_k^{ur})$, then

$$NP(G(w_\star, -)) \approx NP(G(w_\star, -))$$

and $(n, v_p(g_n(w_\star)))$ will not be a vertex.

(2) Consider $n = d_k^{ur} + 1$.

Assume w_* is "closed" to w_k, so that

$$v_p(w_* - w_k) > v_p(w_k' - w_k),$$

where w_k' is any zero of $g_l(w)$, $l = d_k^{ur}, d_k^{ur}+1, d_k^{Iw} - d_k^{ur}$.

$\Rightarrow$ $v_p(w_* - w_k') = v_p(w_k' - w_k)$ other than w_k.

Under this assumption,

we have $v_p(g_l(w_\star)) = v_p(g_l(w_k))$ for $l = d_k^{ur}, d_k^{Iw} - d_k^{ur}$.

$$P_1 = (d_k^{ur}, v_p(g_{d_k^{ur}}(w_\star))), \quad P_2 = (d_k^{Iw} - d_k^{ur}, v_p(g_{d_k^{Iw} - d_k^{ur}}(w_\star))).$$

Ghost duality $\Rightarrow$ the slope of $\overline{P_1P_2}$ is $\frac{k-2}{2}$.

Consider the pt $P(d_k^{ur}+1, v_p(g_{d_k^{ur}+1}(w_\star)))$

$$v_p(g_{d_k^{ur}+1}(w_\star)) = v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1,\hat{k}}(w_\star))$$
$$= v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1,\hat{k}}(w_k)).$$

- P lies on or above $\overline{P_1P_2}$

$$\Leftrightarrow \text{ slope of } \overline{P_1P_2} \geq \frac{k-2}{2}$$
$$\Leftrightarrow v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1,\hat{k}}(w_k)) \geq \frac{k-2}{2}$$
$$\Leftrightarrow v_p(w_\star - w_k) \geq v_p(g_{d_k^{ur}}(w_k)) - v_p(g_{d_k^{ur}+1,\hat{k}}(w_k)) + \frac{k-2}{2}. \quad (*)$$

- Introduce $\Delta'_{k,l} = v_p(g_{\frac{1}{2}d_k^{Iw}+l,\hat{k}}(w_k)) - \frac{k-2}{2}\cdot l$

$$l = -\tfrac{1}{2}d_k^{new}, -\tfrac{1}{2}d_k^{new}+1, \cdots, 0, 1, \cdots, \tfrac{1}{2}d_k^{new}.$$

(Ghost duality $\Leftrightarrow \Delta'_{k,l} = \Delta'_{k,-l}$).

$$(*) \Leftrightarrow v_p(w_\star - w_k) \geq \Delta'_{k,-\frac{1}{2}d_k^{new}} - \Delta'_{k,-\frac{1}{2}d_k^{new}+1}$$
$$= \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}.$$

- Consider the pt $P(d_k^{ur}+2, v_p(g_{d_k^{ur}+1}(w_\star)))$.

$$(*) \xrightarrow{\text{Ghost duality}} P'(d_k^{Iw} - d_k^{ur} - 1, v_p(g_{d_k^{Iw} - d_k^{ur}-1}(w_\star)))$$

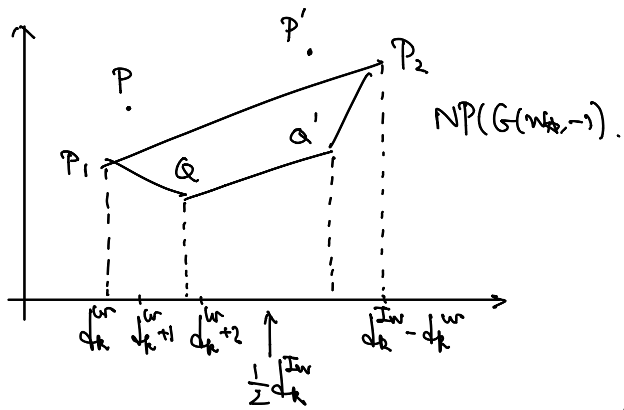
lies on or above $\overline{P_1P_2}$.

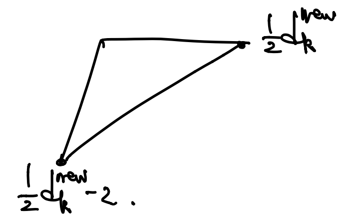
- Consider the pt $Q(d_k^{ur}+2, v_p(g_{d_k^{ur}+2}(w_\star)))$.

A similar computation

$\rightsquigarrow$ Q lies below $\overline{P_1P_2} \Leftrightarrow v_p(w_\star - w_k) < \frac{1}{2}(\Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-2})$

P lies below $\overline{P_1P_2} \Leftrightarrow v_p(w_\star - w_k) < \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}$.





If $\frac{1}{2}\left(\Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-2}\right) > \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}$

$\iff 2\Delta'_{k,\frac{1}{2}d_k^{new}-1} > \Delta'_{k,\frac{1}{2}d_k^{new}} + \Delta'_{k,\frac{1}{2}d_k^{new}-2}$.

$(l, \Delta'_{k,l})$ for $l = \frac{1}{2}d_k^{new}-2, \dots, \frac{1}{2}d_k^{new}$ are not lower vertices.

Def'n Let Δ_k = lower convex hull of the pts $(l, \Delta'_{k,l})$, $|l| \leq \frac{1}{2}d_k^{new}$.
For such l, $(l, \Delta_{k,l})$ is the corresponding pt on Δ_k.

Def'n (Near Steinberg range)
Fix ε and $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$.
For $k \equiv k_\varepsilon\ (p-1)$ we define $L_{w_\star,k}$ to be the largest int (if any) in $\{1, \dots, \frac{1}{2}d_k^{new}\}$ s.t. $v_p(w_\star - w_k) \geq \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}$.
$(L_{w_\star,k}, \Delta_{k,L_{w_\star,k}})$ must be a vertex of Δ_k.

$(\Delta_{k,L_{w_\star,k}} = \Delta'_{k,L_{w_\star,k}})$

Rmk (1) The intuition of $nS_{w_\star,k}$ is that
$w_\star$ is close to w_k s.t. $NP(G(w_\star,-)) \approx NP(G(w_k,-))$
on the interval $\overline{nS_{w_\star,k}}$ where $NP(G(w_\star,-))$ has a long straight line on $\overline{nS_{w_\star,k}}$.

(2) In the def'n of $(w_\star, n)$ being near-Steinberg

the wt w_k may not be unique!

(3) In our previous discussion we assume

$$v_p(w_\star - w_k) > v_p(w_\star - w_{k'}) \text{ for all other ghost zeroes.}$$

<u>Some results on $\Delta'_{k,l}$ or $\Delta_{k,l}$</u> $(l, \Delta'_{k,l})$

<u>Lemma</u> $\Delta'_{k,l+1} - 2\Delta'_{k,l} + \Delta'_{k,l-1} \geq l - 2v_p(l), \ \forall l \geq 1.$

$\Rightarrow \Delta_{k,l} = \Delta'_{k,l}$ for $1 \leq l \leq 2p, \ l \neq p$.

<u>Lemma</u> $\Delta'_{k,l} - \Delta'_{k,l-1} \geq \frac{1}{2}\min\{a+2, p-1-a\} + \frac{1}{2}(p-1)(l-1)$

$$\geq \frac{3}{2} + \frac{p-1}{2}(l-1).$$

(This inequality is very sharp.)

<u>Theorem</u> Fix $w_\star$.

(1) The set of near Steinberg ranges $\mathrm{nS}_{w_\star,k}$ for all k is nested,

i.e. for any two such open intervals,

either they are disjoint

or one is contained in the other.

(2) The x-coordinates of vertices of $NP(G(w_\star, -))$

are exactly those integers which do not lie in any $\mathrm{nS}_{w_\star,k}$

i.e. $\forall n \geq 1$, $(n, w_\star)$ is near-Steinberg

$\Leftrightarrow$ $(n, v_p(g_n(w_\star)))$ is NOT a vertex of $NP(G(w_\star, -))$.

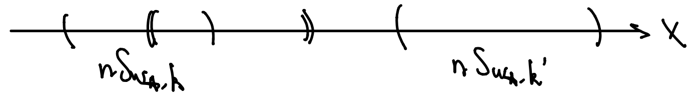
<u>Rmk</u> (1) If $\mathrm{nS}_{w_\star,k_1} \cap \mathrm{nS}_{w_\star,k_2} \neq \phi$

and we assume $L_{w_\star,k_1} \geq L_{w_\star,k_2}$,

then $L_{w_\star,k_1} \geq p^{1+\frac{p-1}{2}(L_{w_\star,k_2}-1)} - L_{w_\star,k_2} \gg L_{w_\star,k_2}$.

(2) Consider a pair $(w_\star, n)$.

Fix $w_\star$, those n s.t. $(n, v_p(g_n(w_\star)))$ is a vertex looking like



Consider another view. Fix n.

<u>Q</u> What is the set of $w_\star$

s.t. $(n, v_p(g_n(w_\star)))$ is not a vertex of $NP(G(w_\star, -))$

$\Updownarrow$ s.t. $(w_\star, n)$ is near Steinberg.

<u>Answer</u>. First $\exists!\ k = k(n)$ s.t. $k \equiv k_\varepsilon \mod p-1$.

$(n = \frac{1}{2} d_k^{Iw})$

then $n \in S_{w_\star, k} \iff v_p(w_\star - w_k) \geq \Delta_{k,1} - \Delta_{k,0} \approx \frac{a+2}{2}$ or $\frac{p-1-a}{2}$

$\approx \frac{p+1}{4}$.

These $w_\star$ corr to a disc at w_k of radius $\frac{p+1}{4}$.

· Next consider: $k' = k \pm (p-1) \Rightarrow \frac{1}{2} d_{k'}^{Iw} = n \pm 1$.

$n \in n S_{w_\star, k'} \iff v_p(w_\star - w_{k'}) \geq \Delta_{k,2} - \Delta_{k,1} \approx \frac{p+1}{2}$.

These corr to two discs centered at k's with radius $p^{-\frac{p+1}{2}}$.